CHAPTER 9
MAGNETIC POTENTIAL

9.1 *Introduction*

We are familiar with the idea that an electric field **E** can be expressed as minus the gradient of a potential function *V*. That is

$$\mathbf{E} = -\mathbf{grad}\, V = -\nabla V. \qquad 9.1.1$$

Note that *V* is not unique, because an arbitrary constant can be added to it. We can define a unique *V* by assigning a particular value of *V* to some point (such as zero at infinity).

Can we express the magnetic field **B** in a similar manner as the gradient of some potential function ψ, so that, for example, $\mathbf{B} = -\mathbf{grad}\,\psi = -\nabla\psi$? Before answering this, we note that there are some differences between **E** and **B** . Unlike **E**, the magnetic field **B** is *sourceless*; there are no sources or sinks; the magnetic field lines are closed loops. The force on a charge *q* in an electric field is $q\mathbf{E}$, and it depends only on where the charge is in the electric field – i.e. on its position. Thus the force is *conservative*, and we understand from any study of classical mechanics that only conservative forces can be expressed as the derivative of a potential function. The force on a charge *q* in a *magnetic* field is $q\mathbf{v} \times \mathbf{B}$. This force (the Lorentz force) does not depend only on the position of the particle, but also on its velocity (speed and direction). Thus the force is not conservative. This suggests that perhaps we cannot express the magnetic field merely as the gradient of a scalar potential function – and this is correct; we cannot.

9.2 *The Magnetic Vector Potential*

Although we cannot express the magnetic field as the gradient of a scalar potential function, we shall define a *vector* quantity **A** whose **curl** is equal to the magnetic field:

$$\mathbf{B} = \mathbf{curl}\,\mathbf{A} = \nabla \times \mathbf{A}\,. \qquad 9.2.1$$

Just as $\mathbf{E} = -\nabla V$ does not define *V* uniquely (because we can add an arbitrary constant to it, so, similarly, equation 9.2.1 does not define **A** uniquely. For, if ψ is some scalar quantity, we can always add $\nabla\psi$ to **A** without affecting **B**, because $\nabla \times \nabla\psi = \mathbf{curl\,grad}\,\psi = 0$.

The vector **A** is called the *magnetic vector potential*. Its dimensions are $\mathrm{MLT^{-1}Q^{-1}}$. Its SI units can be expressed as T m, or $\mathrm{Wb\ m^{-1}}$ or $\mathrm{N\ A^{-1}}$.

It might be briefly noted here that some authors define the magnetic vector potential from **H** = **curl A**, though it is standard SI practice to define it from **B** = **curl A**. Systems of units and definitions other than SI will be dealt with in Chapter 16.

Now in electrostatics, we have $\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}}$ for the electric field near a point charge, and, with $\mathbf{E} = -\mathbf{grad}\ V$, we obtain for the potential $V = \frac{q}{4\pi\epsilon r}$. In electromagnetism we have $\mathbf{dB} = \frac{\mu I}{4\pi r^2}\hat{\mathbf{r}} \times \mathbf{ds}$ for the contribution to the magnetic field near a circuit element $\mathbf{ds}$. Given that $\mathbf{B} = \mathbf{curl\ A}$, can we obtain an expression for the magnetic vector potential from the current element? The answer is yes, if we recognize that $\hat{\mathbf{r}}/r^2$ can be written $-\nabla(1/r)$. (If this isn't obvious, go to the expression for $\nabla\psi$ in spherical coordinates, and put $\psi = 1/r$.) The Biot-Savart law becomes

$$\mathbf{dB} = -\frac{\mu I}{4\pi}\nabla(1/r) \times \mathbf{ds} = \frac{\mu I}{4\pi}\ \mathbf{ds} \times \nabla(1/r). \qquad 9.2.3$$

Since $\mathbf{ds}$ is independent of r, the nabla can be moved to the left of the cross product to give

$$\mathbf{dB} = \nabla \times \frac{\mu I}{4\pi r}\mathbf{ds}. \qquad 9.2.4$$

The expression $\frac{\mu I}{4\pi r}\mathbf{ds}$, then, is the contribution $\mathbf{dA}$ to the magnetic vector potential from the circuit element $\mathbf{ds}$. Of course an isolated circuit element cannot exist by itself, so, for the magnetic vector potential from a complete circuit, the line integral of this must be calculated around the circuit.

9.3 *Long, Straight, Current-carrying Conductor*

By way of example, let us use the expression $\mathbf{dA} = \frac{\mu I}{4\pi r}\mathbf{ds}$, to calculate the magnetic vector potential in the vicinity of a long, straight, current-carrying conductor ("wire" for short!). We'll suppose that the wire lies along the z-axis, with the current flowing in the direction of positive z. We'll work in cylindrical coordinates, and the symbols $\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$ will denote the unit orthogonal vectors. After we have calculated $\mathbf{A}$, we'll try and calculate its **curl** to give us the magnetic field $\mathbf{B}$. We already know, of course, that for a straight wire the field is $\mathbf{B} = \frac{\mu I}{2\pi\rho}\hat{\phi}$, so this will serve as a check on our algebra.

Consider an element $\hat{\mathbf{z}}\,dz$ on the wire at a height z above the xy-plane. (The length of this element is dz; the unit vector $\hat{\mathbf{z}}$ just indicates its direction.) Consider also a point P in the xy-plane at a distance ρ from the wire. The distance of P from the element dz is $\sqrt{\rho^2 + z^2}$. The contribution to the magnetic vector potential is therefore

$$\mathbf{dA} = \hat{\mathbf{z}}\frac{\mu I}{4\pi}\cdot\frac{dz}{(\rho^2+z^2)^{1/2}}. \qquad 9.3.1$$

The total magnetic vector potential is therefore

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\int_0^\infty\frac{dz}{(\rho^2+z^2)^{1/2}}. \qquad 9.3.2$$

This integral is infinite, which at first may appear to be puzzling. Let us therefore first calculate the magnetic vector potential for a finite section of length $2l$ of the wire. For this section, we have

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\int_0^l\frac{dz}{(\rho^2+z^2)^{1/2}}. \qquad 9.3.3$$

To integrate this, let $z = \rho\tan\theta$, whence $\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\int_0^\alpha \sec\theta d\theta$, where $l = \rho\tan\alpha$. From this we obtain $\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\ln(\sec\alpha+\tan\alpha)$, whence

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\ln\left(\frac{\sqrt{l^2+\rho^2}+l}{\rho}\right). \qquad 9.3.4$$

For $l >> \rho$ this becomes

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu I}{2\pi}\cdot\ln\left(\frac{2l}{\rho}\right) = \hat{\mathbf{z}}\frac{\mu I}{2\pi}(\ln 2l - \ln\rho). \qquad 9.3.5$$

Thus we see that the magnetic vector potential in the vicinity of a straight wire is a vector field parallel to the wire. If the wire is of infinite length, the magnetic vector potential is infinite. For a finite length, the potential is given exactly by equation 9.3.4, and, very close to a long wire, the potential is given approximately by equation 9.3.5.

Now let us use equation 9.3.5 together with $\mathbf{B} = \mathbf{curl\ A}$, to see if we can find the magnetic field $\mathbf{B}$. We'll have to use the expression for **curl A** in cylindrical coordinates, which is

$$\mathbf{curl A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(A_\phi + \rho\frac{\partial A_\phi}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{z}}. \qquad 9.3.6$$

In our case, **A** has only a z-component, so this is much simplified:

$$\mathbf{curl A} = \frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\hat{\boldsymbol{\rho}} - \frac{\partial A_z}{\partial\rho}\hat{\boldsymbol{\phi}}. \qquad 9.3.7$$

And since the z-component of **A** depends only on ρ, the calculation becomes trivial, and we obtain, as expected

$$\mathbf{B} = \frac{\mu I}{2\pi\rho}\hat{\boldsymbol{\phi}}. \qquad 9.3.8$$

This is an approximate result for very close to a long wire – but it is exact for any distance for an infinite wire. This may strike you as a long palaver to derive equation 9.3.8 – but the object of the exercise was not to derive equation 9.3.8 (which is trivial from Ampère's theorem), but to derive the expression for **A**. Calculating **B** subsequently was only to reassure us that our algebra was correct.

9.4 *Long Solenoid*

Let us place an infinitely long solenoid of n turns per unit length so that its axis coincides with the z-axis of coordinates, and the current I flows in the sense of increasing ϕ. In that case, we already know that the field inside the solenoid is uniform and is $\mu n I \hat{\mathbf{z}}$ inside the solenoid and zero outside. Since the field has only a z component, the vector potential **A** cam have only a ϕ-component.

We'll suppose that the radius of the solenoid is a. Now consider a circle of radius r (less than a) perpendicular to the axis of the solenoid (and hence to the field **B**). The magnetic flux through this circle (i.e. the surface integral of **B** across the circle) is $\pi r^2 B = \pi r^2 nI$. Now, as everybody knows, the surface integral of a vector field across a closed curve is equal to the line integral of its **curl** around the curve, and this is equal to $2\pi r A_\phi$. Thus, inside the solenoid the vector potential is

$$\mathbf{A} = \tfrac{1}{2}\mu n r I \hat{\boldsymbol{\phi}}. \qquad 9.4.1$$

It is left to the reader to argue that, outside the solenoid ($r > a$), the magnetic vector potential is

$$\mathbf{A} = \frac{\mu n a^2 I}{2r}\hat{\boldsymbol{\phi}}. \qquad 9.4.2$$

9.5 *Divergence*

Like the magnetic field itself, the lines of magnetic vector potential form closed loops (except in the case of the infinitely long straight conducting wire, in which case they are infinitely long straight lines). That is to say **A** has no sources or sinks, or, in other words, its divergence is everywhere zero:

$$\text{div}\,\mathbf{A} = 0. \qquad 9.5.1$$