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SOME APPLICATIONS OF PLANAR GRAPH IN KNOT THEORY*

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Abstract The relationship between a link diagram and its corresponding planar graph is briefly reviewed. A necessary and sufficient condition is given to detect when a planar graph corresponds to a knot. The relationship between planar graph and almost planar Seifert surface is discussed. Using planar graph, we construct an alternating amphicheiral prime knot with crossing number n for any even number $n \geq 4$. This gives an affirmative answer to problem 1.66(B) on Kirby's problem list.

Key words Planar graph; almost planar Seifert surface; amphicheiral knot; alternating knot

2000 MR Subject Classification 57M25

1 Introduction

From a link diagram we can obtain a signed planar graph [1, 2]. Contrarily, we can also reestablish the original link from that signed planar graph. Hence, theoretically we can obtain all messages about a link from its corresponding signed planar graph. Sometimes it is more convenient to study the signed planar graphs than study links. In 1930s, C.Bankwitz studied the alternating knot from planar graph, in 1980s K.Murasugi defined some polynomial invariants with the planar graph viewpoint [3]. In this article, we use the conversion between links and signed planar graphs to obtain some interesting results about links. All nomenclatures undefined follow from [1, 4].

2 The Conversion Between Links and Signed Planar Graphs

Let D be a link diagram of a link L. D divides the plane into a number of regions. Now, we color these regions with white or black. We begin by coloring the unbounded region white, denoted by W, then color the neighboring regions into black, denoted by D. Next, for those regions who share a common edge with the black regions, we color them white, and this process

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continues until all the regions have been colored white or black. The next theorem claims this is a well-defined coloring.

Theorem 2.1 [5] The coloring process given above is well-defined.

Proof The proof is given by induction on the crossing number n of D. First, consider the case of $n = 0$, then, D is just some disjoint disks and the conclusion follows obviously. Assuming that the conclusion is correct when the crossing number is n, we will show that, when crossing number is $n + 1$, the conclusion is still correct. Consider a neighborhood of a crossing point locally, we can get rid of the double-point as follows:

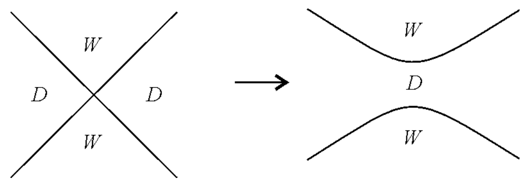

Figure 1

The new diagram has only n crossing points so it can be well colored by induction. Then, the upper and lower region of the right figure in Figure 1 must be painted with the same color, so the diagram with $n + 1$ crossing points can be colored so that there is no common edge between two regions with the same color. Hence, we finish the proof.

With a diagram D we associate a signed planar graph G as follows. We first place a vertex in each of the black regions. If two black regions meet at a crossing point, then we add an edge between two vertices which the black regions correspond. Finally, we assign each edge a sign $+$ or $-$ according to whether the twist of the crossing is positive or negative, see Figure 2. Now, we obtain a signed planar graph from a given link diagram.

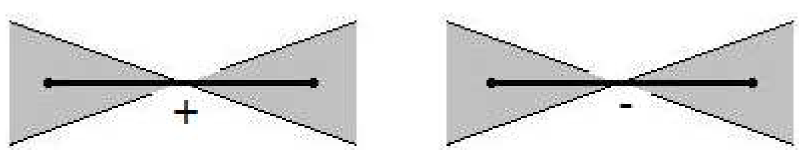

Figure 2

Contrarily, for a given signed planar graph G, we can construct a link diagram. Consider a vertex of G with degree k, then there are k radials starting from that vertex. We place k points around the vertex, such that there exists one point between two contiguous radials. Following the process below, we can get a local crossing from an edge of G. For all the vertices and edges, repeat the above process, then we can obtain a link diagram.

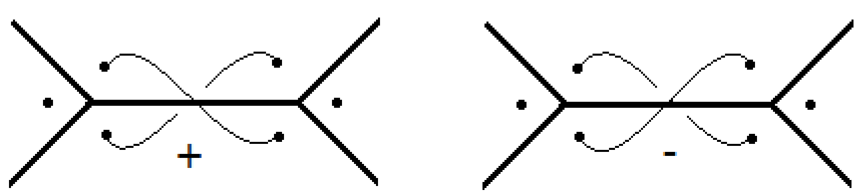

Figure 3

It is not difficult to find that the two processes above are reversed, that is, if we convert a diagram to a signed planar graph, then convert this graph to a new link diagram, the new diagram denotes the same link and vice versa. For the signed planar graph of a diagram, we have some simple conclusions below:

Proposition 2.1 A diagram is alternating, if and only if all the edges of its signed planar graph have the same sign.

Proposition 2.2 If we change the signs for all edges of a signed planar graph, then the link associates to the new graph is the mirror image of the link corresponding to the former diagram.

Proposition 2.3 A diagram is reduced, if and only if the planar graph contains no loop and cut-edge.

Given a signed planar graph G, it is not obvious whether it corresponds to a link or a knot. A natural question is which kind of graph represents a knot. Although this is a fundamental question, we do not find any result about it. So, we give an answer here. As a disconnected signed planar graph of course represents a splittable link, so without loss of generality, we assume that G is connected. Besides, we also assume that the graph G contains no loop and hang-edge, in fact, we can get rid of all loops and hang-edges by the first kind of Reidemeister move. Consider an edge l_1 of G with two end-vertices a_1, a_2. With a_2 as the center, rotate l_1 clockwisely until meeting another edge with one end point a_2, say l_2. Next, we rotate l_2 around the other end-vertex a_3 of l_2 anticlockwisely, then it will meet another edge l_3 with one end point a_3. Then, we turn l_3 around on the other end-vertex a_4 of l_3 clockwisely. Repeating this process until we come back to l_1 and the next rotation is the same as that of the first time. Then, $l_1, l_2, \cdots, l_i, l_1$ form a circuit of G, and we call it a K-circuit.

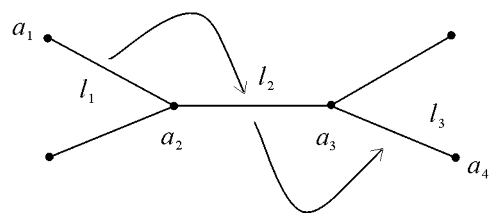

Figure 4

For an edge l of G, we can choose either of the two end-vertices as the center of rotation, and the direction can also be chosen as either clockwise or counterclockwise. Then starting with l, we can obtain 4 K-circuits (maybe some of them are the same). Suppose the size (the number of edges) of G is n, then we can get $4n$ K-circuits totally. However, some of them actually denote the same K-circuit, we use i to denote the number of different K-circuits and we say G contains i K-circuits. Specially, when $i = 1$, we say G is a K-signed planar graph.

Theorem 2.2 A signed planar graph G contains i K-circuits, if and only if the link associated to G contains i connected components.

Proof First, if G contains i K-circuits, we claim that each K-circuit of G represents a connected component of the link L. Consider a fixed K-circuit, suppose it starts from l_1, and the first rotation is around an end-vertex a_2 of l_1 with clockwise rotation, until l_1 meets l_2. As each edge of G locally represents two segments with one crossing, when we finish the first rotation, a segment of l_1 is connected with a segment of l_2. Next, when l_2 is turned to l_3

anticlockwisely, the long segment above is joined with a segment of l_3, see Figure 5. Continuing the process along the chosen K-circuit we can get a growing curve, and when the rotation comes back to l_1, we will obtain a simple closed curve, which is a component of L. It is obvious that different K-circuit represents different component of the link L. So, if G contains i K-circuits, then the associated link L also contains i connected components.

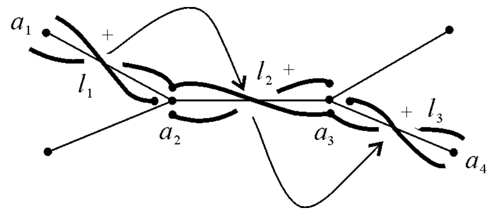

Figure 5

Contrarily, given a diagram of a link with i connected components. From the analysis above, it is found that a component of the link corresponds to an K-circuit of the planar graph D. So, the number of K-circuits is exactly the number of components of L. The proof is finished.

Specifically, when a signed planar graph contains only one K-circuit, we have a simple corollary below:

Corollary 2.1 A signed planar graph corresponds to a knot, if and only if it is an K-signed planar graph.

3 Almost Planar Seifert Surface and Planar Graph

For any tame knot $K \subset R^3$, there exists a Seifert surface S in R^3, such that $\partial S = K$. Usually S cannot be embedded in R^2, unless K is unknot and S is chosen as a disk. Although S cannot be planar in general, we are interested in some special kinds of Seifert surface, say almost planar Seifert surface.

Definition 3.1 A Seifert surface S in R^3 is called an almost planar Seifert surface if there exists a projection $f : S \to R^2$ such that there are only finitely many points of $f(S)$ whose preimages contain more than one point and $f(K)$ is a diagram of K.

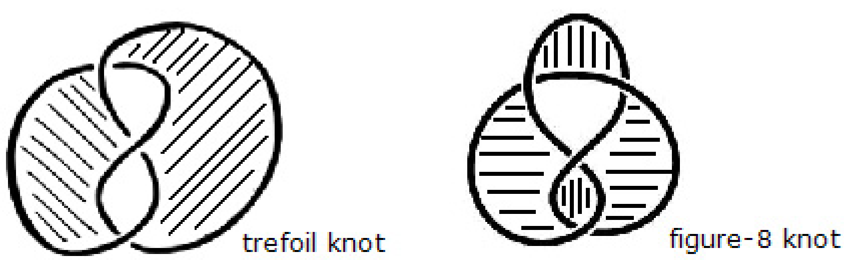

Figure 6

From the definition, it follows that every multiple point of $f(S)$ is exactly a double point of $f(K)$. For example, the Seifert surface of right-hand trefoil in Figure 6 is an almost planar Seifert surface, because those multiple points of $f(S_{\text{trefoil}})$ are just the 3 double points of the diagram. For the Seifert Surface of figure-8 knot in Figure 6, however, it is not an almost planar Seifert surface. In fact, suppose there exists a projection f such that $f(S_{\text{figure}-8})$ contains only finite number of multiple points, because this Seifert surface can be regarded as two copies of full-twisted annules $S_1^1 \times I, S_2^1 \times I$ with $I_1 \times I \sim I \times I_2$, here I_i $(i = 1, 2)$ denotes a segment of S_i^1 $(i = 1, 2)$. Notice that $S_1^1 \times t_1$ is perpendicular to $t_2 \times S_2^1$, hence $f(S_1^1 \times t_1)$ is perpendicular to $f(t_2 \times S_2^1)$ at p on R^2. Obviously, $f(S_1^1 \times t_1)$ and $f(t_2 \times S_2^1)$ have at least another intersection point q, then, it is easy to find that a neighborhood of q contains infinite multiple points, see Figure 7 below. Therefore, the seifert surface of figure-8 knot in Figure 6 is not an almost planar Seifert surface.

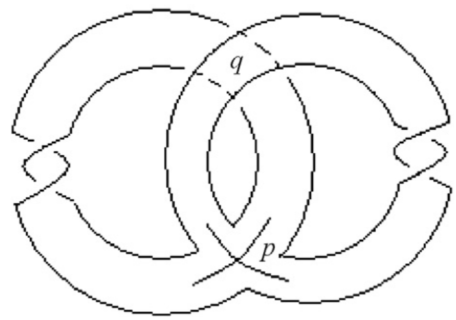

Figure 7

Now, we give another description of almost planar Seifert surface. It is well-known that every knot in S^3 can be represented as a closed braid. According to Seifert's algorithm, after killing all the double points of a closed braid, we get some disjoint Seifert circles on the plane. They are all around a center point from the innermost to the outermost. However, for an almost planar Seifert surface, if one gets rid of all the double points of the associated diagram, one gets some Seifert circles which are mutually separate. This means that the interior of any Seifert circle contains no other Seifert circles. So, if there exists a diagram of K such that all Seifert circles obtained from Seifert's algorithm are mutually separate, then this Seifert surface is an almost planar Seifert surface. From the graph viewpoint, we have a theorem below:

Theorem 3.1 A knot K has an almost planar Seifert surface, if and only if there exists a diagram of K such that its corresponding signed planar graph is a bipartite graph.

Proof For necessity, if there is an almost planar Seifert surface S with $\partial S = K$, then from the analysis above, we know that there is a diagram of K such that all Seifert circles are mutually separate. The associated planar graph is exactly the Seifert graph on the plane. As Seifert surface is orientable so there is no odd cycle in the planar graph. From the theory of graphs, we know that a graph is a bipartite graph if and only if it contains no odd cycle. So, the conclusion follows.

For sufficiency, with a given bipartite graph G, it contains no odd cycle. Then, we can assign each vertex of G with a positive sign or negative sign, such that the end-vertices of any edge have different signs. Transform each vertex into a small disk with two sides colored white and black respectively, and make the upper side of the disk black (white) if the sign of the associated vertex is positive (negative). Next, convert each edge into a half-twist band with ends connected to the disks obtained from its end-vertices. It is seen that the surface we get is an orientable surface with boundary K. As multiple points of the projection of the surface are one to one corresponding to the edges of G, so it is an almost planar Seifert surface. The proof is finished.

It is obvious that almost planar Seifert surface also can be defined for links and the statement above still holds. Specifically, we can get some message about the genus of K when K has an almost planar Seifert surface, or has a singed planar bipartite graph, according to Theorem 3.1.

Corollary 3.1 If there exists a diagram for a link L such that its associated planar graph G is a bipartite graph, and suppose G divides the plane into f regions, then $g(L) \leq \frac{f-n}{2}$, here n denotes the number of components of L.

Proof We use v, e to denote the number of vertices and edges of G, respectively. From the proof of Theorem 3.1, we can get a Seifert surface F of L from G, and $g_F(L) = 1 - \frac{v-e+n}{2} = \frac{f-n}{2}$, where $f = 2 - v + e$. Hence, $g(L) \leq g_F(L) = \frac{f-n}{2}$.

Corollary 3.2 Under the same assumption of Corollary 3.1, if all the signs of edges of G are the same, then, $g(L) = \frac{f-n}{2}$.

Proof According to Proposition 2.1 all the signs of edges of G are the same implies that the associated diagram is an alternating diagram. As the genus of an alternating link is realized by the Seifert surface obtained by applying Seifert's algorithm to a reduced alternating projection[6], then the result follows.

4 The Construction of Prime Alternating Amphicheiral Knots

An alternating knot is any knot that has a projection such that if one traverses the knot in a fixed direction then under crossings and over crossings appear alternately. Alternating knots is an important kind of knots and people have made a deep study of them. In 1983, William Menasco proved that, if $K_1\#K_2$ is an alternating knot, then it appears composite in any alternating projection [1]. In 1986, Louis Kauffman, Morwen Thistlethwaite, Kunio Murasugi proved that any reduced diagram of an alternating link has the fewest possible crossings [7–9]. From these results, it is concluded that $c(K_1\#K_2) = c(K_1) + c(K_2)$ for $K_1\#K_2$ an alternating knot. In 1990, William Menasco and Morwen Thistlethwaite proved the Tait Flyping Conjecture [10], which states that, given any two reduced alternating diagrams D_1 and D_2 of an oriented prime alternating link, D_1 may be transformed to D_2 by a sequence of moves called flypes. See the figure below:

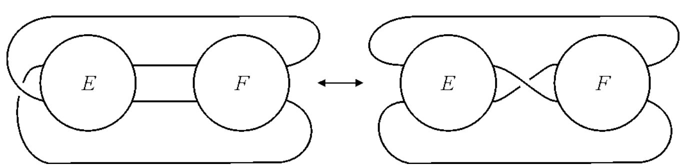

Figure 8

About 100 years ago, Tait once conjectured that amphicheiral knots have even crossing numbers. However, in 1998, a 15-crossing nonalternating amphicheiral knot was discovered by Hoste et al [11]. Although this conjecture is not true in general, one can prove that alternating amphicheiral knots have even crossing number [1]. A nature question is that, for any given even number n, is there an alternating amphicheiral prime knot with crossing number n? This is exactly the Problem 1.66(B) on Kirby's problem list [12]. In Theorem 4.2, we will give an affirmative answer to it.

Before constructing these knots, we consider the geometric dual graph of a planar graph first. For a signed planar graph G in R^2, we can define a dual signed planar graph G' of G in R^2 and the sign of each edge of G' is defined opposite to that of its dual edge in G. It is easy to generalize this definition to a signed graph on S^2, which could be regarded as one point compactification of R^2. From the process of the conversion between links and signed planar graphs, it is seen that, if two signed graphs are isotopic on S^2 (that is, two planar graphs are isotopic and all signs of corresponding edges are the same), then the associated links are equivalent. For signed planar graph and its dual graph on S^2, we have

Theorem 4.1 The associated links of a signed graph and its dual signed graph on S^2 are equivalent.

Proof We use L and L' to denote two links corresponding to G and G', respectively. In fact, G and G' provide two dual dissections of S^2 and we can realize the isotopy from L to L' locally step by step in some regions of the dissection. For example, we can move an arc l of L to the position of l', which is an arc of L', see Figure 8. As the sign of an edge of G is opposite to that of its dual edge in G', so the crossing corresponding to the edge is preserved after the move. For each edge of G, we make a similar movement. This realizes the isotopy from L to L'. The result follows.

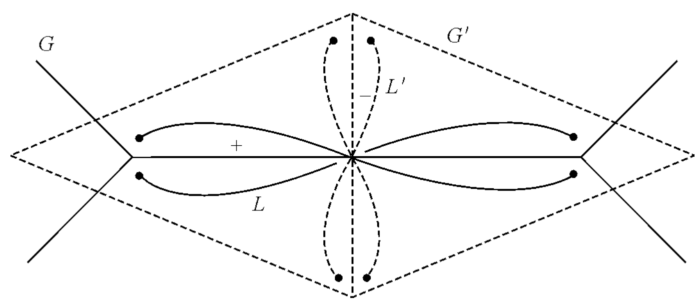

Figure 9

Corollary 4.1 If there exists a diagram D for a knot K such that the associated planar graph G is isotopic to its dual planar graph G' on S^2, and the signs of corresponding edges of G and G' are opposite, then K is an amphicheiral knot.

Proof As G is isotopic to G' and all the signs on edges are opposite, so K is the mirror image of the knot corresponding to G', say K'. By Theorem 4.1, we know that K and K' are the same knot so K is equivalent to its mirror image, the proof is finished.

From the corollary above, it is found that, if the diagram D is alternating and G is isotopic to its dual planar graph G' as planar graphs, then K is an amphicheiral knot. With this conclusion, we can construct an alternating amphicheiral prime knot with crossing number n for every even $n \geq 4$. In fact, for every even $n \geq 4$, we only need to construct a planar graph that is isotopic to its dual graph, and all the edges are assigned the same. As what we construct is an amphicheiral knot, so edges all with positive sign or negative sign are the same. Consider the following planar graphs:

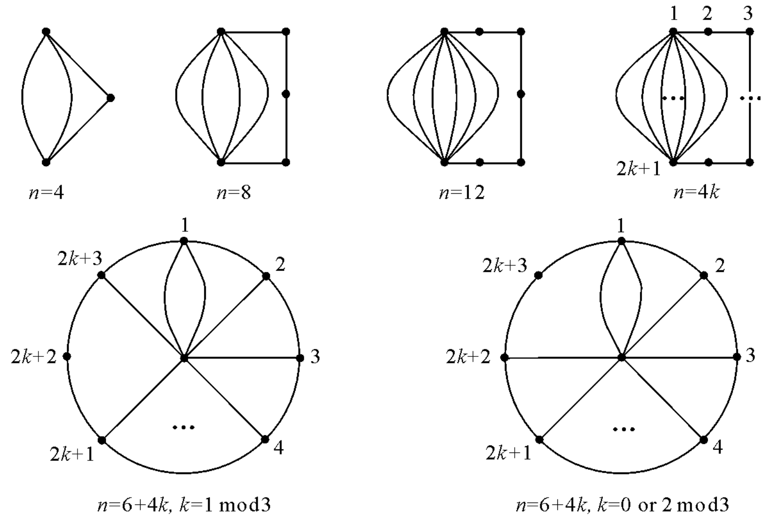

Figure 10

Theorem 4.2 Each planar graph above corresponds to an alternating amphicheiral prime knot.

Proof First, we can check all the graphs above correspond to knots with Corollary 2.1 inductively. In fact, for $n = 4k,\ k = 1, 2, 3 \cdots$, it is not difficult to find that they all correspond to knots inductively. For $n = 4k + 6,\ k = 1$ mod 3, we can check this as follows: according to the process of transforming a planar graph to a diagram, we place four points around vertex 1, see Figure 11. Let us begin our trip with the point that locates outside of the circle. The next vertex when we arrive at a point that locates outside of the circle is vertex 4. Then followed by $7, 11, \cdots$, till we arrive at vertex $2k + 2$. Hence, when we come back to vertex 1, we arrive at the point below the starting point. Then, we begin our second round around the circle. This time when we come back to the vertex 1, we will arrive at the point locating on the left-side of the vertex. It is found that, when we finish our third round, we will come back to the starting point and all points around vertices of the graph have been arrived for one time. So, we can conclude that all these planar graphs correspond to knots, not links. Similarly, we can check those planar graphs with $n = 4k + 6, k = 0$ or 2 mod 3.

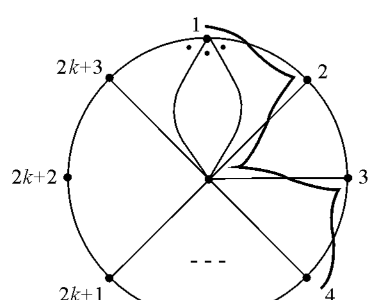

Figure 11

Second, as all the edges are signed with the same sign, so they correspond to alternating diagrams. By Proposition 2.3, as all the planar graphs above contain no cut-edge or loop, so all the diagrams are reduced. Hence, the number of edges of each graph is exactly the crossing number of the knot associated to it. In contrast, it is found that all graphs on S^2 are isotopic to their dual graphs respectively by induction. So, from Corollary 4.1, we know that each planar graph above corresponds to an alternating amphicheiral knot.

Finally, we claim that all knots obtained from planar graphs above are prime. Recall the result of William Menasco, which states that, if $K_1 \# K_2$ is an alternating knot, then it appears composite in any alternating projection. Consequently, if any one of the knots is not prime, then its associated planar graph must contain cut-edge or cut-vertex, but it can be seen from planar graphs above that none of them contains cut-edge or cut-vertex. Therefore, all the knots constructed are alternating amphicheiral prime knots.

Remark 4.1 Similar to the construction of the first row of Figure 10, we can also construct planar graphs with $n = 4k + 2,\ k = 1, 2, 3 \cdots$. But these graphs represent alternating amphicheiral links rather than knots, which coincide to links constructed in [13], where the author did not identify which of them are knots and whether they are prime. For example, in [13], the three planar graphs on P.222 represent one knot and two links.

Remark 4.2 Recently, Aparna Dar also constructed alternating amphicheiral prime knots with every even crossing number via closed 3-braids [14]. His method is different from ours.

Remark 4.3 It is natural to ask whether all alternating amphicheiral prime knots can be realized from Corollary 4.1. It is equivalent to ask whether for any alternating amphicheiral prime knot there exists a reduced alternating diagram such that its associated planar graph is isotopic to its dual graph on S^2. This question was named as Mirror Conjecture in [15]. This conjecture was shown to be false by Oliver Dasbach and Stefan Hougardy, see [16] for more details.

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