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Journal of Functional Analysis 202 (2003) 327–341
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Sub-Bergman Hilbert spaces in the unit disk, II

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Received 28 May 2002; accepted 3 September 2002

Communicated by D. Sarason

Abstract

For a finite Blaschke product B let T_B denote the analytic multiplication operator (also called a Toeplitz operator) on the Bergman space of the unit disk. We show that the defect operators $(I - T_BT_B^*)^{1/2}$ and $(I - T_B^*T_B)^{1/2}$ both map the Bergman space to the Hardy space and the Hardy space to the Dirichlet space.

MSC: 47B35; 46E20

Keywords: Defector operator; Toeplitz operator; Sub-Bergman Hilbert space

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. We will be concerned with three classical Hilbert spaces of analytic functions in $\mathbb{D}$, namely, the Bergman space A^2, the Hardy space H^2, and the Dirichlet space $\mathscr{D}$.

Throughout the paper we let B denote a finite Blaschke product. The operator of multiplication by B on A^2 will be denoted by T_B. Historically, the operator T_B is also called the (analytic) Toeplitz operator with symbol B, which explains the use of the notation T_B.

The operators T_B and T_B^* are contractions. Actually,

$$||T_B|| = ||T_B^*|| = 1.$$

So we can consider the defect operators

$$T_1 = (I - T_BT_B^*)^{1/2}, \quad T_2 = (I - T_B^*T_B)^{1/2},$$

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doi:10.1016/S0022-1236(02)00086-1

where I is the identity operator on A^2. Such square roots are very important in operator theory, especially in the areas of factorization and canonical models.

The main results of the paper are Theorems A and B below. Theorem A describes the range of the operators T_1 and T_2 above as the Hardy space.

Theorem A. $T_1(A^2) = T_2(A^2) = H^2$.

When we restrict the operators T_1 and T_2 to the subspace H^2 of A^2, we find that the image must be the Dirichlet space.

Theorem B. $T_1(H^2) = T_2(H^2) = \mathscr{D}$.

Note that when $B = z^N$, the operators

$$I - T_B T_B^*, \quad I - T_B^* T_B$$

are both positive and diagonal with respect to the standard basis of A^2, and their eigenvalues are easily computable. This, along with the description of A^2, H^2, and $\mathscr{D}$ in terms of Taylor coefficients, easily shows that the theorems above are true in this case.

The general case will be proved in the framework of sub-Bergman Hilbert spaces. This framework was established in [3] as an extension of the theory of sub-Hardy Hilbert spaces developed by de Branges, Rovnyak, Sarason, and some of their students and collaborators. Sarason's book [2] contains the main achievements in this area.

We thank Michael Stessin for many helpful conversations on Toeplitz operators induced by finite Blaschke products.

2. Preliminaries

Recall that the Bergman space A^2 of $\mathbb{D}$ consists of analytic functions f in $\mathbb{D}$ such that

$$||f||_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. We will need to use two other forms of the norm in A^2. The first involves the Taylor coefficients of f. Thus for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have

$$||f||_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

Another calculation using the Taylor expansion shows that $|f|^2_{A^2}$ is comparable to

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^2 \, dA(z).$$

The Hardy space H^2 of $\mathbb{D}$ consists of analytic functions f in $\mathbb{D}$ such that

$$||f||^2_{H^2} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt < \infty.$$

It is well known that $\sup_{0<r<1}$ above can be replaced by $\sup_{\sigma<r<1}$ for any constant $\sigma \in (0, 1)$. In terms of the Taylor expansion of f,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have

$$||f||^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2.$$

We will also need to use the following form of the norm on H^2:

$$||f||^2_{H^2} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|^2} \, dA(z).$$

Since

$$\log \frac{1}{|z|^2} \sim (1 - |z|^2)$$

as $|z| \to 1^-$, we conclude that an analytic function f in $\mathbb{D}$ belongs to the Hardy space if and only if

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) \, dA(z) < \infty;$$

and the square root of this quantity is equivalent to the norm on H^2.

The Dirichlet space $\mathscr{D}$ consists of analytic functions f in $\mathbb{D}$ with

$$||f||^2_{\mathscr{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

When

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is the Taylor series of f, we have

$$||f||_{\mathscr{D}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2.$$

In particular, we see that $\mathscr{D} \subset H^2 \subset A^2$.

It should be clear how to polarize the above norms to get the corresponding inner products on A^2, H^2, and $\mathscr{D}$.

We will also need the space H^∞, consisting of all bounded analytic functions in $\mathbb{D}$. It is easy to see that H^∞ is dense in H^2 and in A^2. However, the space H^∞ is not contained in the Dirichlet space $\mathscr{D}$. The set of all polynomials is dense in each of the spaces A^2, H^2, and $\mathscr{D}$.

Since the Bergman space A^2 is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, we can consider the orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto A^2. This is usually called the Bergman projection and it has the following integral representation:

$$P(g)(z) = \int_{\mathbb{D}} \frac{1}{(1 - z\bar{w})^2} g(w)\, dA(w), \quad g \in L^2(\mathbb{D}, dA),$$

where the integral kernel above is called the Bergman kernel.

Given any function $\varphi \in L^\infty(\mathbb{D})$, we can define a linear operator T_φ on A^2 as follows:

$$T_\varphi f = P(\varphi f), \quad f \in A^2.$$

It is obvious that T_φ is bounded with $||T_\varphi|| \leqslant ||\varphi||_\infty$. The operator T_φ is called the Toeplitz operator on A^2 with symbol φ.

It is easy to check that $T_\varphi^* = T_{\bar{\varphi}}$ for each $\varphi \in L^\infty(\mathbb{D})$. Three Toeplitz operators will be of interest to us in the paper, namely, T_B, $T_{\bar{B}}$, and $T_{|B|^2}$. Observe that $T_{\bar{B}} = T_B^*$ and $T_{\bar{B}} T_B = T_{|B|^2}$.

The next estimate is undoubtedly well known, but we include a proof here for the lack of a specific reference.

Lemma 1. *If φ is an analytic self-map of the disk $\mathbb{D}$, then φ is a finite Blaschke product if and only if there exists a constant $C > 0$ such that*

$$C^{-1}(1 - |z|^2) \leqslant 1 - |\varphi(z)|^2 \leqslant C(1 - |z|^2)$$

for all $z \in \mathbb{D}$.

Proof. First observe that for every analytic self-map φ of $\mathbb{D}$ there exists a positive constant $C > 0$ with

$$1 - |z|^2 \leqslant C(1 - |\varphi(z)|^2)$$

for all $z \in \mathbb{D}$. This follows easily from the classical Schwarz lemma.

Next observe that if $B = B_1 B_2$ is a product of two Blaschke products, then

$$1 - |B|^2 = 1 - |B_1|^2 + |B_1|^2(1 - |B_2|^2) \leqslant (1 - |B_1|^2) + (1 - |B_2|^2).$$

The inequality

$$1 - |B(z)|^2 \leqslant C(1 - |z|^2)$$

for finite Blaschke products then follows from induction and the above observation.

Finally, if there exists a constant $C > 0$ such that

$$1 - |\varphi(z)|^2 \leqslant C(1 - |z|^2)$$

for all $z \in \mathbb{D}$, then it is clear that

$$\lim_{z \to \zeta} |\varphi(z)| = 1$$

for every point $\zeta \in \partial\mathbb{D}$. In particular, φ is an inner function. Since the above convergence is uniform, the function φ does not collapse at any boundary point, which implies that B must be a finite Blaschke product; see Section II-6 of [1]. $\square$

Lemma 2. *For any* $0 \leqslant t < \infty$ *the integral operator*

$$Tf(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - z\bar{w}|^{2+t}} f(w)\, dA(w)$$

is bounded on $L^2(\mathbb{D}, dA)$.

Proof. See Theorem 4.2.3 of [4]. $\square$

Since T_B is just the operator of multiplication by B on A^2, its action on A^2 is evident. We now describe the action of $T_{\bar{B}}$ and $T_{|B|^2}$ on A^2.

Proposition 3. *Suppose the zeros of B are distinct. If* $f \in A^2$ *and F is any anti-derivative of* f *in* $\mathbb{D}$, *then*

$$T_{\bar{B}} f(z) = \frac{f(z)}{B(z)} - \frac{F(z)B'(z)}{B(z)^2} + \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(z - a_k)^2},$$

where $a_1, \ldots, a_N$ *are the zeros of B in* $\mathbb{D}$.

Proof. Since the desired result is a pointwise formula, an easy approximation argument shows that we can assume f to be analytic on the closed unit disk. Let

$$\partial = \partial_w + \partial_{\bar{w}}$$

and recall that

$$dA(w) = \frac{1}{2\pi i}\, d\bar{w} \wedge dw.$$

An application of the Stokes formula yields

$$\begin{aligned} T_{\bar{B}} f(z) &= -\frac{1}{2\pi i} \int_{\mathbb{D}} \partial \left(\frac{\bar{B}(w) F(w)}{(1 - z\bar{w})^2}\, d\bar{w} \right) \\ &= -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\bar{B}(w) F(w)}{(1 - z\bar{w})^2}\, d\bar{w} \\ &= \frac{1}{2\pi i} \frac{d}{dz} \int_{\partial\mathbb{D}} \frac{F(w)\, dw}{B(w)(w - z)}. \end{aligned}$$

By the residue theorem,

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{F(w)\, dw}{B(w)(w - z)} = \frac{F(z)}{B(z)} + \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(a_k - z)}.$$

It follows that

$$T_{\bar{B}} f(z) = \frac{f(z)}{B(z)} - \frac{F(z) B'(z)}{B^2(z)} + \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(z - a_k)^2},$$

completing the proof of the proposition. □

Corollary 4. *Suppose all the zeros of B, $a_1, \ldots, a_N$, are simple. If $f \in A^2$ and F is any anti-derivative of Bf in $\mathbb{D}$, then*

$$T_{|B|^2} f(z) = f(z) - \frac{F(z) B'(z)}{B(z)^2} + \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(z - a_k)^2}.$$

Proof. Simply replace f by Bf in Proposition 3. □

Note that if the zeros of B are not distinct, for example, if a_k is a zero of B of multiplicity m with $m > 1$, then the term

$$\frac{F(a_k)}{B'(a_k)(a_k - z)}$$

in the proposition and corollary above should be replaced by the residue of the function

$$g(w) = \frac{F(w)}{B(w)(w - z)}$$

at the point a_k, which is a linear combination of

$$\frac{1}{a_k - z}, \ldots, \frac{1}{(a_k - z)^m},$$

and hence is a rational function whose poles are at the zeros of B.

In the formulas (and their revised version if there are multiple zeros) in both Proposition 3 and Corollary 4, the poles in the second term are cancelled by the poles of the third term, so we do not have to worry about the zeros that appear in the denominator. If we denote by G the sum of the second and third term, then it is clear that the membership of G in A^2, H^2, or $\mathscr{D}$ is equivalent to that of the function F. This observation will be used several times later without being explicitly mentioned.

3. Sub-Bergman Hilbert spaces

Let H be a Hilbert space and T a contraction on H. Following [2] we use $\mathscr{H}(T)$ to denote the Hilbert space whose underlying set is the range of the positive operator $(I - TT^*)^{1/2}$ and whose inner product is given by

$$\langle (I - TT^*)^{1/2}x, (I - TT^*)^{1/2}y \rangle_{\mathscr{H}(T)} = \langle x, y \rangle_H,$$

where

$$x, y \in H \ominus \ker(I - TT^*)^{1/2}.$$

When $T = T_B$ on A^2, we denote the resulting space $\mathscr{H}(T)$ by $\mathscr{H}(B)$. Similarly, we use $\mathscr{H}(\bar{B})$ to denote the Hilbert space $\mathscr{H}(T_{\bar{B}})$. Since the spaces $\mathscr{H}(B)$ and $\mathscr{H}(\bar{B})$ are Hilbert spaces contained in A^2, it is natural for us to call them sub-Bergman Hilbert spaces.

Note that the inner products on $\mathscr{H}(B)$ and $\mathscr{H}(\bar{B})$ are drastically different from the inner product on A^2. In fact, it was shown in [3] that neither $\mathscr{H}(B)$ nor $\mathscr{H}(\bar{B})$ is closed in the A^2-metric, but both spaces contain H^∞.

In terms of sub-Bergman Hilbert spaces, we see that the assertions in Theorem A of the Introduction are equivalent to

$$\mathscr{H}(B) = \mathscr{H}(\bar{B}) = H^2.$$

Theorem B then follows from this and some computations involving the operators $I - T_B T_{\bar{B}}$ and $I - T_{\bar{B}} T_B$.

We will need the following result from [3] about sub-Bergman Hilbert spaces.

Lemma 5. *The space $\mathcal{H}(\bar{B})$ consists of analytic functions in $\mathbb{D}$ of the form*

$$f(z) = \int_{\mathbb{D}} \frac{1 - |B(w)|^2}{(1 - z\bar{w})^2} g(w)\, dA(w),$$

where g is analytic in $\mathbb{D}$ and

$$\int_{\mathbb{D}} |g(z)|^2 (1 - |B(z)|^2)\, dA(z) < \infty.$$

We will also need to use the following result.

Lemma 6. *We always have $H^\infty \subset \mathcal{H}(\bar{B}) \subset \mathcal{H}(B)$.*

Proof. The inclusion $H^\infty \subset \mathcal{H}(\bar{B})$ is established in [3]. The inclusion $\mathcal{H}(\bar{B}) \subset \mathcal{H}(B)$ follows from Douglas's criterion (see I-4 and I-5 of [2]) and the operator inequality

$$T_B T_{\bar{B}} \leqslant T_{\bar{B}} T_B,$$

which is a consequence of the subnormality of the analytic Toeplitz operator T_B. Note that this argument was due to the referee of the paper [3]. □

4. Proof of Theorem A

We prove Theorem A in this section.

Lemma 7. *The space $\mathcal{H}(\bar{B})$ is contained in H^2.*

Proof. Given $f \in \mathcal{H}(\bar{B})$, Lemmas 1 and 5 tell us that f can be represented as

$$f(z) = \int_{\mathbb{D}} \frac{1 - |B(w)|^2)}{(1 - z\bar{w})^2} g(w)\, dA(w),$$

where g is analytic in $\mathbb{D}$ and

$$\int_{\mathbb{D}} |g(z)|^2 (1 - |z|^2)\, dA(z) < \infty.$$

Differentiating under the integral sign, we obtain

$$f'(z) = 2 \int_{\mathbb{D}} \frac{1 - |B(w)|^2}{(1 - z\bar{w})^3} \bar{w}\, g(w)\, dA(w).$$

Using Lemma 1 and the fact that

$$(1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \leqslant 2|1-z\bar{w}|,$$

we can find a positive constant C such that

$$(1-|z|^2)^{1/2}|f'(z)| \leqslant C \int_{\mathbb{D}} \frac{1}{|1-z\bar{w}|^2}(1-|w|^2)^{1/2}|g(w)|\, dA(w).$$

By Lemma 2,

$$\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)\, dA(z) < \infty,$$

which shows that f belongs to H^2. □

Theorem 8. $\mathscr{H}(\bar{B}) = H^2$.

Proof. Let H_B denote the Hilbert space of analytic functions f in $\mathbb{D}$ such that

$$||f||_B^2 = \int_{\mathbb{D}} |f(z)|^2(1-|B(z)|^2)\, dA(z) < \infty.$$

The inner product on H_B is the polarization of the above norm. By Lemma 1 and the fact that

$$||f||_{H^2}^2 \sim |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)\, dA(z),$$

the operator $S : H^2 \to H_B$ defined by

$$Sf(z) = \frac{d}{dz}(zf(z)) = f(z) + zf'(z)$$

is invertible. In particular, the image of the unit ball of H^2 under the mapping S must contain a ball with center 0 and radius c in H_B, where c is some positive constant.

Consider the operator

$$Tg(z) = \int_{\mathbb{D}} \frac{1-|B(w)|^2}{(1-z\bar{w})^2} g(w)\, dA(w).$$

By Lemmas 5 and 7, T is a bounded linear operator from H_B into H^2. Furthermore, according to Lemma 6, the range of $T : H_B \to H^2$ (which is $\mathscr{H}(\bar{B})$) contains H^∞. To show that $T(H_B) = H^2$, it suffices to show that the operator $T : H_B \to H^2$ is bounded below.

Let c be the positive constant from the first paragraph of this proof. Then for any unit vector g in H_B, we have

$$\begin{aligned}||Tg||_{H^2} &= \sup\{|\langle Tg, f\rangle_{H^2}| : ||f||_{H^2} \leqslant 1\}\\ &= \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{Sf(w)}(1-|B(w)|^2)\,dA(w)\right| : ||f||_{H^2} \leqslant 1\right\}\\ &\geqslant c\sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\,dA(w)\right| : ||h||_B \leqslant 1\right\}\\ &= c,\end{aligned}$$

where the second equality above follows from a use of Fubini's theorem. The shows that the operator $T : H_B \to H^2$ is bounded below and hence must be onto. □

Theorem 9. $\mathscr{H}(B) = H^2$.

Proof. By Lemma 6 and Theorem 8, it suffices to prove $\mathscr{H}(B) \subset H^2$.

Given $f \in \mathscr{H}(B)$, we apply Proposition 3.8 of [3] and Theorem 8 above to get $T_{\bar{B}}f \in \mathscr{H}(\bar{B}) = H^2$. The desired result then follows from the following lemma. □

Lemma 10. *If f is in A^2, then $T_{\bar{B}} \in H^2$ if and only if $f \in H^2$; and $T_{\bar{B}}f \in \mathscr{D}$ if and only if $f \in \mathscr{D}$.*

Proof. This is a direct consequence of Proposition 3 (and the remarks following Corollary 4) and the fact that any anti-derivative of a function in A^2 must belong to the Dirichlet space $\mathscr{D}$ which is contained in H^2. □

5. Proof of Theorem B

Theorem B will be proved in this section. Actually, we will prove that the operators $I - T_B T_{\bar{B}}$ and $I - T_{\bar{B}} T_B$ both map A^2 to $\mathscr{D}$, which, when combined with Theorem A, will then prove Theorem B.

Theorem 11. $(I - T_{\bar{B}} T_B)(A^2) = \mathscr{D}$.

Proof. Consider the operator

$$Tf(z) = \int_{\mathbb{D}} \frac{1-|B(w)|^2}{(1-z\bar{w})^2} f(w)\,dA(w).$$

In the previous section, we showed that T maps H_B, the Hilbert space of analytic functions f in $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f(z)|^2(1-|B(z)|^2)\,dA(z)<\infty,$$

onto H^2. It is easy to see that the operator

$$I - T_{\bar{B}}T_B = I - T_{|B|^2}$$

is the restriction of T to A^2.

Differentiating inside the integral and applying Lemma 1, we obtain a constant $C>0$ such that

$$\left|\frac{d}{dz}Tf(z)\right| \leqslant C\int_{\mathbb{D}} \frac{1-|w|^2}{|1-z\bar{w}|^3}|f(w)|\,dA(w).$$

This, together with Lemma 2, shows that T maps A^2 boundedly into the Dirichlet space $\mathcal{D}$.

To prove that $I - T_{|B|^2}$ maps A^2 onto $\mathcal{D}$, we recall from Corollary 4 that

$$(I - T_{|B|^2})f(z) = \frac{F(z)B'(z)}{B(z)^2} - \sum_{k=1}^{N}\frac{F(a_k)}{B'(a_k)(z-a_k)^2},$$

where F is any anti-derivative of Bf. Note that if B has multiple zeros, then a revised version of the above formula should be used; see the remarks following Corollary 4. Since A^2 is dense in H_B, an easy approximation argument shows that the above formula for $I - T_{|B|^2}$ also holds for T, that is,

$$Tf(z) = \frac{F(z)B'(z)}{B(z)^2} - \sum_{k=1}^{N}\frac{F(a_k)}{B'(a_k)(z-a_k)^2}$$

for all $f \in H_B$.

Given a function $g \in \mathcal{D} \subset H^2$, there exists a function $f \in H_B$ such that $g = Tf$. By the representation of T in the previous paragraph, any anti-derivative F of Bf belongs to $\mathcal{D}$. Since differentiation carries the Dirichlet space to the Bergman space, we conclude that $Bf \in A^2$. But B is a finite Blaschke product, we must have $f \in A^2$. This shows that the operator $I - T_{|B|^2}$ maps A^2 onto $\mathcal{D}$. □

Corollary 12. $(I - T_{\bar{B}}T_B)^{1/2}(H^2) = \mathcal{D}$.

Proof. This is a direct consequence of Theorems 8 and 11. □

Recall that the operator $T : H_B \to H^2$ is invertible. In particular, the restriction of T to A^2 must be one to one. It follows that the operator $I - T_{|B|^2} : A^2 \to \mathscr{D}$ is invertible. The following result tells us how to reverse this operator on a large part of $\mathscr{D}$.

Theorem 13. *Suppose* $g \in \mathscr{D}$ *and* g *vanishes at all the zeros of* $B'(z)$ (*counting multiplicities*). *If*

$$f(z) = 2g(z) + \frac{B(z)g'(z)}{B'(z)} - \frac{B(z)g(z)B''(z)}{B'(z)^2},$$

then $f \in A^2$ *and* $(I - T_{|B|^2})f = g$.

Proof. We may assume that the zeros of B are all simple. The general case then follows from an approximation argument.

Recall that

$$T_{|B|^2}f(z) = f(z) - \frac{F(z)B'(z)}{B(z)^2} + \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(z - a_k)^2},$$

where $f \in A^2$ and F is any anti-derivative of Bf. Note that any anti-derivative of Bf can be used here, so in the calculations below, we will not attach an arbitrary constant to the indefinite integrals.

Suppose f and g satisfy the assumptions in the theorem. Then it is easy to check that

$$f(z) = \frac{d}{dz}\left(\frac{h'(z)}{B'(z)}\right) = g(z) + \frac{d}{dz}\left(\frac{B(z)g(z)}{B'(z)}\right),$$

where $h = BG$ and G is any anti-derivative of g. This shows that $f \in A^2$. Also, we can integrate by parts to find an anti-derivative of Bf as follows.

$$F(z) = \int B(z)f(z)\,dz = \frac{B(z)h'(z)}{B'(z)} - \int h'(z)\,dz = \frac{B(z)h'(z)}{B'(z)} - h(z).$$

Plugging in $h = BG$ and simplifying the result, we obtain

$$F(z) = \frac{B^2(z)g(z)}{B'(z)}.$$

In particular, F vanishes at all the zeros of B. Combining this with what we had in the previous paragraph, we then obtain

$$T_{|B|^2}f = f - g$$

or

$$(I - T_{|B|^2})f = g,$$

finishing the proof of the theorem. □

Theorem 14. $(I - T_B T_{\bar{B}})(A^2) = \mathscr{D}$.

Proof. It follows from Proposition 3 that

$$(I - T_B T_{\bar{B}})f(z) = \frac{B'(z)}{B(z)} F(z) - B(z) \sum_{k=1}^{N} \frac{F(a_k)}{B'(a_k)(z - a_k)^2},$$

where $a_1, \ldots, a_N$ are the zeros of B and F is any anti-derivative of f. Once again, if B has multiple zeros, the formula above needs to be modified, but the following argument still works in this case. Since anti-differentiation takes the Bergman space to the Dirichlet space, the above formula clearly shows that the operator $I - T_B T_{\bar{B}}$ carries A^2 into the Dirichlet space $\mathscr{D}$.

To prove that $I - T_B T_{\bar{B}}$ maps A^2 onto $\mathscr{D}$, we are going to make use of the intertwining relation

$$T_{\bar{B}}(I - T_B T_{\bar{B}})^{1/2} = (I - T_{\bar{B}} T_B)^{1/2} T_{\bar{B}}.$$

See I-7 in [2].

Given $g \in \mathscr{D}$, Theorem 9 tells us that there exists $h \in A^2$ such that

$$g = (I - T_B T_{\bar{B}})^{1/2} h.$$

Applying the intertwining relation mentioned earlier, we obtain

$$T_{\bar{B}} g = (I - T_{\bar{B}} T_B)^{1/2} T_{\bar{B}} h.$$

By Lemma 10, $T_{\bar{B}} g$ belongs to $\mathscr{D}$. Since the operator $(I - T_{\bar{B}} T_B)^{1/2}$ is one to one on A^2 and maps H^2 exactly to $\mathscr{D}$, we conclude that the function $T_{\bar{B}} h$ belongs to H^2, which in turn gives $h \in H^2$ according to Lemma 10. Applying Theorem 9 again, we can find a function $f \in A^2$ such that

$$h = (I - T_B T_{\bar{B}})^{1/2} f.$$

Combining this with the earlier representation of g, we obtain

$$g = (I - T_B T_{\bar{B}}) f,$$

completing the proof of the theorem. $\square$

Note that we can also find an explicit formula for the inverse of the invertible operator

$$I - T_B T_{\bar{B}} : A^2 \to \mathscr{D}$$

on a large part of $\mathscr{D}$. In fact, if $g \in \mathscr{D}$ and g vanishes at the zeros of $B'(z)$ including multiplicity, then

$$(I - T_B T_{\bar{B}}) f = g,$$

where

$$f = F', \quad F = \frac{Bg}{B'}.$$

We omit the details.

Corollary 15. $(I - T_B T_{\bar{B}})^{1/2}(H^2) = \mathscr{D}$.

Proof. This is clearly a consequence of Theorems 9 and 14. $\square$

We have now completed the proof of Theorems A and B stated in the Introduction.

6. Further remarks

For simplicity of presentation we still write

$$T_1 = (I - T_B T_{\bar{B}})^{1/2}, \quad T_2 = (I - T_{\bar{B}} T_B)^{1/2}.$$

Carefully examining the proofs earlier, we find that the mappings

$$T_k : A^2 \to H^2, \quad T_k : H^2 \to \mathscr{D}$$

are all bounded invertible operators. As a result, we realize that the Hardy space H^2 can be equipped with two inner products such that the associated norms are equivalent to $\| \|_{H^2}$ and the associated reproducing kernels are given by

$$K_1(z, w) = \frac{1 - B(z)\overline{B(w)}}{(1 - z\bar{w})^2}$$

and

$$K_2(z, w) = \int_{\mathbb{D}} \frac{1 - |B(u)|^2}{(1 - z\bar{u})^2 (1 - u\bar{w})^2} \, dA(u),$$

respectively. It would be interesting to have an explicit form for these inner products on H^2.

If φ is a more general function in the closed unit ball of H^∞, in particular, if φ is a more general inner function, it would be nice to know more about the associated sub-Bergman spaces $\mathscr{H}(T_\varphi)$ and $\mathscr{H}(T_{\bar{\varphi}})$.

References

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