Sub-Bergman Hilbert Spaces on the Unit Disk

KEHE ZHU

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let H^p and A^p denote the Hardy and Bergman spaces on $\mathbb{D}$, respectively. Our main concern will be the Bergman space A^2, which is a Hilbert space with the following inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z),$$

where dA is the normalized area measure on $\mathbb{D}$.

Suppose T is a contraction on a Hilbert space H. Following [6] we shall use $\mathcal{H}(T)$ to denote the Hilbert space whose underlying set is the range of the operator $(I - TT^*)^{1/2}$ and whose inner product is given by

$$\left\langle (I - TT^*)^{1/2}x, (I - TT^*)^{1/2}y \right\rangle_{\mathcal{H}(T)} = \langle x, y \rangle_H, \quad x, y \in H \ominus \ker(I - TT^*)^{1/2}.$$

If the contraction T is a Toeplitz operator on H^2 or A^2 induced by an analytic function φ, we then denote the resulting space by $\mathcal{H}(\varphi)$. Similarly, if T is the Toeplitz operator on H^2 or A^2 induced by a conjugate analytic symbol $\overline{\varphi}$, then we denote the resulting space by $\mathcal{H}(\overline{\varphi})$. In the context of Hardy spaces, $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$ are called sub-Hardy Hilbert spaces by Sarason in [6]. We thus arrive at the title of the present paper.

The theory of sub-Hardy Hilbert spaces was developped by de Branges, Rovnyak, Sarason, and some of their students and collaborators. Sarason's recent monograph [6] presents most of the main developments in this area.

The purpose of this paper is to examine some of the problems considered in [6] in the context of Bergman spaces. Our approach will be via the general theory of reproducing kernels. As a by-product of our analysis we shall obtain a sharper version of a result of Hedenmalm's about extremal functions for invariant subspaces of the Bergman space.

Indiana University Mathematics Journal ©, Vol. 45, No. 1 (1996)

2. Preliminaries on reproducing kernels. In this section we recall several fundamental results from the theory of (analytic) reproducing kernels which will be needed later for studying sub-Bergman Hilbert spaces.

A function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is called a reproducing kernel on $\mathbb{D}$ if there exists a (unique) Hilbert space H of functions on $\mathbb{D}$ such that $K_w = K(\,\cdot\,, w)$ belongs to H for every $w \in \mathbb{D}$ and

$$f(w) = \langle f, K_w \rangle, \qquad f \in H, w \in \mathbb{D}.$$

The following theorem of Moore [5] characterizes reproducing kernels on $\mathbb{D}$ in terms of the positivity of certain matrices. Note that we do not attempt to give original references for results we use. Our main reference for the general theory of reproducing kernels is Saitoh's book [5].

Theorem 2.1 *A function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is a reproducing kernel on $\mathbb{D}$ if and only if the matrix $(K(z_i, z_j))_{n \times n}$ is positive-definite, where n is any positive integer and $z_1, \cdots, z_n$ are arbitrary points in $\mathbb{D}$. Equivalently, the function K is a reproducing kernel on $\mathbb{D}$ if and only if*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(z_i, z_j) c_i \overline{c_j} \geq 0$$

for all positive integers n and all $z_1, \cdots, z_n \in \mathbb{D}$ and $c_1, \cdots, c_n \in \mathbb{C}$.

If K is a reproducing kernel on $\mathbb{D}$, then it can be shown that K has the following properties; see [5].

(1) $K(z, z) \geq 0$ for all $z \in \mathbb{D}$.
(2) $K(z, w) = \overline{K(w, z)}$ for all z and w in $\mathbb{D}$.
(3) If H is the Hilbert space corresponding to K and $\{e_n\}$ is an orthonormal basis for H, then

$$K(z, w) = \sum e_n(z) \overline{e_n(w)}, \qquad z, w \in \mathbb{D}.$$

(4) $|K(z, w)|^2 \leq K(z, z) K(w, w)$ for all $z, w \in \mathbb{D}$.

The last property above is usually useful in checking that a certain function K is not a reproducing kernel; we shall call this property the Cauchy-Schwarz inequality for reproducing kernels.

As a consequence of Moore's theorem we see that the sum of two reproducing kernels is again a reproducing kernel. It can also be checked using Moore's theorem that the product of two reproducing kernels is again a reproducing kernel; see Lemma 11.

Let H^∞ denote the space of bounded analytic functions in $\mathbb{D}$, equipped with the sup-norm. We use $(H^\infty)_1$ to denote the closed unit ball of H^∞.

The following theorem of Beatrous and Burbea [5] gives a non-trivial way of generating new reproducing kernels from old ones. This result will be the basis for our later analysis.

Theorem 2.2 *Let K be a reproducing kernel on $\mathbb{D}$ and H be the corresponding reproducing Hilbert space. For $\varphi \in (H^\infty)_1$ the function*

$$K^*(z,w) = \left(1 - \varphi(z)\overline{\varphi(w)}\right) K(z,w)$$

is a reproducing kernel on $\mathbb{D}$ if and only if φ is a contractive multiplier of H.

That φ is a contractive multiplier of H means that for every function $f \in H$ we have $\varphi f \in H$ and $\|\varphi f\| \leq \|f\|$.

As an example, we observe that if φ is an element of the closed unit ball of H^∞, then φ is a contractive multiplier of A^2. It follows that the function

$$K_\varphi(z,w) = \frac{1 - \varphi(z)\overline{\varphi(w)}}{(1 - z\overline{w})^2}$$

is a reproducing kernel on $\mathbb{D}$. We shall see later that the reproducing Hilbert space corresponding to this kernel is the sub-Bergman Hilbert space $\mathcal{H}(\varphi)$ defined in the Introduction. The same remark applies to the Hardy space situation. In particular, we see that we could have defined sub-Hardy or sub-Bergman Hilbert spaces in terms of reproducing kernels instead of ranges of Toeplitz operators.

3. Sub-Bergman Hilbert spaces. In this section we study sub-Bergman Hilbert spaces on the unit disk. In particular, we shall show that every bounded analytic function is a multiplier of the sub-Bergman Hilbert spaces $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$. We shall also prove that each one of these spaces contains H^∞. This is in sharp contrast with the situation of sub-Hardy spaces; see [6].

For a bounded function φ in $\mathbb{D}$ recall that the Toeplitz operator $T_\varphi : A^2 \to A^2$ is defined by

$$T_\varphi(f) = P(\varphi f), \qquad f \in A^2,$$

where P is the Bergman projection. It is clear that T_φ is a bounded linear operator on A^2 with $\|T_\varphi\| \leq \|\varphi\|_\infty$. See [7] for basic properties of Toeplitz operators. Recall that for a function φ in the closed unit ball of H^∞, the sub-Bergman Hilbert spaces $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$ are defined in the introduction in terms of certain operator ranges.

We begin with the calculation of reproducing kernels for the sub-Bergman Hilbert spaces. First note that if T is a contraction on a Hilbert space H with reproducing kernels K_w^H, $w \in \mathbb{D}$, then the reproducing kernels for $\mathcal{H}(T)$ (see Introduction for definition) are given by $(I - TT^*)K_w^H$, $w \in \mathbb{D}$; see I-3 in [6] for an explanation.

Proposition 3.1 *Suppose φ is a function in the closed unit ball of H^∞. Then the reproducing kernel of $\mathcal{H}(\varphi)$ is*

$$K_\varphi(z,w) = \frac{1-\varphi(z)\overline{\varphi(w)}}{(1-z\overline{w})^2}.$$

Proof. Let K_w be the reproducing kernels of A^2. Then the reproducing kernels of $\mathcal{H}(\varphi)$ are given by $(I - T_\varphi T_{\overline{\varphi}})K_w$, $w \in \mathbb{D}$. Since $T_{\overline{\varphi}}K_w = \overline{\varphi(w)}K_w$, we arrive at the desired result. □

Corollary 3.2 *Let n be a positive integer. Then $\mathcal{H}(z^n) = H^2$ with inner product given by*

$$\langle f, g\rangle = \sum_{k=0}^{n-1} \frac{a_k \bar{b}_k}{k+1} + \frac{1}{n}\sum_{k=n}^{\infty} a_k \bar{b}_k,$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Proof. According to the proposition above, the reproducing kernel of $\mathcal{H}(z^n)$ is given by

$$K_n(z,w) = \frac{1}{1-z\overline{w}} \sum_{k=0}^{n-1} (z\overline{w})^k.$$

A direct calculation shows that this is the reproducing kernel of H^2 with the inner product

$$\langle f, g\rangle = \sum_{k=0}^{n-1} \frac{a_k \bar{b}_k}{k+1} + \frac{1}{n}\sum_{k=n}^{\infty} a_k \bar{b}_k,$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Since a reproducing kernel uniquely determines the corresponding reproducing Hilbert space, we obtain the desired results. □

As a consequence of the corollary above we see that for every integer n the operator $(I - T_{z^n} T_{z^n}^*)^{1/2}$ maps A^2 onto H^2. In particular, the operator $(I - T_z T_{\overline{z}})^{1/2}$ is a unitary from A^2 onto H^2. It would be interesting to find a direct proof of this.

Proposition 3.3 *For $\varphi \in (H^\infty)_1$ the reproducing kernel of $\mathcal{H}(\overline{\varphi})$ is given by*

$$K_{\overline{\varphi}}(z, w) = \int_{\mathbb{D}} \frac{1 - |\varphi(u)|^2}{(1 - z\overline{u})^2(1 - u\overline{w})^2}\, dA(u).$$

Proof. Let K_w, $w \in \mathbb{D}$, be the reproducing kernels for A^2. Then the reproducing kernels for $\mathcal{H}(\overline{\varphi})$ are given by $(I - T_{|\varphi|^2})K_w$, $w \in \mathbb{D}$. Specifically, the reproducing kernel of $\mathcal{H}(\overline{\varphi})$ is

$$K_{\overline{\varphi}}(z, w) = T_{1-|\varphi|^2} K_w(z) = \int_{\mathbb{D}} \frac{1 - |\varphi(u)|^2}{(1 - z\overline{u})^2(1 - u\overline{w})^2}\, dA(u). \qquad \square$$

Corollary 3.4 *Let n be a positive integer. Then $\mathcal{H}(\overline{z}^n) = H^2$ with inner product given by*

$$\langle f, g\rangle = \langle f, g\rangle_{A^2} + \frac{1}{n}\langle f, g\rangle_{H^2}, \qquad f,\ g \in \mathcal{H}(\overline{z}^n).$$

Proof. A direct calculation shows that the reproducing kernel of H^2 with inner product

$$\langle f, g\rangle = \langle f, g\rangle_{A^2} + \frac{1}{n}\langle f, g\rangle_{H^2}$$

is

$$K_n(z, w) = \sum_{k=0}^{\infty} \frac{n(k+1)}{k+n+1}(z\overline{w})^k = \int_{\mathbb{D}} \frac{1 - |u|^{2n}}{(1 - z\overline{u})^2(1 - u\overline{w})^2}\, dA(u).$$

Since a reproducing kernel uniquely determines a reproducing Hilbert space, we arrive at the desired result. $\square$

It follows from the corollary above that the range of the operator $(I - T_{|z|^{2n}})^{1/2}$ on A^2 is H^2, where n is any positive integer. Again we do not know a direct proof of this.

Proposition 3.5 *Suppose $\varphi \in (H^\infty)_1$. Then $\mathcal{H}(\overline{\varphi})$ consists of analytic functions of the form*

$$f(z) = \int_{\mathbb{D}} \frac{1 - |\varphi(w)|^2}{(1 - z\overline{w})^2} g(w)\, dA(w),$$

where g is analytic and

$$\int_{\mathbb{D}} |g(z)|^2 (1 - |\varphi(z)|^2)\, dA(z) < +\infty.$$

Proof. Let

$$dA_\varphi(z) = (1 - |\varphi(z)|^2)\, dA(z)$$

and let A^2_φ be the closed subspace of $L^2(\mathbb{D}, dA_\varphi)$ consisting of analytic functions. Consider the operator

$$S_\varphi : A^2_\varphi \to A^2$$

defined by

$$S_\varphi g(z) = P\left((1 - |\varphi|^2)g\right)(z) = \int_{\mathbb{D}} \frac{1 - |\varphi(w)|^2}{(1 - z\overline{w})^2}\, g(w)\, dA(w),$$

where P is the Bergman projection. It is easy to see that S_φ is a contraction. Furthermore, $S^*_\varphi : A^2 \to A^2_\varphi$ is simply the inclusion operator. In fact, for $f \in A^2$ and $g \in A^2_\varphi$ we have

$$\begin{aligned} \left\langle S^*_\varphi f, g\right\rangle_{A^2_\varphi} &= \langle f, S_\varphi g\rangle_{A^2} = \left\langle f, P((1 - |\varphi|^2)g)\right\rangle_{A^2} \\ &= \left\langle f, (1 - |\varphi|^2)g\right\rangle_{A^2} = \langle f, g\rangle_{A^2_\varphi}. \end{aligned}$$

Let $\mathcal{M}(S_\varphi)$ be the image of S_φ in A^2 together with the inner product

$$\langle S_\varphi f, S_\varphi g\rangle = \langle f, g\rangle_{A^2_\varphi}, \qquad f, g \in A^2_\varphi \ominus \ker(S_\varphi).$$

Then $\mathcal{M}(S_\varphi)$ is a Hilbert space with reproducing kernels $S_\varphi S^*_\varphi K_w$, $w \in \mathbb{D}$, where K_w are the reproducing kernels of A^2. Since $S^*_\varphi : A^2 \to A^2_\varphi$ is the inclusion mapping, we conclude that the reproducing kernel of $\mathcal{M}(S_\varphi)$ is given by

$$S_\varphi K_w(z) = \int_{\mathbb{D}} \frac{1 - |\varphi(u)|^2}{(1 - z\overline{u})^2(1 - u\overline{w})^2}\, dA(u),$$

which coincides with the reproducing kernel of $\mathcal{H}(\overline{\varphi})$. By uniqueness, we have $\mathcal{H}(\overline{\varphi}) = \mathcal{M}(S_\varphi)$, completing the proof of the proposition. □

For a bounded analytic function φ in $\mathbb{D}$ we introduce two more sub-Bergman Hilbert spaces. Specifically, we let $\mathcal{M}(\varphi) = \varphi A^2$ together with the inner product

$$\langle \varphi f, \varphi g\rangle_{\mathcal{M}(\varphi)} = \langle f, g\rangle_{A^2};$$

and we let $\mathcal{M}(\overline{\varphi})$ be the range of the Toeplitz operator $T_{\overline{\varphi}}$ on A^2 together with the inner product

$$\langle T_{\overline{\varphi}} f, T_{\overline{\varphi}} g\rangle_{\mathcal{M}(\overline{\varphi})} = \langle f, g\rangle_{A^2},$$

where f and g are from $A^2 \ominus \ker T_{\overline{\varphi}}$. It is easy to check that $\ker T_{\overline{\varphi}}$ is the orthogonal complement of the invariant subspace I_φ of A^2 generated by φ.

Proposition 3.6 *Suppose φ is a bounded analytic function in $\mathbb{D}$. Then $\mathcal{M}(\varphi)$ is closed in A^2 if and only if $\varphi = FB$, where F is an invertible element in H^∞ and B is a Blaschke product whose zero set is the union of finitely many interpolating sequences.*

This is well-known; see [4] for example. The following result is more interesting and we have not seen it before.

Theorem 3.7 *Let φ be a (nonzero) bounded analytic function in $\mathbb{D}$. Then $\mathcal{M}(\overline{\varphi})$ is always dense in A^2. Moreover, the following conditions are equivalent.*

(1) *$\mathcal{M}(\overline{\varphi})$ is closed in A^2.*
(2) *$\mathcal{M}(\overline{\varphi}) = A^2$.*
(3) *$T_{|\varphi|^2}$ is invertible on A^2.*
(4) *$\varphi = FB$, where F is an invertible element in H^∞ and B is a Blaschke product whose zero set is the union of finitely many interpolating sequences.*

Proof. Recall that $\ker(T_{\overline{\varphi}}) = I_\varphi^\perp$, where I_φ is the invariant subspace generated by φ. It follows that $\mathcal{M}(\overline{\varphi})$ is closed if and only if $T_{\overline{\varphi}}$ is bounded below on I_φ, or

$$\langle T^2_{|\varphi|^2} f, f\rangle \geq \epsilon \langle T_{|\varphi|^2} f, f\rangle$$

for some $\epsilon > 0$ and all $f \in A^2$. By the spectral theorem, the inequality $T^2_{|\varphi|^2} \geq \epsilon T_{|\varphi|^2}$ is equivalent to the invertibility of $T_{|\varphi|^2}$. This proves the equivalence of (1) and (3).

Since T_φ is one-to-one on A^2 (unless $\varphi = 0$), its adjoint, $T_{\overline{\varphi}}$, must have dense range. Thus (1) and (2) are equivalent.

On the other hand, the spectral theorem implies that the invertibility of $T_{|\varphi|^2}$ is equivalent to $T_{|\varphi|^2} \geq \delta I$ for some $\delta > 0$. This is equivalent to

$$\int_{\mathbb{D}} |\varphi f|^2 \, dA \geq \delta \int_{\mathbb{D}} |f|^2 \, dA$$

for all $f \in A^2$, which, according to [4], is equivalent to (4). □

Proposition 3.8 *Suppose $\varphi \in (H^\infty)_1$ and $f \in A^2$. Then f belongs to $\mathcal{H}(\varphi)$ if and only if $T_{\overline{\varphi}} f$ belongs to $\mathcal{H}(\overline{\varphi})$ in which case*

$$\|f\|^2_{\mathcal{H}(\varphi)} = \|f\|^2 + \|T_{\overline{\varphi}} f\|^2_{\mathcal{H}(\overline{\varphi})}.$$

Similarly, f belongs to $\mathcal{H}(\overline{\varphi})$ if and only if $\varphi f \in \mathcal{H}(\varphi)$ in which case

$$\|f\|^2_{\mathcal{H}(\overline{\varphi})} = \|f\|^2 + \|\varphi f\|^2_{\mathcal{H}(\varphi)}.$$

Proof. This is a special case of a more general result of Lotto and Sarason's; see I-8 of [6]. □

Proposition 3.9 *Let $\varphi \in (H^\infty)_1$. Then*

$$\mathcal{H}(\varphi) \cap \mathcal{M}(\varphi) = \varphi\mathcal{H}(\overline{\varphi}).$$

Proof. Again, this is a special case of the more general result I-9 in [6]. This is also an easy consequence of Proposition 3.8. □

Proposition 3.10 *Suppose φ and ψ are functions in H^∞ with $\|\varphi\|_\infty \le 1$. Then the Toeplitz operator $T_{\overline{\psi}}$ leaves both $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$ invariant.*

Proof. Same as the Hardy space case; see II-7 in [6]. □

The following lemma will be needed for our determination of the multipliers of both $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$. Although it is probably well-known to experts, we still include a proof, because we do not have a reference for it.

Lemma 3.11 *Suppose K_1 and K_2 are two reproducing kernels on $\mathbb{D}$. Then the product $K = K_1K_2$ is again a reproducing kernel on $\mathbb{D}$.*

Proof. Let H be the Hilbert space corresponding to K_1. Let $\{e_n\}$ be an orthonormal basis for H. Then

$$K_1(z,w) = \sum_{n=1}^\infty e_n(z)\overline{e_n(w)}, \qquad z, w \in \mathbb{D}.$$

Now if $z_1, \cdots, z_N$ are points from $\mathbb{D}$ and $c_1, \cdots, c_N$ are complex numbers, then by Theorem 2.1

$$\sum_{i,j=1}^N K(z_i,z_j)c_i\overline{c_j} = \sum_{n=1}^\infty \sum_{i,j=1}^N K_2(z_i,z_j)(c_ie_n(z_i))\overline{(c_je_n(z_j))} \ge 0,$$

since K_2 is a reproducing kernel. Applying Theorem 2.1 again, we conclude that K is a reproducing kernel. □

Note that the lemma above can also be proved by using Theorem 2.1 and the following well-known fact from linear algebra: The Hadamard product (or entry by entry product) of two positive-definite matrices is again positive-definite.

Note also that we are concerned with the usual product of two kernel functions here. This is not to be confused with the well-known "product" $K_1 * K_2$, which is defined on $\mathbb{D} \times \mathbb{D}$ as follows:

$$(K_1 * K_2)((z_1, z_2), (w_1, w_2)) = K_1(z_1, w_1)K_2(z_2, w_2).$$

The reproducing Hilbert space corresponding to $K_1 * K_2$ is the tensor product of H_1 and H_2 (see [5]), where H_1 and H_2 are the reproducing Hilbert spaces corresponding to K_1 and K_2, respectively.

Theorem 3.12 *Suppose φ and ψ are functions in H^∞ and $\|\varphi\|_\infty \le 1$. Then ψ is a multiplier on both $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$. Moreover, the norm of M_ψ on $\mathcal{H}(\varphi)$ or $\mathcal{H}(\overline{\varphi})$ does not exceed $\|\psi\|_\infty$.*

Proof. Without loss of generality we may assume $\|\psi\|_\infty = 1$. To show that ψ is a contractive multiplier on $\mathcal{H}(\varphi)$, according to Theorem 2.2, it suffices to show that the function

$$K(z, w) = \frac{(1 - \psi(z)\overline{\psi(w)})(1 - \varphi(z)\overline{\varphi(w)})}{(1 - z\overline{w})^2}$$

is a reproducing kernel. Since the functions

$$\frac{1 - \psi(z)\overline{\psi(w)}}{1 - z\overline{w}}, \qquad \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}}$$

are both reproducing kernels (of sub-Hardy Hilbert spaces [6]), the desired result then follows from Lemma 3.11.

That ψ is a multiplier of $\mathcal{H}(\overline{\varphi})$ follows from Proposition 3.8 and the just proved fact that ψ is a multiplier of $\mathcal{H}(\varphi)$. Indeed, if $f \in \mathcal{H}(\overline{\varphi})$ then $\varphi f \in \mathcal{H}(\varphi)$ and hence $\psi\varphi f \in \mathcal{H}(\varphi)$, which in turn gives $\psi f \in \mathcal{H}(\overline{\varphi})$ with

$$\begin{aligned} \|\psi f\|^2_{\mathcal{H}(\overline{\varphi})} &= \|\psi f\|^2 + \|\psi\varphi f\|^2_{\mathcal{H}(\varphi)} \\ &\le \|\psi\|^2_\infty \left(\|f\|^2 + \|\varphi f\|^2_{\mathcal{H}(\varphi)}\right) \\ &= \|\psi\|^2_\infty \|f\|^2_{\mathcal{H}(\overline{\varphi})}, \end{aligned}$$

completing the proof of the theorem. □

Corollary 3.13 *Let φ be a non-constant function in the unit ball of H^∞. Then both $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$ are dense and non-closed subspaces in A^2.*

Proof. Just observe that if I is a closed subspace of A^2 invariant under both T_z and $T_{\overline{z}}$, then I is either $\{0\}$ or A^2. □

Theorem 3.14 *Let φ be a function in the closed unit ball of H^∞. Then both $\mathcal{H}(\varphi)$ and $\mathcal{H}(\overline{\varphi})$ contain H^∞.*

Proof. Let A^2_φ be the space defined in the proof of Proposition 3.5. Consider the closed subspace M of A^2_φ spanned by the functions $z, z^2, \cdots$. Since $f(0) = 0$ for every $f \in M$ and the constant function 1 is in A^2_φ, we see that M is a proper subspace of A^2_φ. Let g be a unit vector in $A^2_\varphi \ominus M$. Then

$$\int_{\mathbb{D}} \frac{1-|\varphi(u)|^2}{(1-z\overline{u})^2} g(u)\, dA(u) = \int_{\mathbb{D}} (1-|\varphi(u)|^2) g(u)\, dA(u) = \langle g, 1\rangle_{A^2_\varphi}$$

for every $z \in \mathbb{D}$. Thus by Proposition 3.5 the space $\mathcal{H}(\overline{\varphi})$ contains the constant function with value $\langle g, 1\rangle_{A^2_\varphi}$.

If $\langle g, 1\rangle_{A^2_\varphi} = 0$ for every unit vector g in $A^2_\varphi \ominus M$, then it will follow that $1 \in M$, which is clearly impossible. Thus $\mathcal{H}(\overline{\varphi})$ contains a non-zero constant function. Since $\mathcal{H}(\overline{\varphi})$ is invariant under multiplication by any bounded analytic function, we conclude that $\mathcal{H}(\overline{\varphi})$ contains H^∞.

To show that $\mathcal{H}(\varphi)$ contains H^∞, it again suffices to show that $\mathcal{H}(\varphi)$ contains the constant function 1, since $\mathcal{H}(\varphi)$ is invariant under multiplication by every bounded analytic function. By Proposition 3.8 we have $1 \in \mathcal{H}(\varphi)$ if and only if $T_{\overline{\varphi}}1 \in \mathcal{H}(\overline{\varphi})$. But $T_{\overline{\varphi}}1 = \overline{\varphi(0)}$. The desired result then follows from the just proved fact that $\mathcal{H}(\overline{\varphi})$ contains H^∞. □

The referee pointed out that $\mathcal{H}(\overline{\varphi}) \subset \mathcal{H}(\varphi)$ which follows by definition of these spaces together with the subnormality of the Toeplitz operator T_φ. Therefore, the proof of Theorem 3.14 is completed after establishing that $H^\infty \subset \mathcal{H}(\overline{\varphi})$.

4. Hedenmalm's estimate for extremal functions. Let I be an invariant subspace of A^2. Let $m = m_I$ be the smallest nonnegative integer so that there exists $f \in I$ with $f^{(m)}(0) \neq 0$. The extremal function of I, denoted G_I, is defined to be the unique solution to the following problem:

$$\sup\{ \mathit{Re}\, f^{(m)}(0) : \|f\| \le 1, f \in I\}.$$

These functions were introduced and studied by Hedenmalm in [2]. In particular, the following result was proved in [2] as a consequence of the proof of the so-called "expansive multiplier property" for the extremal functions.

Proposition 4.1 *Every extremal function G in A^2 satisfies $|G(z)| \le (1 - |z|^2)^{-1/2}$ for all $z \in \mathbb{D}$.*

The "expansive multiplier property" depends very much on the underlying space. For example, this property does not even hold for very nicely weighted Bergman spaces on the disk; see [3]. However, we shall present a proof for the above estimate on extremal functions which works in much more general spaces than the Bergman spaces. Our approach also improves Hedenmalm's estimate.

Theorem 4.2 *Let G be the extremal function for an invariant subspace I of A^2. Suppose $K_{I^\perp}$ is the reproducing kernel for the orthogonal complement of I in A^2. Then*

$$|G(z)|^2 \leq \frac{1}{1-|z|^2} - (1-|z|^2)K_{I^\perp}(z,z)$$

for all $z \in \mathbb{D}$.

Proof. Without loss of generality we may assume $m_I = 0$; the general case will then follow from an approximation argument.

Since I is an invariant subspace of A^2, the function z is a contractive multiplier on I. By Theorem 2.2 the function $(1-z\overline{w})K_I(z,w)$ is a reproducing kernel, where K_I is the reproducing kernel of I. Applying the Cauchy-Schwarz inequality to this new kernel, we obtain

$$\frac{|K_I(z,w)|^2}{K_I(z,z)K_I(w,w)} \leq \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$$

for all z and w not in Z_I. Set $w = 0$ and observe that

$$G(z) = K_I(z,0)/\sqrt{K_I(0,0)}$$

in the case $m_I = 0$. This leads to

$$|G(z)|^2 \leq (1-|z|^2)K_I(z,z)$$

for all $z \in \mathbb{D}$. Since

$$K_I(z,z) + K_{I^\perp}(z,z) = \frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

the desired result follows. □

5. Some remarks. Our approach here works in much more general situations than that of the Bergman space. For example, the results here can easily be generalized to weighted Bergman spaces (with standard weights) on the unit disk. Some results even generalize to weighted Bergman and Hardy spaces on the unit ball and polydisk of $\mathbb{C}^n$.

The density of polynomials in a reproducing Hilbert space plays an important role in certain factorization problems in operator theory; see [1]. We do not know if the polynomials are dense in $\mathcal{H}(\varphi)$ or $\mathcal{H}(\overline{\varphi})$.

Let I be an invariant subspace of A^2. It would be interesting to consider the Hilbert space corresponding to the reproducing kernel

$$\left(1-\varphi(z)\overline{\varphi(w)}\right) K_I(z,w),$$

where K_I is the reproducing kernel of I and $\varphi \in (H^\infty)_1$. We are able to compute such a space only in very special cases.

Finally, we would like to thank the referee for several useful suggestions and comments.

REFERENCES

[1] G. ADAMS, J. FROELICH, P. MCGUIRE, AND V. PAULSEN, *Analytic reproducing kernels and factorization*, Indiana Univ. Math. J. **43** (1994), 839-856.

[2] H. HEDENMALM, *A factorization theorem for square area-integrable analytic functions*, J. reine angew. Math. **422** (1991), 45-68.

[3] H. HEDENMALM AND K. ZHU, *On the failure of optimal factorization for certain weighted Bergman spaces*, Complex Variables **19** (1992), 165-176.

[4] G. MCDONALD AND C. SUNDBERG, *Toeplitz operators on the disk*, Indiana Univ. Math. J. **28** (1979), 595-611.

[5] S. SAITOH, *Theory of Reproducing Kernels and its Applications*, Pitman Research Notes in Mathematics **189**, New York, 1988.

[6] D. SARASON, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons, Inc., New York, 1994.

[7] K. ZHU, *Operator Theory in Function Spaces*, Dekker, New York, 1990.

This research was supported, in part, by a grant from the National Science Foundation.

Department of Mathematics
State University of New York
Albany, NY 12222
E-MAIL: kzhu@math.albany.edu

Received: August 1st, 1995; revised: July 5th, 1996.