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Extension of the Douady-Hubbard's Theorem on Convergence of Periodic External Rays of the Mandelbrot Set to Polynomials of type E_d

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Abstract

We consider the complex dynamics of a one parameter family of polynomials of type E_d [1], as $f_c(z) = z(z + c)^d$, where $d \geq 1$ is a given integer and $c \in \mathbb{C}$. In the dynamics of quadratic polynomials $P_c(z) = z^2 + c$, Douady and Hubbard [2, 3] have proved that periodic external rays land on a parameter in the Mandelbrot set. This result has been extended to the parameter space of uni-critical polynomials $g_c(z) = z^d + c$[8]. We extend the Douady-Hubbard Theorem to the polynomials of type E_d.

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1 Introduction

We first recall some terminology and definitions in holomorphic dynamics. let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial self-map of the complex plan. For each $z \in \mathbb{C}$, the orbit of z is

$$\mathrm{Orb}_f(z) = \{z, f(z), f(f(z)), \cdots, f^n(z), \cdots\}.$$

The dynamical plane $\mathbb{C}$ is decomposed into two complementary sets:
the *filled Julia set*

$$K(f) = \{c \in \mathbb{C} : \quad \mathrm{Orb}_f(z) \quad \text{is bounded}\},$$

and its complementary, the *basin of infinity*

$$A_f(\infty) = \mathbb{C} - K(f).$$

The boundary of $K(f)$, called the *Julia set*, is denoted by $J(f)$.
Let f is a polynomial of degree d, in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then the point at infinity is a super-attracting fixed point and hence there exists a neighborhood of infinity, U, and a conformal map(the Böttcher map),

$$\varphi : U \to \{z \in \mathbb{C}; |z| > 1\},$$

such that $\varphi(\infty) = \infty, \varphi'(\infty) = 1$ and $\varphi \circ f = (\varphi)^d$.
For $\theta \in \mathbb{R}/\mathbb{Z}$, the dynamical external ray of argument θ is defined as

$$R_\theta = \varphi^{-1}\{re^{2\pi i\theta}; r > 1\}.$$

The dynamical external ray R_θ of $J(f)$ starts at ∞ and either ends at a precritical point or ends by accumulating on some subset of the Julia set $J(f)$. A ray R is said to land or converge, if the accumulation set $\overline{R} - R$ is a singleton subset of $J(f)$. A ray R_θ is called a rational ray if θ is rational, i.e. $\theta \in \mathbb{Q}/\mathbb{Z}$. When f is a quadratic polynomial, say $f(z) = P_c = z^2 + c$, the Mandelbrot set $\mathcal{M}_2$ is defined as the set of parameter value c, for which $K(P_c)$ is connected, that is

$$\mathcal{M}_2 = \{c \in \mathbb{C} : K(P_c) \quad \text{is connected}\}.$$

More generally, for the family $g_c(z) = z^d + c$, the connectedness locus, called the Mandelbrot set $\mathcal{M}_d$, is defined by

$$\mathcal{M}_d = \{c \in \mathbb{C} : K(g_c) \quad \text{is connected}\}.$$

Douady and Hubbard have proved that the Mandelbrot set is connected, that is, there exist the Riemann map $\Phi : \mathbb{C} - \mathcal{M}_2 \to \mathbb{C} - \overline{D}$ such that $\Phi(c) = \varphi_c(c)$. Let $R_\theta(\mathcal{M}_2) = \Phi^{-1}\{re^{2\pi i\theta}; r > 1\}$ be the parameter external ray for the Mandelbrot set.

The following substantial result have been obtained:

Douady-Hubbard's Theorem[2, 3]. *If $\theta \in \mathbb{Q}/\mathbb{Z}$ is rational with odd denominator, then the external ray $R_\theta(\mathcal{M}_2)$ for the Mandelbrot set lands at a well defined polynomial $P_c \in \mathcal{M}_2$, which possesses a parabolic periodic orbit.*

The extension of the theorem to other classes of polynomials constitutes part of today's research in this area. For instance, Douady-Hubbard's Theorem has been extended to the Mandelbrot set $\mathcal{M}_d$ of uni-critical polynomials $g_c(z) = z^d + c$ [8].

the aim of this paper is to extend Douady-Hubbard's Theorem to the class of *polynomials of type* E_d, $f_c(z) = z(z+c)^d$.
For the family $f_c(z) = z(z+c)^d$, the *connectedness locus* $\mathcal{C}_d$, or what is the same, the *Mandelbrot set* is defined by

$$\mathcal{C}_d = \{c \in \mathbb{C} : K(f_c) \quad \text{is connected}\}.$$

For the polynomial $f_c(z) = z(z+c)^d$, a rational ray R_θ will be called *periodic* if θ is periodic under multiplication by the degree $d+1$, so that $(d+1)^n\theta \equiv \theta(mod1)$ for some $n \geq 1$.
We shall see some properties of $\mathcal{C}_d$, among them the following main result:

Main result. *If $\theta \in \mathbb{Q}/\mathbb{Z}$ is rational and periodic, so that $(d+1)^n\theta \equiv \theta(mod1)$, then the external ray $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$ lands at a well defined polynomial f_c, which possesses a parabolic periodic orbit.*

2 Polynomials of type E_d

Let us recall (see[1]) terminology and definitions in monic family of higher degree polynomials. Consider the monic family of complex polynomials $f_c(z) = z(z+c)^d$, where $c \in \mathbb{C}$ and $d \geq 2$. Each f_c has degree $d+1$ and has exactly two critical points: $-c$ with multiplicity $d-1$, and $c_0 = \frac{-c}{d+1}$ with multiplicity one. Moreover, $f_c(-c) = 0$ and 0 is fixed. It is proved (see[1]) that polynomials with these features, can always be expressed in the form f_c:

Definition 2.1 *[1] A monic polynomial f of degree $d \geq 2$ is of type E_d if it satisfies the following properties:*

1. f has two critical points: $-c$ of multiplicity $d-1$ and c_0 of multiplicity one.

2. f has a fixed point at $z = 0$.

3. $f(-c) = 0$.

Proposition 2.1 *Any monic polynomial $f(z)$ of degree $d+1$ which is of type E_d is of the form*

$$f_c(z) = z(z+c)^d.$$

Moreover, if f_c and $f_{c'}$ of type E_d are affine conjugate with $d \geq 2$, then $c = c'$.

The proof is straightforward and is omitted. □

Notations. Denote Σ_d is the rotation group $\Sigma_d = \{\omega \in \mathbf{C} : \omega^d = 1\}$.

3 Boundedness and symmetry

In this section, we first observe that the Mandelbrot set $\mathcal{C}_d$ is bounded. Then we show that it is symmetric with respect to the action of Σ_d.

Proposition 3.1 *For each $d \geq 2$, there exists a real number $1 < \alpha < 2$ such that*

$$\mathcal{C}_d = \{c \in \mathbb{C} : \quad |f_c^n(c_0)| \leq (1 + \frac{d+1}{d}\alpha) \quad \textit{for every} \quad n \in \mathbb{N}\}.$$

Proof. For $|z| > (1 + |c|)$, we have $|f_c(z)| \geq |z|(|z| - |c|)^d > |z|$. It follows that

$$\{z : |z| > (1 + |c|)\} \subset A_c(\infty).$$

Now if α is the unique positive root of the polynomial $g(t) = t^{d+1} - (d+1)t - d$, then for $|c| > \frac{d+1}{d}\alpha$, we have

$$|f_c(c_0)| = (\frac{|c|}{d+1})(\frac{d|c|}{d+1})^d > (1 + |c|).$$

Henceforth, $c_0 \in A_c(\infty)$. □

Corollary. The Mandelbrot set $\mathcal{C}_d$ is a compact subset of the disk $\{c : |c| \leq \frac{d+1}{d}\alpha\}$.

Proposition 3.2 *The set $\mathcal{C}_d$ is invariant under the action of the group Σ_d.*

Proof. Let $c \in \mathcal{C}_d, \omega = e^{2\pi i/d}$ and c_0 a critical point of f_c. Then $\omega c \in \mathcal{C}_d$. Indeed, ωc_0 is a critical point of $f_{\omega c}$, the corresponding critical value being related by

$$f_{\omega c}(\omega c_0) = \omega c_0(\omega c_0 + \omega c)^d = \omega c_0(c_0 + c)^d = \omega f_c(c_0),$$

hence $|f_{\omega c}(\omega c_0)| = |f_c(c_0)|$. In view of the proposition 3.1, the value α depends only on the degrees of the polynomials f_c and $f_{\omega c}$ which are of the same degree. This completes the proof. □

4 Main Result

Now we are going to prove the landing theorem in the connectedness locus. For this purpose, we need slight modifications of some well known results.

Proposition 4.1 *[6] Let f_c is a polynomial with connected Julia set. Then every periodic external ray R_θ lands at a well defined periodic point which is either repelling or parabolic.*

(Stability of repelling orbits:)

Proposition 4.2 *[5] Suppose that for the polynomial f_{c_0}, there is a repelling periodic point z_0 at which some periodic dynamic ray at angle θ lands at z_0. Then for polynomials f_c sufficiently close to f_{c_0}, the corresponding ray lands at the corresponding fixed point of f_c.*

Proposition 4.3 *[5] If z is a parabolic fixed point of f_c with multiplier $f_c'(z) = e^{2\pi ip/q}$, then the rotation number is equal to p/q.*

Finally we have:

Theorem 4.1 *If $\theta \in \mathbb{Q}/\mathbb{Z}$ is rational and periodic, then the external ray $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$ for the Connectedness locus lands at a well defined polynomial f_c, which possesses a parabolic periodic orbit.*

Proof. We must compare external rays $R_\theta(\mathcal{C}_d)$ in parameter space with external rays R_θ for the Julia set $J(f_c)$. Recall from [4] that a polynomial $f_c(z) = z(z+c)^d$ belongs to the external ray $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d))$ in parameter space if and only if the corresponding ray R_θ in the dynamic plane passes through the critical value $f_c(\rho_0) = \upsilon_0$. Let $f_{c_0} \in \mathcal{C}_d$ be any accumulation point for the ray $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$. According to proposition 4.1 the corresponding external ray R_θ necessarily lands at a periodic point $z_0 \in J(f_{c_0})$ which is either parabolic or repelling. Suppose that this point were repelling. Then according to proposition 4.2 for any polynomial $f_c(z) = z(z+c)^d$ sufficiently close to f_{c_0} the corresponding ray $R_\theta(f_c)$ would land at a periodic point $z(f)$ close to z_0. In particular this ray $R_\theta(f_c)$ could not bounce off any pre-critical point for f_c. But if we choose any f_c belonging to $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$, then the ray $R_\theta(f_c)$ does bounce off some pre-critical point of f_c. For the angle θ is periodic with period q, it follows that the forward image $f_c^{q-1}(R_\theta(f_c))$ bounces off the critical point. Since such an $f_c \in R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$ can be chosen arbitrarily close to f_{c_0} this yields a contradiction.
Therefore z_0 must be a parabolic periodic point for f_{c_0}. Since the ray $R_\theta(f_{c_0})$

is fixed by the q-fold iterate $f_{c_0}^q$ it follows from proposition 4.3 that its landing point z_0 must be a fixed point of multiplier $+1$ for $f_{c_0}^q$.
There are only finitely many polynomials $f_c(z) = z(z+c)^d$ possesses a fixed point of multiplier one. Since the set of all limit points of $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$ in $\mathcal{C}_d$ is connected and is contained in this finite set it follows that $R_{(\frac{1}{2d}+\frac{d+1}{d}\theta)}(\mathcal{C}_d)$ must land at a single uniquely defined point $f_{c_0} \in \mathcal{C}_d$. $\square$

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