Solutions to Problems in Goldstein, *Classical Mechanics*, Second Edition

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Chapter 1

Problem 1.1

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV (million electron volt) is a unit of energy, used in modern physics, equal to 1.60 x 10^{-6} erg. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34 x 10^{-17} gm-cm/sec.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90 x 10^{-22} gm, what is its kinetic energy, in electron volts?

Place the nucleus at the origin, and suppose the electron is emitted in the positive y direction, and the neutrino in the positive x direction. Then the resultant of the electron and neutrino momenta has magnitude

$$|\mathbf{p}_{\mathbf{e}+\nu}| = \sqrt{(1.73)^2 + 1^2} = 2 \text{ MeV/c},$$

and its direction makes an angle

$$\theta = \tan^{-1}\frac{1.73}{1} = 60^\circ$$

with the x axis. The nucleus must acquire a momentum of equal magnitude and directed in the opposite direction. The kinetic energy of the nucleus is

$$T = \frac{p^2}{2m} = \frac{4 \text{ MeV}^2 \text{ c}^{-2}}{2 \cdot 3.9 \cdot 10^{-22} \text{ gm}} \cdot \frac{1.78 \cdot 10^{-27} \text{ gm}}{1 \text{ MeV c}^{-2}} = 9.1 \text{ ev}$$

This is much smaller than the nucleus rest energy of several hundred GeV, so the non-relativistic approximation is justified.

Problem 1.2

The *escape velocity* of a particle on the earth is the minimum velocity required at the surface of the earth in order that the particle can escape from the earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for the earth, ignoring the presence of the moon, is 6.95 mi/sec.

If the particle starts at the earth's surface with the escape velocity, it will just manage to break free of the earth's field and have nothing left. Thus after it has escaped the earth's field it will have no kinetic energy left, and also no potential energy since it's out of the earth's field, so its total energy will be zero. Since the particle's total energy must be constant, it must also have zero total energy at the surface of the earth. This means that the kinetic energy it has at the surface of the earth must exactly cancel the gravitational potential energy it has there:

$$\frac{1}{2}mv_e^2 - G\frac{mM_R}{R_R} = 0$$

so

$$\begin{aligned} v = \sqrt{\left(\frac{2GM_R}{R_R}\right)} &= \left(\frac{2\cdot(6.67\cdot 10^{11}\ \text{m}^3\ \text{kg}^{-3}\ \text{s}^{-2})\cdot(5.98\cdot 10^{24}\ \text{kg})}{6.38\cdot 10^6\ \text{m}}\right)^{1/2} \\ &= 11.2\ \text{km/s}\cdot\frac{1\ \text{m}}{1.61\ \text{km}} = 6.95\ \text{mi/s}. \end{aligned}$$

Problem 1.3

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric resistance, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 6800 ft/sec and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Suppose that, at time t, the rocket has mass $m(t)$ and velocity $v(t)$. The total external force on the rocket is then $F = gm(t)$, with $g = 32.1$ ft/s^2, pointed downwards, so that the total change in momentum between t and $t + dt$ is

$$F dt = -gm(t)dt. \tag{1}$$

At time t, the rocket has momentum

$$p(t) = m(t)v(t). \tag{2}$$

On the other hand, during the time interval dt the rocket releases a mass Δm of gas at a velocity v' with respect to the rocket. In so doing, the rocket's velocity increases by an amount dv. The total momentum at time $t + dt$ is the sum of the momenta of the rocket and gas:

$$p(t + dt) = p_r + p_g = [m(t) - \Delta m][v(t) + dv] + \Delta m[v(t) + v'] \tag{3}$$

Subtracting (2) from (3) and equating the difference with (1), we have (to first order in differential quantities)

$$-gm(t)dt = m(t)dv + v'\Delta m$$

or

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\frac{\Delta m}{dt}$$

which we may write as

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\gamma \tag{4}$$

where

$$\gamma = \frac{\Delta m}{dt} = \frac{1}{60} m_0 s^{-1}.$$

This is a differential equation for the function $v(t)$ giving the velocity of the rocket as a function of time. We would now like to recast this as a differential equation for the function $v(m)$ giving the rocket's velocity as a function of its mass. To do this, we first observe that since the rocket is *releasing* the mass Δm every dt seconds, the time derivative of the rocket's mass is

$$\frac{dm}{dt} = -\frac{\Delta m}{dt} = -\gamma.$$

We then have

$$\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt} = -\gamma\frac{dv}{dm}.$$

Substituting into (4), we obtain

$$-\gamma\frac{dv}{dm} = -g - \frac{v'}{m}\gamma$$

or

$$dv = \frac{g}{\gamma}dm + v'\frac{dm}{m}.$$

Integrating, with the condition that $v(m_0) = 0$,

$$v(m) = \frac{g}{\gamma}(m - m_0) + v' \ln\left(\frac{m}{m_0}\right).$$

Now, γ=(1/60)m_0 s^{-1}, while v' =-6800 ft/s. Then

$$v(m) = 1930 \text{ ft/s} \cdot \left(\frac{m}{m_0} - 1\right) + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right)$$

For $m_0 \gg m$ we can neglect the first term in the parentheses of the first term, giving

$$v(m) = -1930 \text{ ft/s} + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right).$$

The escape velocity is $v = 6.95$ mi/s $= 36.7 \cdot 10^3$ ft/s. Plugging this into the equation above and working backwards, we find that escape velocity is achieved when m_0/m=293.

Thanks to Brian Hart for pointing out an inconsistency in my original choice of notation for this problem.

Problem 1.4

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

We have

$$\mathbf{F} = \dot{\mathbf{p}} \tag{5}$$

If m is constant,

$$\mathbf{F} = m\dot{\mathbf{v}}$$

Dotting $\mathbf{v}$ into both sides,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{v} = m\mathbf{v} \cdot \dot{\mathbf{v}} &= \frac{1}{2} m \frac{d}{dt} |\mathbf{v}|^2 \\ &= \frac{dT}{dt} \end{aligned} \tag{6}$$

On the other hand, if m is not constant, instead of $\mathbf{v}$ we dot $\mathbf{p}$ into (5):

$$\begin{aligned} \mathbf{F} \cdot \mathbf{p} &= \mathbf{p} \cdot \dot{\mathbf{p}} \\ &= m\mathbf{v} \cdot \frac{d(m\mathbf{v})}{dt} \\ &= m\mathbf{v} \cdot \left(\mathbf{v}\frac{dm}{dt} + m\frac{d\mathbf{v}}{dt} \right) \\ &= \frac{1}{2} v^2 \frac{d}{dt} m^2 + \frac{1}{2} m^2 \frac{d}{dt}(v^2) \\ &= \frac{1}{2} \frac{d}{dt}(m^2 v^2) = \frac{d(mT)}{dt}. \end{aligned}$$

Problem 1.5

> Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation
>
> $$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2.$$

We have

$$R_x = \frac{1}{M}\sum_i m_i x_i$$

so

$$R_x^2 = \frac{1}{M^2}\left[\sum_i m_i^2 x_i^2 + \sum_{i\neq j} m_i m_j x_i x_j\right]$$

and similarly

$$R_y^2 = \frac{1}{M^2}\left[\sum_i m_i^2 y_i^2 + \sum_{i\neq j} m_i m_j y_i y_j\right]$$

$$R_z^2 = \frac{1}{M^2}\left[\sum_i m_i^2 z_i^2 + \sum_{i\neq j} m_i m_j z_i z_j\right].$$

Adding,

$$R^2 = \frac{1}{M^2}\left[\sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j (\mathbf{r_i}\cdot\mathbf{r_j})\right]. \qquad (7)$$

On the other hand,

$$r_{ij}^2 = r_i^2 + r_j^2 - 2\mathbf{r_i}\cdot\mathbf{r_j}$$

and, in particular, $r_{ii}^2 = 0$, so

$$\begin{aligned}\sum_{i,j} m_i m_j r_{ij}^2 &= \sum_{i\neq j}[m_i m_j r_i^2 + m_i m_j r_j^2 - 2m_i m_j (r_i\cdot r_j)]\\ &= 2\sum_{i\neq j} m_i m_j r_i^2 - 2\sum_{i\neq j} m_i m_j (r_i\cdot r_j). \qquad (8)\end{aligned}$$

Next,

$$M\sum_i m_i r_i^2 = \sum_j m_j\left(\sum_i m_i r_i^2\right) = \sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j r_i^2. \qquad (9)$$

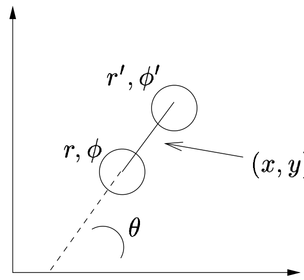

Figure 1: My conception of the situation of Problem 1.8

Subtracting half of (8) from (9), we have

$$M \sum m_i r_i^2 - \frac{1}{2} \sum ij m_i m_j r_{ij}^2 = \sum_i m_i^2 r_i^2 + \sum_{i \neq j} m_i m_j (\mathbf{r_i} \cdot \mathbf{r_j})$$

and comparing this with (7) we see that we are done.

Problem 1.8

Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\cos\theta \, dx + \sin\theta \, dy = 0$$

$$\sin\theta \, dx - \cos\theta \, dy = a(d\phi + d\phi')$$

(where θ, ϕ, and ϕ' have meanings similar to the problem of a single vertical disc, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

My conception of the situation is illustrated in Figure 1. θ is the angle between the x axis and the axis of the two wheels. ϕ and ϕ' are the rotation angles of the two wheels, and $\mathbf{r}$ and $\mathbf{r}'$ are the locations of their centers. The center of the wheel axis is the point just between $\mathbf{r}$ and $\mathbf{r}'$:

$$(x, y) = \frac{1}{2}(r_x + r'_x, r_y + r'_y).$$

If the ϕ wheel rotates through an angle $d\phi$, the vector displacement of its center will have magnitude $ad\phi$ and direction determined by θ. For example, if $\theta = 0$ then the wheel axis is parallel to the x axis, in which case rolling the ϕ wheel clockwise will cause it to move in the negative y direction. In general, referring to the Figure, we have

$$\mathbf{dr} = a\,d\phi[\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}] \tag{10}$$

$$\mathbf{dr}' = a\,d\phi'[\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}] \tag{11}$$

Adding these componentwise we have[1]

$$\begin{aligned} dx &= \frac{a}{2}[d\phi + d\phi']\sin\theta \\ dy &= -\frac{a}{2}[d\phi + d\phi']\cos\theta \end{aligned}$$

Multiplying these by $\sin\theta$ or $-\cos\theta$ and adding or subtracting, we obtain

$$\begin{aligned} \sin\theta\,dx - \cos\theta\,dy &= a[d\phi + d\phi'] \\ \cos\theta\,dx + \sin\theta\,dy &= 0. \end{aligned}$$

Next, consider the vector $\mathbf{r}_{12} = \mathbf{r} - \mathbf{r}'$ connecting the centers of the two wheels. The definition of θ is such that its tangent must just be the ratio of the y and x components of this vector:

$$\begin{aligned} \tan\theta &= \frac{y_{12}}{x_{12}} \\ \to \sec^2\theta\,d\theta &= -\frac{y_{12}}{x_{12}^2}\,dx_{12} + \frac{1}{x_{12}}\,dy_{12}. \end{aligned}$$

Subtracting (11) from (10),

$$\sec^2\theta d\theta = a[d\phi - d\phi']\left(-\frac{y_{12}}{x_{12}^2}\sin\theta - \frac{1}{x_{12}}\cos\theta\right)$$

Again substituting for y_{12}/x_{12} in the first term in parentheses,

$$\sec^2\theta d\theta = -a[d\phi - d\phi']\frac{1}{x_{12}}(\tan\theta\sin\theta + \cos\theta)$$

or

$$\begin{aligned} d\theta &= -a[d\phi - d\phi']\frac{1}{x_{12}}(\sin^2\theta\cos\theta + \cos^3\theta) \\ &= -a[d\phi - d\phi']\frac{1}{x_{12}}\cos\theta. \end{aligned} \tag{12}$$

[1]Thanks to Javier Garcia for pointing out a factor-of-two error in the original version of these equations.

However, considering the definition of θ, we clearly have

$$\cos\theta = \frac{x_{12}}{(x_{12}^2 + y_{12}^2)^{1/2}} = \frac{x_{12}}{b}$$

because the magnitude of the distance between r_1 and r_2 is constrained to be b by the rigid axis. Then (12) becomes

$$d\theta = -\frac{a}{b}[d\phi - d\phi']$$

with immediate solution

$$\theta = C - \frac{a}{b}[\phi - \phi'].$$

with C a constant of integration.

Problem 1.9

A particle moves in the $x - y$ plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

The particle's position is $(x(t), y(t))$, while the position of the moving point is $(f(t), 0)$. Then the vector **d** from the particle to the point has components

$$d_x = x(t) - f(t) \qquad d_y = y(t). \tag{13}$$

The particle's velocity **v** has components

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \tag{14}$$

and for the vectors in (13) and (**??** to be in the same direction, we require

$$\frac{v_y}{v_x} = \frac{d_y}{d_x}$$

or

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

so

$$\frac{dy}{y} = \frac{dx}{x - f(t)} \tag{15}$$

For example, if $f(t) = \alpha t$, then we may integrate to find

$$\ln y(t) = \ln[x(t) - \alpha(t)] + C$$

or

$$y(t) = C \cdot [x(t) - \alpha t]$$

which is a holonomic constraint. But for general $f(t)$ the right side of (15) is not integrable, so the constraint is nonholonomic.

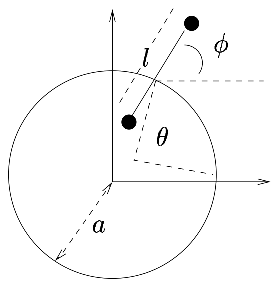

Figure 2: My conception of the situation of Problem 1.10

Problem 1.10

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Set up the kinetic energy in generalized coordinates.

My conception of this one is shown in Figure 2. θ is the angle representing how far around the circle the center of the rod has moved. ϕ is the angle the rod makes with the x axis.

The position of the center of the rod is $(x, y) = (a\cos\theta, a\sin\theta)$. The positions of the masses relative to the center of the rod are $(x_{rel}, y_{rel}) = \pm(1/2)(l\cos\phi, l\sin\phi)$. Then the absolute positions of the masses are

$$(x, y) = (a\cos\theta \pm \frac{l}{2}\cos\phi, a\sin\theta \pm \frac{l}{2}\sin\phi)$$

and their velocities are

$$(v_x, v_y) = (-a\sin\theta\,\dot\theta \mp \frac{l}{2}\sin\phi\,\dot\phi, a\cos\theta\,\dot\theta \pm \frac{l}{2}\cos\phi\,\dot\phi).$$

The magnitudes of these are

$$\begin{aligned}|v| &= a^2\dot\theta^2 + \frac{l^2}{4}\dot\phi^2 \pm al\,\dot\theta\,\dot\phi(\sin\theta\sin\phi + \cos\theta\cos\phi)\\ &= a^2\dot\theta^2 + \frac{l^2}{4}\dot\phi^2 \pm al\,\dot\theta\,\dot\phi\cos(\theta - \phi)\end{aligned}$$

When we add the kinetic energies of the two masses, the third term cancels, and we have

$$T = \frac{1}{2}\sum mv^2 = m(a^2\dot\theta^2 + \frac{l^2}{4}\dot\phi^2).$$

Problem 1.11

Show that Lagrange's equations in the form of Eq. 1-53 can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes known as the *Nielsen* form of the Lagrange equations.

Problem 1.12

A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r},\mathbf{v}) = V(r) + \sigma \cdot \mathbf{L}$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and σ is a fixed vector in space.

(a) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq. (1-58).

(b) Show that the components in the two coordinate systems are related to each other as in Eq. (1-49).

(c) Obtain the equations of motion in spherical polar coordinates.

Problem 1.13

A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right),$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics).

If we take

$$U(r) = \frac{1}{r}\left(1 + \frac{v^2}{c^2}\right) = \frac{1}{r} + \frac{(\dot{r})^2}{c^2 r}$$

then

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2}$$

and

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{d}{dt}\left(\frac{2\dot{r}}{c^2 r}\right) = \frac{2\ddot{r}}{c^2 r} - \frac{2(\dot{r})^2}{c^2 r^2}$$

so

$$Q_r = -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}} = \frac{1}{r^2}\left(1 + \frac{2r\ddot{r} - (\dot{r})^2}{c^2}\right)$$

The Lagrangian for motion in a plane is

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{r}^2\dot{\theta}^2 - \frac{1}{r^2}\left(1 + \frac{2r\ddot{r} - (\dot{r})^2}{c^2}\right).$$

Problem 1.14

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}$$

also satisfies Lagrange's equations, where F is any arbitrary, but differentiable, function of its arguments.

We have

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} \tag{16}$$

and

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}. \tag{17}$$

For the function F we may write

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$$

and from this we may read off

$$\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt} = \frac{\partial F}{\partial q_i}.$$

Then taking the time derivative of (17) gives

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial F}{\partial q_i}$$

so we have

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt} - \frac{d}{dt}\frac{\partial F}{\partial q_i}.$$

The first two terms on the RHS cancel because L satisfies the Euler-Lagrange equations, while the second two terms cancel because F is differentiable. Hence L' satisfies the Euler-Lagrange equations.

Problem 1.16

A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0, c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1-56) is related to L' by a point transformation (cf. Exercise 15 above). What is the significance of the condition on the value of $b^2 - ac$?

Clearly we have

$$\frac{\partial L}{\partial x} = -Kax - Kby \qquad \frac{\partial L}{\partial \dot{x}} = ma\dot{x} + mb\dot{y}$$

so the Euler-Lagrange equation for x is

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \qquad \rightarrow \qquad m(a\ddot{x} + b\ddot{y}) = -K(ax + by).$$

Similarly, for y we obtain

$$m(b\ddot{y} + c\ddot{y}) = -K(bx + cy).$$

These are the equations of motion for a particle of mass m undergoing simple harmonic motion in two dimensions, as if bound by two springs of spring constant K. Normally we would express the Lagrangian in unravelled form, by transforming to new coordinates u_1 and u_2 with

$$u_1 = ax + by \qquad u_2 = bx + cy.$$

The condition $b^2 - ac \neq 0$ is the condition that the coordinate transformation not be degenerate, i.e. that there are actually two distinct dimensions in which the particle experiences a restoring force. If $b^2 = ac$ then we have just a one-dimensional problem.

Problem 1.17

Obtain the Lagrangian equations of motion for a spherical pendulum, i.e. a mass point suspended by a rigid weightless rod.

Denoting the mass of the particle by m, the length of the rod by L, and the angle between the rod and the vertical by θ, we have the particle's linear velocity given in magnitude by $v = L\dot{\theta}$, while its height is $h = -L\cos\theta$ (where the fulcrum of the pendulum is taken as the origin of coordinates). Then

$$L = T - V = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

so the equation of motion is

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \qquad \to \qquad -g\sin\theta = L\ddot{\theta}.$$

Problem 1.18

A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x),$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

We have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= m\dot{x}^2\frac{dV}{dx} - 2V(x)\frac{dV}{dx} \\
\frac{\partial L}{\partial \dot{x}} &= \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m^2(\dot{x})^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}\frac{d}{dt}V(x)
\end{aligned}$$

In the last equation we can use

$$\frac{d}{dt}V(x) = \dot{x}\frac{dV}{dx}.$$

Then the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \to \quad m^2(\dot{x})^2\ddot{x} + 2m\ddot{x}V(x) + m\dot{x}^2\frac{dV}{dx} + 2V(x)\frac{dV}{dx}$$

or

$$\left(m\ddot{x} + \frac{dV}{dx}\right)\left(m\dot{x}^2 + 2V(x)\right) = 0.$$

If we identify $F = -dV/dx$ and $T = m\dot{x}^2/2$, we may write this as

$$(F - m\ddot{x})(T + V) = 0$$

So, this is saying that, at all times, either the difference between F and ma is zero, *or* the sum of kinetic and potential energy is zero.

Problem 1.19

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole).

Let d be the height of m_2 above its lowest possible position, so that $d = 0$ when the string is fully extended beneath the table and m_1 is just about to fall through the hole. Also, let θ be the angular coordinate of m_1 on the table. Then the kinetic energy of m_2 is just $m_2\dot{d}^2/2$, while the kinetic energy of m_1 is $m_1\dot{d}^2/2 + m_1 d^2\dot{\theta}^2/2$, and the potential energy of the system is just the gravitational potential energy of m_2, $U = m_2 g d$. Then the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_1 d^2\dot{\theta}^2 - m_2 g d$$

and the Euler-Lagrange equations are

$$\frac{d}{dt}(m_1 d^2\dot{\theta}) = 0$$
$$(m_1 + m_2)\ddot{d} = -m_2 g + m_1 d\dot{\theta}^2$$

From the first equation we can identify a first integral, $m_1 d^2\dot{\theta} = l$ where l is a constant. With this we can substitute for $\dot{\theta}$ in the second equation:

$$(m_1 + m_2)\ddot{d} = -m_2 g + \frac{l^2}{m_1 d^3}$$

Because the sign of the two terms on the RHS is different, this is saying that, if l is big enough (if m_1 is spinning fast enough), the centrifugal force of m_1 can balance the downward pull of m_2, and the system can be in equilibrium.

Problem 1.20

Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig. 1-4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Taking the origin at the fulcrum of the first pendulum, we can write down the coordinates of the first mass point:

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ y_1 &= -l_1 \cos\theta_1 \end{aligned}$$

The coordinates of the second mass point are defined relative to the coordinates of the first mass point by exactly analogous expressions, so relative to the coordinate origin we have

$$\begin{aligned} x_2 &= x_1 + l_2 \sin\theta_2 \\ y_2 &= y_1 - l_2 \cos\theta_2 \end{aligned}$$

Differentiating and doing a little algebra we find

$$\begin{aligned} \dot{x}_1^2 + \dot{y}_1^2 &= l_1^2\dot{\theta}_1^2 \\ \dot{x}_2^2 + \dot{y}_2^2 &= l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 - 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned}$$

The Lagrangian is

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_2\cos\theta_1 + m_2 g l_2 \cos\theta_2$$

with equations of motion

$$\frac{d}{dt}\Big[(m_1+m_2)l_1^2\dot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1-\theta_2)\Big] = -(m_1+m_2)gl_2\sin\theta_1$$

and

$$\frac{d}{dt}\Big[l_2\dot{\theta}_2 - l_1\dot{\theta}_1\cos(\theta_1-\theta_2)\Big] = -g\sin\theta_2.$$

If $\dot{\theta}_1 = 0$, so that the fulcrum for the second pendulum is stationary, then the second of these equations reduces to the equation we derived in Problem 1.17.

Problem 1.21

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\Psi(\mathbf{r}, t),$$
$$\Phi \to \Phi - \frac{1}{c}\frac{\partial \Psi}{\partial t},$$

where Ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

The Lagrangian for a particle in an electromagnetic field is

$$L = T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t)$$

If we make the suggested gauge transformation, this becomes

$$\begin{aligned}
&\to T - q\left[\Phi(\mathbf{x}(t)) - \frac{1}{c}\left.\frac{\partial\Psi}{\partial t}\right|_{\mathbf{x}=\mathbf{x}(t)}\right] + \frac{q}{c}\left[\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \mathbf{v}\cdot\nabla\Psi(\mathbf{x}(t))\right] \\
&= T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \frac{q}{c}\left[\frac{\partial\Psi}{\partial t} + \mathbf{v}\cdot\nabla\Psi(\mathbf{x}(t))\right] \\
&= T - q\Phi(\mathbf{x}(t)) + \frac{q}{c}\mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{v}(t) + \frac{q}{c}\frac{d}{dt}\Psi(\mathbf{x}(t)) \\
&= L + \frac{q}{c}\frac{d}{dt}\Psi(\mathbf{x}(t)).
\end{aligned}$$

So the transformed Lagrangian equals the original Lagrangian plus a total time derivative. But we proved in Problem 1.15 that adding the total time derivative of any function to the Lagrangian does not affect the equations of motion, so the motion of the particle is unaffected by the gauge transformation.

Problem 1.22

Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for fall from rest is $v = mg/k$.

The Lagrangian for the particle is

$$L = \frac{1}{2}m\dot{z}^2 - mgz$$

and the dissipation function is $k\dot{z}^2/2$, so the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \frac{\partial F}{\partial \dot{z}} \rightarrow m\ddot{z} = mg - k\dot{z}.$$

This says that the acceleration goes to zero when $mg = k\dot{z}$, or $\dot{z} = mg/k$, so the velocity can never rise above this terminal value (unless the initial value of the velocity is greater than the terminal velocity, in which case the particle will slow down to the terminal velocity and then stay there).