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Toeplitz operators and $\mathcal{H}_\infty$ calculus

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Abstract

Let A be the generator of a strongly continuous, exponentially stable, semigroup on a Hilbert space. Furthermore, let the scalar function g be bounded and analytic on the left-half plane, i.e., $g(-s) \in \mathcal{H}_\infty$. By using the Toeplitz operator associated to g, we construct an infinite-time admissible output operator $g(A)$. If g is rational, then this operator is bounded, and equals the "normal" definition of $g(A)$. Although in general $g(A)$ may be unbounded, we always have that $g(A)$ multiplied by the semigroup is a bounded operator for every positive time instant. Furthermore, when there exists an admissible output operator C such that (C, A) is exactly observable, then $g(A)$ is bounded for all g with $g(-s) \in \mathcal{H}_\infty$, i.e., there exists a bounded $\mathcal{H}_\infty$-calculus. Moreover, we rediscover some well-known classes of generators also having a bounded $\mathcal{H}_\infty$-calculus.

Keywords: Toeplitz operators; Functional calculus; Admissible output operators; Strongly continuous semigroups

1. Introduction

Functional calculus is a sub-field of mathematics with a long history. It started in the thirties of the last century with the work by von Neumann for self-adjoint operators [10], and was further extended by many researchers, see e.g. [6] and [2]. For an overview, see the book by Markus Haase, [5]. The basic idea behind functional calculus for the operator A is to construct a mapping from an algebra of (scalar) functions to the class of (bounded) operators, such that

- The function identically equals to one is mapped to the identity operator;

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- If $f(s) = (s-a)^{-1}$, then $f(A) = (sI - A)^{-1}$;
- Furthermore, the operator associated to $f_1 \cdot f_2$ equals $f(A) f_2(A)$.

Before we explain the contribution of this paper, we introduce some notation. By X we denote the separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and by A we denote an unbounded operator from its domain $D(A) \subset X$ to X. We assume that A generates an exponentially stable semigroup on X, which we denote by $(T(t))_{t \geqslant 0}$.

By $\mathcal{H}_\infty^-$ we denote the space of all bounded, analytic functions defined on the half-plane $\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$. It is clear that this function class is an algebra under pointwise multiplication and addition. Hence this could serve as a class for which one could build a functional calculus. However, it is known that there exists a generator of exponential stable semigroup, which does not have a functional calculus with respect to $\mathcal{H}_\infty^-$. For a proof of this and many more, we refer to [1,5], and the references therein. Although a bounded functional calculus is not possible, an unbounded functional calculus is always possible.

Theorem 1.1. *Under the assumptions stated above, we have that for all $g \in \mathcal{H}_\infty^-$ there exists an operator $g(A)$ which is bounded from the domain of A to X, and which is admissible, i.e.,*

$$\int_0^\infty \left\| g(A) T(t) x_0 \right\|^2 dt \leqslant \gamma_A \|g\|_\infty^2 \|x_0\|^2, \quad x_0 \in X.$$

The mapping $g \mapsto g(A)$ satisfies the conditions of a functional calculus. Furthermore, for all $t > 0$, we have that $g(A)T(t)$ can be extended to a bounded operator, and

$$\left\| g(A) T(t) \right\| \leqslant \frac{\gamma}{\sqrt{t}}.$$

Apart from proving this theorem, we shall also rediscover some classes of generators for which $g(A)$ is bounded for all $g \in \mathcal{H}_\infty^-$, i.e., for which there is a bounded functional calculus.

For the proof of the above result, we need beside the Hardy space $\mathcal{H}_\infty^-$ also the Hardy spaces $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$. $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ denote the Laplace transform, $\mathcal{L}$, of functions in $L^2((0,\infty), X)$ and $L^2((-\infty, 0), X)$, respectively. It is known that this transformation is an isometry. Every function in $\mathcal{H}_\infty^-$, $\mathcal{H}_2(X)$ and $\mathcal{H}_2^\perp(X)$ has a unique extension to the imaginary axis on which these functions are bounded, and square integrable, respectively. Furthermore, the norm of $g \in \mathcal{H}_\infty^-$ equals the (essential) supremum over the imaginary axis of the boundary function. Let $f(t)$ be a function in $L^2((0,\infty), X)$ with Laplace transform $F(s)$, and let $f_{\text{ext}}(t)$ be the function in $L^2((-\infty,\infty), X)$ defined by

$$f_{\text{ext}}(t) = \begin{cases} f(t) & t \geqslant 0, \\ 0 & t < 0. \end{cases}$$

Then the Fourier transform $\hat{f}_{\text{ext}}$ of $f_{\text{ext}}(t)$ satisfies $\hat{f}_{\text{ext}}(\omega) = F(i\omega)$, for almost all $\omega \in \mathbb{R}$. Here $F(i\cdot)$ denote the boundary function of the Laplace transform $F(s)$.

We define the following Toeplitz operator on $L^2((0,\infty); X)$.

Definition 1.2. Let g be an element of $\mathcal{H}_\infty^-$. Associated to this function we define the mapping M_g as

$$M_g f = \mathcal{L}^{-1}\big(\Pi(gF)\big), \quad f \in L^2\big((0,\infty), X\big), \tag{1}$$

where F denotes the Laplace transform of f. Π denotes the projection onto $\mathcal{H}_2(X)$.

It is clear that this is a linear bounded map from $L^2((0,\infty); X)$ into itself, and

$$\|M_g\| \leqslant \|g\|_\infty. \tag{2}$$

Furthermore, it follows easily from (1) that if K is a bounded mapping on X, then its commutes with M_g, i.e.,

$$KM_g = M_g K. \tag{3}$$

It is easy to see that $\mathcal{H}_\infty^-$ is an algebra under the multiplication and addition. In particular $g_1 g_2 \in \mathcal{H}_\infty^-$ whenever $g_1, g_2 \in \mathcal{H}_\infty^-$. Furthermore, we have the following result.

Lemma 1.3. *Let g_1 and g_2 be elements of $\mathcal{H}_\infty^-$. Then*

$$M_{g_1 g_2} = M_{g_1} M_{g_2}. \tag{4}$$

In particular, if g is invertible in $\mathcal{H}_\infty^-$, then M_g is (boundedly) invertible and $(M_g)^{-1} = M_{g^{-1}}$.

Proof. We use the fact that any $g \in \mathcal{H}_\infty^-$ maps $\mathcal{H}_2^\perp$ into $\mathcal{H}_2^\perp$.

$$\begin{aligned} M_{g_1} M_{g_2} f &= \mathcal{L}^{-1}\big(\Pi g_1\big(\Pi(g_2 F)\big)\big) \\ &= \mathcal{L}^{-1}\big(\Pi(g_1 g_2 F)\big) + \mathcal{L}^{-1}\big(\Pi\big(g_1(I - \Pi)(g_2 F)\big)\big) \\ &= \mathcal{L}^{-1}\big(\Pi(g_1 g_2 F)\big) + 0, \end{aligned}$$

where we have used the above mentioned fact that $g_1(I - \Pi)$ maps into $\mathcal{H}_2^\perp$, and so $\Pi g_1(I - \Pi) = 0$. Since by definition $\mathcal{L}^{-1}(\Pi(g_1 g_2 F))$ equals $M_{g_1 g_2} f$, we have proved the first assertion.

The last assertion follows directly, since $M_1 = I$. □

By σ_τ we denote the shift with $\tau \geqslant 0$, i.e.,

$$\big(\sigma_\tau(f)\big)(t) = f(t + \tau), \quad t \geqslant 0. \tag{5}$$

This is also a linear bounded map from $L^2((0,\infty); X)$ into itself. This mapping commutes with M_g as is shown next.

Lemma 1.4. *For all $\tau > 0$ and all g in $\mathcal{H}_\infty^-$, we have that*

$$\sigma_\tau(M_g f) = M_g(\sigma_\tau f), \quad f \in L^2\big((0,\infty), X\big). \tag{6}$$

Proof. We use the following well-known equality. If h is Fourier transformable, then the Fourier transform of $h(\cdot + \tau)$ equals $e^{i\omega\tau}\hat{h}(\omega)$, where $\hat{h}$ denotes the Fourier transform of h.

Let $h \in L^2((0,\infty); X)$, then

$$\mathcal{L}(\sigma_\tau h) = \widehat{(\sigma_\tau h)_{\text{ext}}} = \widehat{\sigma_\tau h_{\text{ext}}} - \hat{q} = e^{i\omega\tau}\widehat{h_{\text{ext}}} - \hat{q} = e^{i\omega\tau}\mathcal{L}(h) - \hat{q}, \tag{7}$$

with $q \in L^2((-\infty,0); X)$. In particular, we find for every $h \in L^2((0,\infty); X)$ that

$$\mathcal{L}(\sigma_\tau h) = \Pi\big(\mathcal{L}(\sigma_\tau h)\big) = \Pi\big(e^{i\omega\tau}\mathcal{L}(h)\big) - 0 = \mathcal{L}(M_{e^{i\cdot\tau}} h), \tag{8}$$

where we have used that $e^{i\omega\tau}$ is the boundary function corresponding to $e^{is\tau} \in \mathcal{H}_\infty^-$.

Using (7) we see that

$$M_g(\sigma_\tau f) = \mathcal{L}^{-1}\big(\Pi\big(ge^{i\cdot\tau}\mathcal{L}(f)\big)\big) - \mathcal{L}^{-1}\big(\Pi(g\hat{q})\big) = \mathcal{L}^{-1}\big(\Pi\big(ge^{i\cdot\tau}\mathcal{L}(f)\big)\big), \tag{9}$$

since $\hat{q} \in \mathcal{H}_2^\perp(X)$ and $g \in \mathcal{H}_\infty^-$. Using Lemma 1.3, we find that

$$M_g(\sigma_\tau f) = \mathcal{L}^{-1}\big(\Pi\big(ge^{i\cdot\tau}\mathcal{L}(f)\big)\big) = M_{e^{i\cdot\tau}g} f = M_{e^{i\cdot\tau}} M_g f. \tag{10}$$

Now using (8), we see that

$$M_g(\sigma_\tau f) = \sigma_\tau(M_g f). \qquad \square \tag{11}$$

2. Output maps and admissible output operators

In this section we study admissible operators which commute with the semigroup. We begin by defining well-posed output maps.

Definition 2.1. Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on the Hilbert space X, and let Y be another Hilbert space. We say that the mapping $\mathcal{O}$ is a well-posed (infinite-time) output map if

- $\mathcal{O}$ is a bounded linear mapping from X into $L^2((0,\infty); Y)$, and
- For all $\tau \geqslant 0$ and all $x_0 \in X$, we have that $\sigma_\tau \mathcal{O} x_0 = \mathcal{O}(T(\tau)x_0)$.

Closely related to well-posed output mappings are admissible operators, which are defined next.

Definition 2.2. Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on the Hilbert space X. Let $D(A)$ be the domain of its generator A. The linear mapping C from $D(A)$ to Y, another Hilbert space, is said to be an (infinite-time) admissible output operator for $(T(t))_{t\geqslant 0}$ if $CT(\cdot)x_0 \in L^2((0,\infty), Y)$ for all $x_0 \in D(A)$ and there exists an m independent of x_0 such that

$$\int_0^\infty \|CT(t)x_0\|_Y^2\, dt \leqslant m\|x_0\|_X^2. \tag{12}$$

If C is (infinite-time) admissible, then for all $x_0 \in X$ we can uniquely define an $L^2((0,\infty), Y)$-function. We denote this function by $CT(\cdot)x_0$. Hence $\mathcal{O} : X \to L^2((0,\infty); Y)$ defined by $\mathcal{O}x_0 = CT(\cdot)x_0$ is a well-posed output map. From [11] we know that the converse holds as well.

Lemma 2.3. *If $\mathcal{O}$ is a well-posed output mapping, then there exists a (unique) linear bounded mapping from $D(A)$ to Y, C, such that $\mathcal{O}x_0 = CT(\cdot)x_0$ for all x_0.*

In the sequel of this section we concentrate on admissible output operators which commute with the semigroup, i.e., C a linear operator from $D(A)$ to X and

$$CT(t)x_0 = T(t)Cx_0 \quad \text{for all } t \geqslant 0 \text{ and } x_0 \in D(A). \tag{13}$$

For these operators we have the following results.

Lemma 2.4. *Let C be the admissible output operator associated with the well-posed output map $\mathcal{O}$. Then (13) holds if and only if for all $t \geqslant 0$ there holds $\mathcal{O}T(t) = T(t)\mathcal{O}$, i.e., $(\mathcal{O}T(t)x_0)(\cdot) = T(t)(\mathcal{O}x_0)(\cdot)$ for all $x_0 \in X$ with equality in $L^2((0,\infty), X)$.*

Theorem 2.5. *Let C be a bounded linear operator from $D(A)$ to X, which is admissible for the exponentially stable semigroup $(T(t))_{t\geqslant 0}$ and which commutes with this semigroup. Then the following holds*

1. *For all $x_0 \in D(A)$, we have that $CA^{-1}x_0 = A^{-1}Cx_0$.*
2. *For all $t > 0$, the operator $CT(t) : D(A) \to X$ can be extended to a bounded operator on X. Furthermore, $\|CT(t)\| \leqslant \gamma t^{-1/2}$ for some γ independent of t.*

Proof. The first assertion follows easily from (13) by using Laplace transforms. We concentrate on the second assertion.

Let $x_0 \in D(A)$ and $x_1 \in X$, then for $t > 0$ we have that

$$\begin{aligned} t\langle x_1, CT(t)x_0\rangle &= \int_0^t \langle x_1, CT(t)x_0\rangle \, d\tau \\ &= \int_0^t \langle x_1, CT(\tau)T(t-\tau)x_0\rangle \, d\tau \\ &= \int_0^t \langle x_1, T(\tau)CT(t-\tau)x_0\rangle \, d\tau \\ &= \int_0^t \langle T(\tau)^* x_1, CT(t-\tau)x_0\rangle \, d\tau \\ &\leqslant \sqrt{\int_0^t \|T(\tau)^* x_1\|^2 \, d\tau} \sqrt{\int_0^t \|CT(t-\tau)x_0\|^2 \, d\tau}. \end{aligned}$$

Using the fact that the semigroup, and hence its adjoint, are uniformly bounded, and the fact that C is (infinite-time) admissible, we find that

$$t\langle x_1, CT(t)x_0\rangle \leqslant \sqrt{t}M\|x_1\|m\|x_0\|.$$

Since this holds for all $x_1 \in X$, we conclude that

$$t\|CT(t)x_0\| \leqslant \sqrt{t}mM\|x_0\|.$$

This inequality holds for all $x_0 \in D(A)$. The domain of a generator is dense, and hence we have proved the second assertion. □

Remark 2.6. By the exponential stability of the semigroup, we see that for t large we can improve the estimate. Let the semigroup satisfy $\|T(t)\| \leqslant M_\omega e^{-\omega t}$. For $t > 1$ we have

$$\|CT(t)\| \leqslant \|CT(1)T(t-1)\| \leqslant \gamma M_\omega e^{-\omega(t-1)}.$$

From Theorem 2.5 it is clear that if the semigroup is surjective, then any admissible C which commutes with the semigroup is bounded. However, this does not hold for a general semigroup as is shown in the following example. Furthermore, this example also shows that the estimate in the previous theorem cannot be improved.

Example 2.7. Let $\{\phi_n,\ n \in \mathbb{N}\}$ be an orthonormal basis of X, and define for $t \geqslant 0$ the operator

$$T(t)\sum_{n=1}^{N}\alpha_n\phi_n = \sum_{n=1}^{N}e^{-n^2t}\alpha_n\phi_n. \tag{14}$$

It is not hard to show that this defines an exponentially stable C_0-semigroup on X. The infinitesimal generator A is given by

$$A\sum_{n=1}^{N}\alpha_n\phi_n = \sum_{n=1}^{N}-n^2\alpha_n\phi_n,$$

with domain

$$D(A) = \left\{x = \sum_{n=1}^{\infty}\alpha_n\phi_n \in X \;\middle|\; \sum_{n=1}^{\infty}\left|n^2\alpha_n\right|^2 < \infty\right\}.$$

We define C as the square root of $-A$, i.e.

$$C\sum_{n=1}^{N}\alpha_n\phi_n = \sum_{n=1}^{N}n\alpha_n\phi_n \tag{15}$$

with domain

$$D(C) = \left\{x = \sum_{n=1}^{\infty}\alpha_n\phi_n \in X \;\middle|\; \sum_{n=1}^{\infty}|n\alpha_n|^2 < \infty\right\}.$$

A straightforward calculation gives that for $x_0 = \sum_{n=1}^{N} \alpha_n \phi_n$, we have that

$$\int_0^\infty \left\| CT(t)x_0 \right\|^2 dt = \int_0^N \sum_{n=1}^{N} \left| n e^{-n^2 t} \alpha_n \right|^2 dt = \frac{1}{2} \sum_{n=1}^{N} |\alpha_n|^2 = \frac{1}{2} \|x_0\|^2.$$

Since the finite sums lie dense, we conclude that C is admissible. It is easy to see that C commutes with the semigroup, and thus from Theorem 2.5 we have that

$$\left\| CT(t) \right\| \leqslant \frac{\gamma}{\sqrt{t}} \tag{16}$$

for some γ independent of t.

Next choose $x_0 = \phi_n$ and $t = n^{-2}$. Using (14) and (15) we see that

$$CT(t)x_0 = n e^{-1} \phi_n = \frac{e^{-1}}{\sqrt{t}} x_0.$$

So there exists a sequence t_n, $n \in \mathbb{N}$ such that $t_n \to \infty$ and $\inf_n \sqrt{t_n} \|CT(t_n)\| > 0$. Thus the estimate (16) cannot be improved.

The Lebesgue extension of an admissible operator is defined by

$$C_L x = \lim_{t \to 0} \frac{1}{t} C \int_0^t T(\tau) x \, d\tau,$$

where

$$D(C_L) = \{x \in X \mid \text{limit exists}\}.$$

A similar extension can be defined using the resolvent. The Lambda extension of an admissible operator is defined by

$$C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda I - A)^{-1} x,$$

where

$$D(C_\Lambda) = \{x \in X \mid \text{limit exists}\}.$$

The precise relation between these extensions is still not completely understood [7], but for admissible operators which commute with the semigroup, we have that both extensions are closed operators.

Lemma 2.8. *Let C be an admissible operator which commutes with the semigroup, then the same holds for its Lebesgue and Lambda extension. Furthermore, these extensions are closed operators.*

Proof. Since A^{-1} and CA^{-1} are bounded, we find for $x_0 \in D(C_L)$

$$\begin{aligned} A^{-1}C_L x_0 &= A^{-1}\lim_{t\downarrow 0}\frac{1}{t}C\int_0^t T(\tau)x_0\,d\tau = \lim_{t\downarrow 0}\frac{1}{t}A^{-1}C\int_0^t T(\tau)x_0\,d\tau \\ &= \lim_{t\downarrow 0}\frac{1}{t}CA^{-1}\int_0^t T(\tau)x_0\,d\tau = CA^{-1}\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(\tau)x_0\,d\tau \\ &= CA^{-1}x_0 = C_L A^{-1}x_0, \end{aligned}$$

where we have used that $\int_0^t T(\tau)x_0 d\tau \in D(A)$ and C commutes with A^{-1}. This proves the first assertion.

Using once more that CA^{-1} and A^{-1} are bounded, we have for $x_0 \in D(C_L)$

$$\begin{aligned} CA^{-1}\int_0^t T(\tau)x_0\,d\tau &= \int_0^t CA^{-1}T(\tau)x_0\,d\tau \\ &= \int_0^t T(\tau)CA^{-1}x_0\,d\tau \\ &= \int_0^t T(\tau)A^{-1}C_L x_0\,d\tau = A^{-1}\int_0^t T(\tau)C_L x_0\,d\tau. \end{aligned} \tag{17}$$

Let x_n be a sequence in $D(C_L)$ which converges to $x \in X$, such that $C_L x_n$ converges to $z \in X$. Then by (17) we find that

$$CA^{-1}\int_0^t T(\tau)x\,d\tau = A^{-1}\int_0^t T(\tau)z\,d\tau. \tag{18}$$

Since $\int_0^t T(\tau)x\,d\tau \in D(A)$, we obtain

$$A^{-1}\int_0^t T(\tau)z\,d\tau = CA^{-1}\int_0^t T(\tau)x\,d\tau = A^{-1}C\int_0^t T(\tau)x\,d\tau. \tag{19}$$

Hence we have that

$$\int_0^t T(\tau)z\,d\tau = C\int_0^t T(\tau)x\,d\tau.$$

Since $t^{-1}\int_0^t T(\tau)z\,d\tau$ converges to z for $t \downarrow 0$, we conclude from the above equality that $x \in D(C_L)$ and $C_L x = z$.

The proof for C_Λ goes very similarly. Basically in the above proof $\int_0^t T(\tau)x\,d\tau$ is replaced everywhere by $(\lambda I - A)^{-1}x$. □

By Weiss [13] we have that C_Λ is an extension of C_L. We claim that for admissible C's which commute with the semigroup they are equal.

3. $\mathcal{H}_\infty$-calculus

For $g \in \mathcal{H}_\infty^-$ we define the following mapping from X to $L^2((0,\infty); X)$

$$\mathfrak{O}_g x_0 = M_g\big(T(t)x_0\big). \tag{20}$$

Hence we have taken in Definition 1.2 $f(t) = T(t)x_0$. Since $T(t)$ is an exponentially stable semigroup, we know that $T(t)x_0 \in L^2((0,\infty); X)$.

It is clear that $\mathfrak{O}_g$ is a linear bounded operator from X into $L^2((0,\infty); X)$. Furthermore, from (6) we have that

$$\sigma_\tau(\mathfrak{O}_g x_0) = M_g\big(\sigma_\tau\big(T(t)x_0\big)\big) = M_g\big(T(t+\tau)x_0\big) = \mathfrak{O}_g\big(T(\tau)x_0\big), \tag{21}$$

where we have used the semigroup property. Hence $\mathfrak{O}_g$ is a well-posed output map, and so by Lemma 2.3 we conclude that $\mathfrak{O}_g$ can be written as

$$\mathfrak{O}_g x_0 = g(A)T(t)x_0 \tag{22}$$

for some infinite-time admissible operator $g(A)$ which is bounded from the domain of A to X.

Since for all $t, \tau \in [0,\infty)$ there holds $T(\tau)T(t) = T(t)T(\tau)$, we conclude from (20) and (3) that

$$\mathfrak{O}_g T(t) = T(t)\mathfrak{O}_g, \quad t \geqslant 0.$$

Hence by (22), we see that $g(A)$ is an admissible operator which commutes with the semigroup. Theorem 2.5 implies that for $t > 0$, $g(A)T(t)$ can be extended to a bounded operator and

$$\big\|g(A)T(t)\big\| \leqslant \frac{\gamma}{\sqrt{t}}. \tag{23}$$

Note that for $t \in [0,1]$ this γ can be chosen as $\sup_{t\in[0,1]} \|T(t)\| \cdot \|g\|_\infty$.

The Laplace transform of $\mathfrak{O}_g$ equals $g(A)(sI - A)^{-1}$. Combining this with the definition of $\mathfrak{O}_g$, implies that

$$\big\|g(A)(sI - A)^{-1}\big\| \leqslant \frac{M\|g\|_\infty}{\sqrt{\operatorname{Re}(s)}}\|x_0\|, \tag{24}$$

where we have taken the norm in X, see also Weiss [12].

Since we have written this admissible operator as the function g working on the operator A, there is likely to be a relation with functional calculus of Phillips, [5, Section 3.3]. This is presented next. The proof is based on the fact that after taking the Laplace transform a convolution product becomes a normal product. The proof is left to the reader.

Lemma 3.1. *If $g \in \mathcal{H}_\infty^-$ is the inverse Fourier transform of the function h, with $h \in L^1(-\infty, \infty)$ with support in $(-\infty, 0)$, then $g(A)$ is bounded,*

$$g(A)x_0 = \int_0^\infty T(t)h(-t)x_0\,dt, \tag{25}$$

and so $g(A)$ corresponds to the classical definition of the function of an operator.

So if g is the Fourier transform of an absolutely integrable function, then $g(A)$ is bounded. We would like to know when it is bounded for every g. For this, we extend the definition of $\mathfrak{O}_g$.

Let C be an admissible output operator for the semigroup $(T(t))_{t\geqslant 0}$. By definition, we know that $CT(\cdot)x_0 \in L^2((0,\infty); Y)$ for all $x_0 \in X$. We define

$$(C \circ \mathfrak{O}_g)x_0 = M_g\bigl(CT(t)x_0\bigr). \tag{26}$$

It is clear that this is a bounded mapping from X to $L^2((0,\infty); Y)$.

As before we have that

$$\sigma_\tau\bigl((C \circ \mathfrak{O}_g)(x_0)\bigr) = (C \circ \mathfrak{O}_g)\bigl(T(\tau)x_0\bigr). \tag{27}$$

And so we can write $(C \circ \mathfrak{O}_g)x_0$ as $\tilde{C}_g T(\cdot)x_0$ for some infinite-time admissible $\tilde{C}_g$. We have that

Lemma 3.2. *The infinite-time admissible operator $\tilde{C}_g$ satisfies*

$$\tilde{C}_g x_0 = Cg(A)x_0, \quad \textit{for } x_0 \in D\bigl(A^2\bigr). \tag{28}$$

Proof. For $x_0 \in D(A^2)$, we introduce $x_1 = Ax_0$. Then the following equalities hold in $L^2((0,\infty); Y)$.

$$\begin{aligned}
\tilde{C}_g T(t)x_0 &= (C \circ \mathfrak{O}_g)x_0 \\
&= M_g\bigl(CT(t)x_0\bigr) \\
&= M_g\bigl(CT(t)A^{-1}x_1\bigr) \\
&= M_g\bigl(CA^{-1}T(t)x_1\bigr) \\
&= CA^{-1}M_g\bigl(T(t)x_1\bigr) \\
&= CA^{-1}g(A)T(t)x_1 \\
&= Cg(A)T(t)A^{-1}x_1 = Cg(A)T(t)x_0,
\end{aligned}$$

where we have used (3). Since both functions are continuous at zero, we find that (28) holds. □

Based on this result, we denote $\tilde{C}_g$ by $Cg(A)$.

Using this, we can prove the following theorems.

Theorem 3.3. *The mapping $g \mapsto g(A)$ forms an (unbounded) $\mathcal{H}_\infty^-$-calculus.*

Proof. It only remains to show that $(g_1g_2)(A) \subset g_1(A)g_2(A)$. By Lemma 1.3 we have that

$$\mathfrak{O}_{g_1g_2}x_0 = M_{g_1g_2}\big(T(t)x_0\big) = M_{g_1}M_{g_2}\big(T(t)x_0\big).$$

For $x_0 \in D(A)$ the last expression equals $M_{g_1}(g_2(A)T(t)x_0)$, see (22). Since $g_2(A)$ commutes with the semigroup, we find that

$$\mathfrak{O}_{g_1g_2}x_0 = M_{g_1}\big(T(t)g_2(A)x_0\big).$$

Using (22) twice, we obtain

$$(g_1g_2)(A)T(t)x_0 = \mathfrak{O}_{g_1g_2}x_0 = g_1(A)T(t)g_2(A)x_0.$$

This is an equality in $L^2((0,\infty); X)$. However, if we take $x_0 \in D(A^2)$, then this holds point-wise, and so for $x_0 \in D(A^2)$.

$$(g_1g_2)(A)x_0 = g_1(A)g_2(A)x_0.$$

This concludes the proof. □

Theorem 3.4. *If there exists an admissible C such that (C, A) is exactly observable, i.e., there exists an $m_1 > 0$ such that for all $x_0 \in X$ there holds*

$$\int_0^\infty \big\|CT(t)x_0\big\|^2\,dt \geqslant m_1\|x_0\|^2,$$

then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Furthermore, if m_2 is the admissibility constant, see Eq. (12)*, then*

$$\big\|g(A)\big\| \leqslant \sqrt{\frac{m_2}{m_1}}\|g\|_\infty. \tag{29}$$

Proof. Let $x_0 \in D(A^2)$

$$\begin{aligned}
m_1\big\|g(A)x_0\big\|^2 &\leqslant \big\|CT(t)g(A)x_0\big\|^2_{L^2((0,\infty);Y)} \\
&= \big\|Cg(A)T(t)x_0\big\|^2_{L^2((0,\infty);Y)} \\
&= \|C \circ \mathfrak{O}_g x_0\|^2_{L^2((0,\infty);Y)} \\
&\leqslant \|g\|^2_\infty \big\|CT(t)x_0\big\|^2_{L^2((0,\infty);Y)} \\
&\leqslant m_2\|g\|^2_\infty\|x_0\|^2.
\end{aligned}$$

Since $D(A^2)$ is dense, we obtain the result. □

As a corollary we obtain the well-known von Neumann inequality. Recall that the operator A is dissipative if

$$\langle x_0, Ax_0\rangle + \langle Ax_0, x_0\rangle \leqslant 0 \quad \text{for all } x_0 \in D(A). \tag{30}$$

Corollary 3.5. *If A is a dissipative operator and its corresponding semigroup is exponentially stable, then A has a bounded $\mathcal{H}_\infty^-$ calculus and for all $g \in \mathcal{H}_\infty^-$*

$$\|g(A)\| \leqslant \|g\|_\infty. \tag{31}$$

Proof. Since A is dissipative and since its semigroup is exponentially stable, we have that A^{-1} is bounded and dissipative. We define Q via

$$\langle x_1, Qx_2\rangle = -\langle A^{-1}x_1, x_2\rangle - \langle x_1, A^{-1}x_2\rangle, \quad x_1, x_2 \in X. \tag{32}$$

It is easy to see that Q is bounded, self-adjoint and by the dissipativity of A^{-1} we have that $Q \geqslant 0$. Define on the domain of A the operator C as $C = \sqrt{Q}A$, then from (32) we find that

$$-\langle Cx_1, Cx_2\rangle = \langle x_1, Ax_2\rangle + \langle Ax_1, x_2\rangle, \quad x_1, x_2 \in D(A). \tag{33}$$

Combining this Lyapunov equation with the exponential stability, gives that for all $x_0 \in D(A)$

$$\int_0^\infty \|CT(t)x_0\|^2\, dt = \|x_0\|^2. \tag{34}$$

Thus we see that the constants m_1 and m_2 in Theorem 3.4 can be chosen to be one, and so (29) gives the results. □

If A generates an exponentially stable semigroup and if there exists an admissible C for which (C, A) is exactly observable, then it is not hard to show that the semigroup is similar to a contraction semigroup. Using this, one can also obtain the above result by Theorem G of [1]. The following result has been proved by McIntosh in [9] using a different approach, see also the remark following the proof.

Theorem 3.6. *Assume that A generates an exponentially stable semigroup. If $(-A)^{\frac{1}{2}}$ is admissible for $(T(t))_{t\geqslant 0}$ and $(-A^*)^{\frac{1}{2}}$ is admissible for the adjoint semigroup $(T(t)^*)_{t\geqslant 0}$, then $g(A)$ is bounded for every $g \in \mathcal{H}_\infty^-$. Thus this semigroup has a bounded $\mathcal{H}_\infty^-$-calculus.*

Proof. Since $A^{1/2}$ is admissible, Lemma 3.2 gives that $A^{1/2} \circ g(A)$ is also admissible. Consider for $x_1 \in D(A^*)$ and $x_0 \in D(A^2)$ the following

$$\begin{aligned}&\langle x_1, g(A)x_0\rangle - \langle x_1, g(A)T(t)x_0\rangle\\ &\quad= \int_0^t \langle x_1, (-A)T(\tau)g(A)x_0\rangle\, d\tau\end{aligned}$$

$$= \int_0^t \left\langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} g(A) T(\tau) x_0 \right\rangle d\tau$$

$$= \int_0^t \left\langle (-A^*)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)^* x_1, g(A)(-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right) x_0 \right\rangle d\tau$$

$$\leqslant \sqrt{\int_0^t \left\| (-A^*)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)^* x_1 \right\|^2 d\tau} \sqrt{\int_0^t \left\| g(A)(-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right) x_0 \right\|^2 d\tau}$$

$$\leqslant \sqrt{\int_0^t \left\| (-A^*)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)^* x_1 \right\|^2 d\tau} \, \|g\|_\infty \sqrt{\int_0^\infty \left\| (-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right) x_0 \right\|^2 d\tau}$$

$$\leqslant m_1 \|x_1\| m_2 \|g\|_\infty \|x_0\|,$$

where m_1 and m_0 are the admissibility constant of $(-A^*)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$, respectively. Furthermore, we used (2).

Since the sets $D(A^*)$ and $D(A^2)$ are dense in X, we obtain that

$$\left\| g(A) \right\| \leqslant m_1 m_2 \|g\|_\infty + \left\| g(A) T(t) \right\|. \tag{35}$$

By Theorem 2.5 we know that $g(A)T(t)$ is bounded, and so we conclude that $(T(t))_{t \geqslant 0}$ has a bounded $\mathcal{H}_\infty^-$-calculus. □

In McIntosh [9] the above theorem was proved using square function estimates. The admissibility of $(-A)^{\frac{1}{2}}$ can be written as

$$m\|x_0\|^2 \geqslant \int_0^\infty \left\| (-A)^{\frac{1}{2}} T(t) x_0 \right\|^2 dt$$

$$= \int_0^\infty \left\| (-tA)^{\frac{1}{2}} T(t) x_0 \right\|^2 \frac{dt}{t}.$$

The latter is the "square function estimate" for $\psi(s) = (-s)^{\frac{1}{2}} e^s$, and so the admissibility condition can be seen as a square function estimate, see also [8]. The other condition used in [9] is that the operator A is sectorial on a sector larger than the sector on which the scalar functions are defined. Since we have as function class $\mathcal{H}_\infty^-$ and since our operators A are assumed to generate an exponential semigroup, this condition seems not to satisfied. However, the admissibility assumptions made in the theorem imply that A generates a bounded analytic semigroup, and so the condition of McIntosh is satisfied.

Lemma 3.7. *Let A generate an exponentially stable semigroup and let $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ be admissible operators for $(T(t))_{t\geqslant 0}$ and $(T(t)^*)_{t\geqslant 0}$, respectively. Then A generates a bounded analytic semigroup.*

Proof. The proof is similar to the proof of Theorem 2.5. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$. Then for $t > 0$ we find

$$
\begin{aligned}
t\langle x_1, AT(t)x_0\rangle &= \int_0^t \langle x_1, AT(t)x_0\rangle \, d\tau \\
&= -\int_0^t \big\langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(t)x_0\big\rangle \, d\tau \\
&= -\int_0^t \big\langle (-A^*)^{\frac{1}{2}} T(\tau)^* x_1, (-A)^{\frac{1}{2}} T(t-\tau)x_0\big\rangle \, d\tau \\
&\leqslant \sqrt{\int_0^t \big\| (-A^*)^{\frac{1}{2}} T(\tau)^* x_1\big\|^2 \, d\tau} \sqrt{\int_0^t \big\|(-A)^{\frac{1}{2}} T(t-\tau)x_0\big\|^2 \, d\tau} \\
&\leqslant m_1\|x_1\| m_2\|x_0\|,
\end{aligned}
$$

where we used that $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are admissible. Since the domain of A^* and A are dense, we obtain that

$$
\big\|AT(t)\big\| \leqslant \frac{M}{t}, \quad t > 0.
$$

By Theorem II.4.6 of [3], we conclude that A generates a bounded analytic semigroup. □

Similarly, we can show that if there exist $\alpha, \beta > 0$ such that $(-A)^{\alpha}$ and $(-A^*)^{\beta}$ are admissible operators for $(T(t))_{t\geqslant 0}$ and $(T(t)^*)_{t\geqslant 0}$, respectively, then A generates an immediately differentiable semigroup.

From [9] we know that if the conditions of Theorem 3.6 hold, then is the semigroup similar to a contraction (or $(-A)^{\frac{1}{2}}$ is exactly observable). We show this next. Note that similar results have also been derived by Grabowski and Callier. Unfortunately, this has only been published in an internal report, [4].

Lemma 3.8. *Under the conditions of Theorem* 3.6 *we have that $(-A)^{\frac{1}{2}}$ is exactly observable, and thus $(T(t))_{t\geqslant 0}$ is similar to a contraction.*

Proof. In idea the proof is the same as that of Theorem 3.6. Let $x_1 \in D(A^*)$ and $x_0 \in D(A)$. We have that

$$\langle x_1, x_0\rangle = \int_0^\infty \big\langle x_1, (-A)T(\tau)x_0\big\rangle\, d\tau$$
$$= \int_0^\infty \big\langle (-A^*)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} T(\tau)x_0\big\rangle\, d\tau$$
$$= \int_0^\infty \left\langle (-A^*)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)^* x_1, (-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)x_0\right\rangle d\tau. \tag{36}$$

Hence

$$\big|\langle x_1, x_0\rangle\big| \leqslant \sqrt{\int_0^\infty \left\| (-A^*)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)^* x_1\right\|^2 d\tau} \sqrt{\int_0^\infty \left\| (-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)x_0\right\|^2 d\tau}$$
$$\leqslant m_1\|x_1\| \sqrt{\int_0^\infty \left\| (-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)x_0\right\|^2 d\tau}.$$

Since the domain of A^* is dense we conclude that

$$\|x_0\| = \sup_{x_1\neq 0} \frac{|\langle x_1, x_0\rangle|}{\|x_1\|} \leqslant m_1 \sqrt{\int_0^\infty \left\| (-A)^{\frac{1}{2}} T\left(\frac{\tau}{2}\right)x_0\right\|^2 d\tau}. \tag{37}$$

Thus $(-A)^{\frac{1}{2}}$ is exactly observable. □

We remark that with the above result, Theorem 3.6 follows also from Theorem 3.4. However, we decided to present this independent proof.

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