Convex hypersurfaces and L^p estimates for Schrödinger equations

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Abstract

This paper is concerned with Schrödinger equations whose principal operators are homogeneous elliptic. When the corresponding level hypersurface is convex, we show the L^p–L^q estimate of the solution operator in the free case. This estimate, combined with the results of fractionally integrated groups, allows us to further obtain the L^p estimate of solutions for the initial data belonging to a dense subset of L^p in the case of integrable potentials.
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1. Introduction

In this paper, we take interest in L^p, $1 \leqslant p < \infty$, estimates of solutions for the following Schrödinger equation:

$$\frac{\partial u}{\partial t} = (iP(D) + V)u, \quad u(0,\cdot) = u_0 \in L^p(\mathbf{R}^n), \tag{$*$}$$

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where $D = -i(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $P : \mathbf{R}^n \to \mathbf{R}$ is a homogeneous elliptic polynomial of order m (m must be even, except $n = 1$), and V is a suitable potential function. In the sequel, we may assume without loss of generality that $P(\xi) > 0$ for $\xi \neq 0$. Otherwise, we have $P(\xi) < 0$ for $\xi \neq 0$, for which the following hypersurface Σ should be replaced by

$$\{\xi \in \mathbf{R}^n | P(\xi) = -1\}.$$

In order to obtain L^p estimates of the solution of $(*)$, we will first treat L^p–L^q estimates of $e^{itP(D)}$, which is the solution operator of $(*)$ with $V = 0$. To this end, we need to consider the compact hypersurface

$$\Sigma = \{\xi \in \mathbf{R}^n | P(\xi) = 1\}.$$

When the Gaussian curvature of Σ is nonzero everywhere, it is known that L^p–L^q estimates of $e^{itP(D)}$ ($t \neq 0$) can be deduced from Miyachi [12]. In fact, Miyachi gave some remarks on these estimates in a more general case where P is a positive and smooth homogeneous function, provided the nonvanishing Gaussian curvature on Σ. Also, dropping the homogeneity of P, Balabane and Emami-Rad [4] studied these estimates under a suitable nondegenerate condition. However, one can check that the nondegenerate condition is equivalent to the nonzero Gaussian curvature if P is homogeneous.

As we know, the nonvanishing Gaussian curvature plays a crucial rule in estimating many oscillatory integrals [16]. This is the reason why one needs such a condition in [4,12]. However, there exist many hypersurfaces Σ whose Gaussian curvatures may vanish at some points (although we have observed that if $m = 2$ then Σ has nonzero Gaussian curvature everywhere under our assumptions on P). These examples are easily available, for instance, the hypersurfaces Σ associated with polynomials $\xi_1^m + \cdots + \xi_n^m$ ($m = 4, 6, \ldots$) or $\xi_1^4 + 6\xi_1^2 \xi_2^2 + \xi_2^4$.

On the other hand, an important subclass of hypersurfaces with vanishing Gaussian curvature at some points is the class of convex hypersurfaces of finite type [5]. The main purpose of this paper is to investigate the L^p estimate of the solution of $(*)$ when Σ is a convex hypersurface of finite type. Roughly speaking, this means that P allows to be degenerate on a subset of $\mathbf{R}^n$.

This paper is organized as follows.

In Section 2, we study L^p–L^q estimates of the solution operator $e^{itP(D)}$ ($t \neq 0$) and the resolvent operator $(\lambda - iP(D))^{-1}$ (Re $\lambda \neq 0$) when Σ is a convex hypersurface of finite type. The method used is quite different from those in the previous papers [4,12], due to the nature of the vanishing Gaussian curvature. Our proof depends heavily on a decay estimate for the kernels $\mathscr{F}^{-1}(e^{\pm iP})$, in which we need to use a powerful theorem in [5]. Since the proof is involved and very technical, we will present it in Section 3.

In Section 4, we show that the operator $iP(D) + V$ with suitable integrable potential V generates an integrated group on $L^p(\mathbf{R}^n)$. As we know, the semigroup of operators is a useful abstract tool to treat Cauchy problems. However, the Cauchy problem $(*)$ in $L^p(\mathbf{R}^n)$ ($p \neq 2$) cannot be treated by classical semigroups of operators

(i.e. C_0-semigroups). In fact, the Schrödinger operator $iP(D)$ generates C_0-semigroups in $L^p(\mathbf{R}^n)$ if and only if $p = 2$ (see [10,12]). Thus, several generalizations of C_0-semigroups, such as smooth distribution semigroups [3], integrated semigroups [1,9], and regularized semigroups [7,8] were introduced and applied to different general differential operators [9,18]. In our case, we use fractionally integrated groups to deal with the Cauchy problem $(*)$ in $L^p(\mathbf{R}^n)$, which will lead to better results than using smooth distribution semigroups (see [3]). Moreover, when P is nondegenerate, we will show how our results present an improvement over Theorems $2'$ and 6 in [4].

Throughout this paper, Σ denotes the hypersurface $\{\xi \in \mathbf{R}^n |\, P(\xi) = 1\}$. Assume that $P : \mathbf{R}^n \to [0, \infty)$ is always a homogeneous elliptic polynomial of order m where $n \geqslant 2$, m is even and $\geqslant 4$.

2. L^p-L^q Estimates for Schrödinger equations without potentials

We start with the concept of finite type. S denotes the smooth hypersurface $\{\xi \in \mathbf{R}^n |\, \phi(\xi) = 0\}$, where $\phi \in C^\infty(\mathbf{R}^n)$ and $\nabla\phi(\xi) \neq 0$ for $\xi \in S$. We say that S is of finite type if any one-dimensional tangent line has at most a finite order of contact with S. The precise definition is as follows.

$\mathbf{S}^{n-1}$ denotes the unit sphere in $\mathbf{R}^n$. Let

$$\nabla_\eta = \sum_{j=1}^n \eta_j \partial/\partial x_j \quad \text{for } \eta = (\eta_1, \ldots, \eta_n) \in \mathbf{S}^{n-1},$$

which is the directional derivative in direction η, and let ∇_η^j be the jth power of this derivative.

Definition 2.1. Let k be an integer. The smooth hypersurface S is of type k if there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^k |\nabla_\eta^j \phi(\xi)| \geqslant \delta \quad \text{for } \xi \in S \text{ and } \eta \in \mathbf{S}^{n-1}.$$

Moreover, we say that S is convex if

$$S \subset \{\eta \in \mathbf{R}^n |\, \langle \eta - \xi, \nabla\phi(\xi) \rangle \geqslant 0\} \quad \text{for } \xi \in S$$

or

$$S \subset \{\eta \in \mathbf{R}^n |\, \langle \eta - \xi, \nabla\phi(\xi) \rangle \leqslant 0\} \quad \text{for } \xi \in S.$$

It is clear that $k \geqslant 2$, and that if S is of type k it is also of type $k'(>k)$. For the hypersurface Σ (i.e. $\{\xi \in \mathbf{R}^n |\, P(\xi) = 1\}$), since

$$\langle \xi, \nabla P(\xi) \rangle = mP(\xi) = m \quad \text{for } \xi \in \Sigma,$$

it follows that $\nabla P(\xi) \neq 0$ for $\xi \in \Sigma$, and thus Σ is smooth. Also, a simple computation leads to

$$\nabla_\eta^m(P(\xi) - 1) = m! P(\eta) \quad \text{for } \xi \in \Sigma \text{ and } \eta \in \mathbf{S}^{n-1}.$$

Hence, we have

Proposition 2.2. Σ *is a smooth compact hypersurface of type less than or equal to m.*

A simple example of polynomials whose level hypersurface Σ is of type m is $\xi_1^m + \cdots + \xi_n^m$ ($m = 4, 6, \ldots$). We notice that there exist polynomials P whose level hypersurfaces Σ are of type $k(<m)$. For example, when $P(\xi) = \xi_1^6 + 5\xi_1^2\xi_2^4 + \xi_2^6$, the corresponding hypersurface Σ is of type 4, but $m = 6$.

We now turn to the Cauchy problem $(*)$ with $V = 0$. In this case, for every initial data $u_0 \in \mathscr{S}(\mathbf{R}^n)$ (the Schwartz space), the solution is given by

$$u(t, \cdot) = e^{itP(D)}u_0 := \mathscr{F}^{-1}(e^{itP}) * u_0,$$

where $\mathscr{F}$ (or $^\wedge$) denotes the Fourier transform, $\mathscr{F}^{-1}$ its inverse, and $\mathscr{F}^{-1}(e^{itP})$ is understood in the distributional sense. Therefore, to obtain L^p–L^q estimates of $e^{itP(D)}$ ($t \neq 0$), the key result is to show estimates of the kernels $\mathscr{F}^{-1}(e^{\pm iP})$.

In the sequel, denote by p' the conjugate index of p, and $\|\cdot\|_{L^p-L^q}$ the norm in $\mathscr{L}(L^p, L^q)$ (the space of all bounded linear operators from L^p to L^q). Let

$$h(m, n, k) = \frac{m-2}{2(m-1)} + \frac{(m-k)(n-1)}{k(m-1)} \quad \text{for } 2 \leqslant k \leqslant m,$$

$\tau = n/h(m, n, k)$, and $q(p) = q(m, n, k, p)$ where

$$\frac{1}{q(m, n, k, p)} = \frac{1}{\tau p} + \frac{1}{\tau' p'} \quad \text{for } 1 \leqslant p < 2.$$

We first remark that when $2 \leqslant k \leqslant m$,

$$\frac{2(m-1)}{m-2} \leqslant \tau \leqslant \frac{2n(m-1)}{m-2}.$$

Since $m \geqslant 4$, it follows that $\tau \in (2, 3n]$. Next, we remark that

$$\frac{1}{2} > \frac{1}{\tau p} + \frac{1}{\tau' p'} > \frac{1}{p'} \quad \text{for } 1 \leqslant p < 2$$

and thus $2<q(p)<p'$. Moreover, denote by I_p ($1\leqslant p\leqslant 2$) the following subset of $[2, \infty]$:

$$I_p = \begin{cases} (q(p), \infty] & \text{if } 1\leqslant p<\tau', \\ \left(q(p), \dfrac{p(2-\tau')}{p-\tau'}\right) & \text{if } \tau'\leqslant p<2, \\ \{2\} & \text{if } p=2. \end{cases}$$

Theorem 2.3. *Suppose Σ is a convex hypersurface of type k. Then $\mathscr{F}^{-1}(e^{\pm iP})\in C^\infty(\mathbf{R}^n)$ and*

$$(\mathscr{F}^{-1}(e^{\pm iP}))(x) = O(|x|^{-h(m,n,k)}) \quad as \ |x|\to\infty.$$

The proof is lengthy and is given in the next section.

Theorem 2.4. *Suppose Σ is a convex hypersurface of type k. If $p\in[1,2]$ and $q\in I_p$, then there exists a constant $C>0$ such that*

$$||e^{itP(D)}||_{L^p-L^q}\leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)} \quad for \ t\neq 0.$$

Proof. By Theorem 2.3, $\mathscr{F}^{-1}(e^{\pm iP})\in L^s(\mathbf{R}^n)$ for $s>\tau$. Since P is homogeneous, one has

$$\mathscr{F}^{-1}(e^{itP})(x) = |t|^{-n/m}\mathscr{F}^{-1}(e^{itP/|t|})(|t|^{-1/m}x) \quad for \ t\neq 0 \ and \ x\in\mathbf{R}^n,$$

and thus

$$||\mathscr{F}^{-1}(e^{itP})||_{L^s} = |t|^{-n/ms'}||\mathscr{F}^{-1}(e^{itP/|t|})||_{L^s}\leqslant C|t|^{-n/ms'} \quad for \ t\neq 0,$$

where the constant C is independent of t. The remainder of the proof will be divided into several steps.

Step 1. When $1\leqslant p<\tau'$ and $\frac{\tau'p}{\tau'-p}<q\leqslant\infty$, it follows from Young's inequality that

$$||e^{itP(D)}||_{L^p-L^q}\leqslant||\mathscr{F}^{-1}(e^{itP})||_{L^s}\leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)} \quad for \ t\neq 0,$$

where $\frac{1}{s} = 1+\frac{1}{q}-\frac{1}{p}$, which implies $s>\tau$.

Step 2. Since $P(D)$ is self-adjoint in $L^2(\mathbf{R}^n)$, $||e^{itP(D)}||_{L^2-L^2} = 1$ for $t\geqslant 0$ by Stone's theorem. When $1\leqslant p<2$ and $q(p)<q\leqslant p'$, we deduce from the Riesz–Thorin

interpolation theorem that

$$||e^{itP(D)}||_{L^p-L^q} \leqslant ||e^{itP(D)}||_{L^1-L^s}^{1-2/p'}||e^{itP(D)}||_{L^2-L^2}^{2/p'} \leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)} \quad \text{for } t \neq 0,$$

where $s = \frac{q(p'-2)}{p'-q} > \tau$.

Step 3. When $1 \leqslant p < \tau'$ and $q(p) < q \leqslant \infty$, we notice $q(p) < \frac{\tau'p}{\tau'-p}$. Since $L^{s_2}(\mathbf{R}^n) \subset L^{s_1}(\mathbf{R}^n) + L^{s_3}(\mathbf{R}^n)$ for $1 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \infty$, the desired estimate is a direct consequence of the conclusions in Steps 1 and 2.

Step 4. When $\tau' \leqslant p < 2$ and $q(p) < q < \frac{p(2-\tau')}{p-\tau'}$, we put $\lambda = \frac{2(p-p_0)}{p(2-p_0)}$, where $p_0 \in [1, \tau')$ such that $q < \frac{2}{\lambda} < \frac{p(2-\tau')}{p-\tau'}$. A simple computation leads to

$$\frac{\lambda}{2} < \frac{1}{q} < \frac{1}{q(p)} = \frac{1-\lambda}{q(p_0)} + \frac{\lambda}{2}.$$

Consequently, there exists a unique $q_0 > q(p_0)$ such that $\frac{1}{q} = \frac{1-\lambda}{q_0} + \frac{\lambda}{2}$. Also, $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{2}$. The desired estimate now can be deduced from the Riesz–Thorin interpolation theorem and the conclusion in Step 1.   □

The subsequent theorem deals with L^p-L^q estimates of the resolvent of $iP(D)$.

Theorem 2.5. *Suppose Σ is a convex hypersurface of type k. If $p \in [1, 2]$, $q \in I_p$, and $\frac{1}{p} - \frac{1}{q} < \frac{m}{n}$, then there exists a constant $C > 0$ such that*

$$||(\lambda - iP(D))^{-1}||_{L^p-L^q} \leqslant C|\text{Re } \lambda|^{\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)-1} \quad \text{for } \text{Re } \lambda \neq 0.$$

Proof. For $\text{Re } \lambda > 0$ and $f \in \mathscr{S}(\mathbf{R}^n)$, one has

$$(\lambda - iP(D))^{-1}f = \mathscr{F}^{-1}((\lambda - iP)^{-1}\hat{f})$$

$$= \int_0^\infty e^{-\lambda t}\mathscr{F}^{-1}(e^{itP}\hat{f})\, dt$$

$$= \int_0^\infty e^{-\lambda t}e^{itP(D)}f\, dt.$$

It follows therefore from Theorem 2.4 that

$$||(\lambda - iP(D))^{-1}||_{L^p-L^q} \leqslant C \int_0^\infty e^{-(\text{Re } \lambda)t}t^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)}\, dt$$

$$= C|\text{Re } \lambda|^{\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)-1}.$$

For Re $\lambda < 0$ and $f \in \mathscr{S}(\mathbf{R}^n)$, one has

$$(\lambda - iP(D))^{-1}f = -(-\lambda + iP(D))^{-1}f = \int_0^\infty e^{\lambda t} e^{-itP(D)} f \, dt,$$

and thus the desired estimate follows from Theorem 2.4. □

Since $p' \in I_p$ for $p \in [1, 2]$, we have

Corollary 2.6. *Suppose Σ is a convex hypersurface of type k. If $p \in [1, 2]$ then*

$$\|e^{itP(D)}\|_{L^p - L^{p'}} \leqslant C |t|^{\frac{n}{m}\left(1 - \frac{2}{p}\right)} \quad \text{for } t \neq 0.$$

If, in addition, $p > \frac{2n}{n+m}$, then

$$\|(\lambda - iP(D))^{-1}\|_{L^p - L^{p'}} \leqslant C |\operatorname{Re} \lambda|^{-\frac{n}{m}\left(\frac{2}{p} - 1\right) - 1} \quad \text{for } \operatorname{Re} \lambda \neq 0.$$

3. The proof of Theorem 2.3

By our assumptions on P, $\phi := P^{1/m}$ is a positive and smooth homogeneous function of degree 1, and $\Sigma = \{\xi \in \mathbf{R}^n \mid \phi(\xi) = 1\}$. Let $\varphi \in C^\infty(\mathbf{R})$ such that supp $\varphi \subset [1, \infty)$ and $\varphi(t) = 1$ for $t \geqslant 2$. Obviously, in order to estimate $\mathscr{F}^{-1}(e^{iP})$ (similarly for $\mathscr{F}^{-1}(e^{-iP})$), it suffices to estimate $\mathscr{F}^{-1}((\varphi \circ \phi)e^{iP})$ (cf. [4, p. 363]). Consider the integral

$$
\begin{aligned}
K_\varepsilon(x) &:= \int_{\mathbf{R}^n} e^{-\varepsilon \phi(y) + iP(y) + i\langle x, y \rangle} \varphi(\phi(y)) \, dy \\
&= \int_0^\infty e^{-\varepsilon t + it^m} t^{n-1} \varphi(t) \left(\int_\Sigma \frac{e^{itr\langle \eta, \xi \rangle}}{|\nabla \phi(\xi)|} \, d\sigma(\xi) \right) dt \quad \text{for } \varepsilon > 0,
\end{aligned}
$$

where $r = |x|$, $x = r\eta$, and $d\sigma$ is the induced surface measure on Σ. We will show that

$$\mathscr{F}^{-1}((\varphi \circ \phi)e^{iP})(x) = (2\pi)^{-n} \lim_{\varepsilon \to 0} K_\varepsilon(x)$$

uniformly for x in compact subsets of $\mathbf{R}^n$, and $K_\varepsilon(x)$ decays as $|x|^{-h(m,n,k)}$. From this we have $\mathscr{F}^{-1}((\varphi \circ \phi)e^{iP}) \in C(\mathbf{R}^n)$ since it is clear that $K_\varepsilon \in C(\mathbf{R}^n)$.

Denote by Π the Gaussian map

$$\xi \in \Sigma \mapsto \frac{\nabla \phi(\xi)}{|\nabla \phi(\xi)|} \in \mathbf{S}^{n-1}.$$

Since Σ is a compact convex hypersurface, Π is a homeomorphism from Σ to $\mathbf{S}^{n-1}$. Given $\eta \in \mathbf{S}^{n-1}$, let $\xi_\pm = \Pi^{-1}(\pm\eta)$. Then

$$\langle \eta, \xi_\pm \rangle = \pm \left\langle \frac{\nabla\phi(\xi_\pm)}{|\nabla\phi(\xi_\pm)|}, \xi_\pm \right\rangle = \pm \frac{\phi(\xi_\pm)}{|\nabla\phi(\xi_\pm)|} = \pm \frac{1}{|\nabla\phi(\xi_\pm)|},$$

Noting that $\pm\eta$ is the outward unit normal to Σ at $\xi_\pm$, by Theorem B in [5] (also cf. [6]) we have

$$\int_\Sigma \frac{e^{i\lambda\langle\eta,\xi\rangle}}{|\nabla\phi(\xi)|}\,d\sigma(\xi) = e^{i\lambda\langle\eta,\xi_+\rangle}H_+(\lambda) + e^{i\lambda\langle\eta,\xi_-\rangle}H_-(\lambda) + H_\infty(\lambda) \quad \text{for } \lambda > 0.$$

Here $H_\pm \in C^\infty((0,\infty))$,

$$|H_\pm^{(j)}(\lambda)| \leqslant C_j \lambda^{-j-(n-1)/k} \quad \text{for } j \in \mathbf{N}_0,$$

and

$$|H_\infty(\lambda)| \leqslant C_j \lambda^{-j} \quad \text{for } j \in \mathbf{N},$$

where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and constants C_j depend only on the hypersurface Σ. Hence,

$$\begin{aligned}
K_\varepsilon(x) = &\int_0^\infty e^{-\varepsilon t + it^m + itr\langle\eta,\xi_+\rangle} t^{n-1}\varphi(t)H_+(tr)\,dt \\
&+ \int_0^\infty e^{-\varepsilon t + it^m + itr\langle\eta,\xi_-\rangle} t^{n-1}\varphi(t)H_-(tr)\,dt \\
&+ \int_0^\infty e^{-\varepsilon t + it^m} t^{n-1}\varphi(t)H_\infty(tr)\,dt \\
:= &\, J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon.
\end{aligned}$$

In the remainder of this section, for the sake of convenience, we will denote by C a generic constant independent of r, t and ε.

We consider first the integral J_3^ε. It is obvious that

$$|J_3^\varepsilon| \leqslant C \int_1^\infty t^{n-1}(tr)^{-(n+1)}\,dt \leqslant Cr^{-(n+1)}.$$

Since $r = |x|$, it follows from the dominated convergence theorem that J_3^ε $(\varepsilon \to 0)$ converges uniformly for x in compact subsets of $\mathbf{R}^n \setminus \{0\}$, and decays as $|x|^{-h(m,n,k)}$, where we notice that $h(m,n,k) \leqslant n$.

Next, consider the integral J_1^ε. Let

$$\begin{cases} u(t) = -\varepsilon t + it^m + itr\langle\eta,\xi_+\rangle, \\ v(t) = t^{n-1}\varphi(t)H_+(tr) \end{cases}$$

for $t > 0$. Since $u'(t) \neq 0$ for $t > 0$, we can define $D_{\#} f = -gf'$ and $D_{*} f = (gf)'$ for $f \in C^1((0, \infty))$, where $g = -1/u'$. By induction on j we find that

$$g^{(j)}(t) = \sum_{l=l_0}^{j} a_l t^{l(m-1)-j} g(t)^{l+1} \quad \text{for } j \in \mathbf{N}_0,$$

where constants a_l depend only on l and m, and $l_0 \in \mathbf{N}_0$ such that $l_0 \geqslant j/(m-1)$. Since there exists a constant $c > 1$ such that $c^{-1} \leqslant |\nabla \phi(\xi)| \leqslant c$ for $\xi \in \Sigma$, we have

$$|g(t)| = 1/|u'(t)| \leqslant 1/r \langle \eta, \xi_+ \rangle = |\nabla \phi(\xi_+)|/r \leqslant c/r \quad \text{for } t > 0.$$

Also, $|g(t)| \leqslant \frac{1}{m} t^{1-m}$ for $t > 0$. Hence,

$$|g^{(j)}(t)| \leqslant Cr^{-1} t^{-j} \quad \text{for } j \in \mathbf{N}_0.$$

On the other hand, one sees

$$\left| \frac{d^j}{dt^j} (H_+(tr)) \right| \leqslant Ct^{-j} (tr)^{-(n-1)/k} \quad \text{for } j \in \mathbf{N}_0,$$

and thus by Leibniz's formula

$$|v^{(j)}(t)| \leqslant Cr^{-(n-1)/k} t^{-j+n-1-(n-1)/k} \quad \text{for } j \in \mathbf{N}_0. \tag{3.1}$$

Since it is not hard to show

$$D_{*}^{j} v = \sum_{\alpha} a_{\alpha} g^{(\alpha_1)} \cdots g^{(\alpha_j)} v^{(\alpha_{j+1})} \quad \text{for } j \in \mathbf{N},$$

where the sum runs over all $\alpha = (\alpha_1, \ldots, \alpha_{j+1}) \in \mathbf{N}_0^{j+1}$ such that $|\alpha| = j$ and $0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_j$, it follows that

$$|(D_{*}^{j} v)(t)| \leqslant Cr^{-j-(n-1)/k} t^{-j+n-1-(n-1)/k} \quad \text{for } j \in \mathbf{N}_0, \tag{3.2}$$

where we used the notation $D_{*}^{0} v = v$. Noting that $D_{\#}^{n} e^{u} = e^{u}$ we have

$$J_1^{\varepsilon} = \int_0^{\infty} (D_{\#}^{n} e^{u})(t) v(t) \, dt = \int_0^{\infty} e^{u(t)} (D_{*}^{n} v)(t) \, dt.$$

Consequently,

$$|J_1^{\varepsilon}| \leqslant Cr^{-n-(n-1)/k} \int_1^{\infty} t^{-1-(n-1)/k} \, dt \leqslant Cr^{-n-(n-1)/k}.$$

Thus, the dominated convergence theorem yields that J_1^{ε} converges uniformly for x in compact subsets of $\mathbf{R}^n \backslash \{0\}$, and decays as $|x|^{-h(m,n,k)}$.

We now consider the integral J_2^ε. In this case, we put

$$\begin{cases} u(t) = -\varepsilon t + it^m - it\bar{r}, \\ v(t) = t^{n-1}\varphi(t)H_-(tr) \end{cases}$$

for $t > 0$, where

$$\bar{r} := -r\langle \eta, \xi_- \rangle = r/|\nabla\phi(\xi_-)|.$$

Since $t_0 := (\bar{r}/m)^{1/(m-1)}$ is the unique critical point of the oscillatory integral J_2^ε, we write

$$J_2^\varepsilon = \left\{ \int_{2t_0}^\infty + \int_{t_0/2}^{2t_0} + \int_0^{t_0/2} \right\} e^{u(t)} v(t)\, dt$$

$$:= J_{2,1}^\varepsilon + J_{2,2}^\varepsilon + J_{2,3}^\varepsilon.$$

From integration by parts one gets

$$J_{2,1}^\varepsilon = -\frac{e^{u(2t_0)}}{u'(2t_0)} \sum_{j=0}^{n-1} (D_*^j v)(2t_0) + \int_{2t_0}^\infty e^{u(t)}(D_*^n v)(t)\, dt.$$

Since

$$|u'(t)| \geqslant mt^{m-1} - \bar{r} \geqslant (2^{m-1} - 1)\bar{r} \geqslant (2^{m-1} - 1)r/c \quad \text{for } t \geqslant 2t_0$$

and since

$$|u'(t)| \geqslant mt^{m-1} - \bar{r} \geqslant m(1 - 2^{1-m})t^{m-1} \quad \text{for } t \geqslant 2t_0,$$

the estimate (3.2) still holds for $t \geqslant 2t_0$. Hence,

$$|J_{2,1}^\varepsilon| \leqslant Cr^{-1} \sum_{j=0}^{n-1} r^{-j-(n-1)/k}(2t_0)^{-j+n-1-(n-1)/k}$$

$$+ C \int_{2t_0}^\infty r^{-n-(n-1)/k} t^{-1-(n-1)/k}\, dt$$

$$\leqslant Cr^{(n-m-m(n-1)/k)/(m-1)} \sum_{j=0}^{n-1} r^{-jm/(m-1)}.$$

But

$$(n - m - m(n-1)/k)/(m-1) \leqslant -h(m, n, k),$$

$J_{2,1}^{\varepsilon}$ converges uniformly for x in compact subsets of $\mathbf{R}^n \backslash \{0\}$, and decays as $|x|^{-h(m,n,k)}$. Since $|u'(t)| \geqslant C t^{m-1}$ and $\geqslant Cr$ for $0 < t \leqslant t_0/2$, a slight modification of the above method leads to the same conclusion for $J_{2,3}^{\varepsilon}$. We omit the details.

In order to deal with $J_{2,2}^{\varepsilon}$, set $w(t) = t^m - t\bar{r}$ and $v(t)$ is defined as in $J_{2,1}^{\varepsilon}$. Obviously,

$$\lim_{\varepsilon \to 0} J_{2,2}^{\varepsilon} = J_{2,2} := \int_{t_0/2}^{2t_0} e^{iw(t)} v(t)\, dt$$

uniformly for x in compact subsets of $\mathbf{R}^n \backslash \{0\}$. It remains to show that $J_{2,2}$ decays as $|x|^{-h(m,n,k)}$. Let $y = (t - t_0)/t_0$, which maps $[t_0/2, 2t_0]$ onto $[-1/2, 1]$, and let $\lambda = m(m-1)t_0^m$. Then

$$\begin{aligned}
\Phi(y) &:= \frac{1}{\lambda}(w(t_0(y+1)) - w(t_0)) \\
&= \frac{1}{m(m-1)}((y+1)^m - my - 1) \\
&= \frac{1}{m(m-1)} \sum_{l=2}^{m} \binom{m}{l} y^l \quad \text{for } y \in [-1/2, 1].
\end{aligned}$$

Consequently,

$$\Phi''(y) = (y+1)^{m-2} \geqslant 2^{2-m} \quad \text{for } y \in [-1/2, 1]$$

and

$$J_{2,2} = t_0 e^{iw(t_0)} \int_{-1/2}^{1} e^{i\lambda\Phi(y)} v(t_0(y+1))\, dy.$$

It follows thus from van der Corput's theorem (cf. [16, p. 334]) that

$$\begin{aligned}
|J_{2,2}| &\leqslant C\lambda^{-1/2} t_0 \left(|v(2t_0)| + \int_{-1/2}^{1} |t_0 v'(t_0(y+1))|\, dy \right) \\
&\leqslant C t_0^{1-m/2} \left(|v(2t_0)| + \sup_{t_0/2 \leqslant t \leqslant 2t_0} |t_0 v'(t)| \right).
\end{aligned}$$

Since (3.1), in which H_+ is replaced by H_-, still holds,

$$|J_{2,2}| \leqslant C r^{-(n-1)/k} t_0^{n-m/2-(n-1)/k} \leqslant C r^{-h(m,n,k)},$$

as desired.

Finally, we only need to show that $K_{\varepsilon}(x)$ converges uniformly for x in some neighborhood of the origin. Let U be the ball $\{x \in \mathbf{R}^n \mid |x| \leqslant m/2M\}$, where M is a

constant such that $|\xi| \leqslant M$ for $\xi \in \Sigma$. For a given $x \in U$, let

$$u(t) = -\varepsilon t + it^m + it\langle x, \xi \rangle \quad \text{for } t \geqslant 1.$$

Then

$$K_\varepsilon(x) = \int_\Sigma \left(\int_1^\infty e^{u(t)} t^{n-1} \varphi(t)\, dt \right) \frac{d\sigma(\xi)}{|\nabla\phi(\xi)|}.$$

Since

$$|u'(t)| \geqslant |mt^{m-1} + \langle x, \xi \rangle| \geqslant mt^{m-1} - M|x| \geqslant (m/2)t^{m-1} \geqslant m/2,$$

as in the case of the estimate of J_1^ε we obtain

$$\left| \int_1^\infty e^{u(t)} t^{n-1} \varphi(t)\, dt \right| \leqslant C,$$

and therefore the claim follows.

Moreover, an analogous method as above leads to

$$D^\alpha \mathscr{F}^{-1}(e^{\pm iP}) = \mathscr{F}^{-1}(\xi^\alpha e^{\pm iP(\xi)}) \in C(\mathbf{R}^n) \quad \text{for } \alpha \in \mathbf{N}_0^n,$$

i.e. $\mathscr{F}^{-1}(e^{\pm iP}) \in C^\infty(\mathbf{R}^n)$ and

$$D^\alpha(\mathscr{F}^{-1}(e^{\pm iP}))(x) = O(|x|^{-h(m,n,k)+|\alpha|/(m-1)}) \; (|x| \to \infty) \quad \text{for } \alpha \in \mathbf{N}_0^n.$$

4. L^p Estimates for Schrödinger equations

It was Balabane and Emami-Rad [4] who first applied smooth distribution semigroups to higher order Schrödinger equations, and showed the L^p estimate of solutions. Arendt and Kellermann [2] showed that a smooth distribution semigroup is equivalent to an integrally integrated semigroup. However, it is known that the fractionally integrated semigroup is a generalization of the integrally integrated semigroup and is a more suitable tool for elliptic differential operators in $L^p(\mathbf{R}^n)$ (cf. [9,17]). We start with its definition.

Definition 4.1. Let A be a linear operator on a Banach space X and $\beta \geqslant 0$. Then a strongly continuous family $T : [0, \infty) \to \mathscr{L}(X)$ is called a β-times integrated semigroup on X with generator A if there exist constants $C, \omega \geqslant 0$ such that $\|T(t)\| \leqslant Ce^{\omega t}$ for $t \geqslant 0$, $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A), and

$$(\lambda - A)^{-1} x = \lambda^\beta \int_0^\infty e^{-\lambda t} T(t) x\, dt \quad \text{for } \lambda > \omega \text{ and } x \in X.$$

If A and $-A$ both are generators of β-times integrated semigroups on X, we say A is the generator of a β-times integrated group on X.

Here is a sufficient condition for an operator to be the generator of integrated semigroups, which is due to Hieber [9, p. 30] (see [2] for a special case).

Lemma 4.2. *Let A be a linear operator on a Banach space X. Suppose there exist constants $C, \omega \geqslant 0$ and $\gamma \geqslant -1$ such that for $\operatorname{Re} \lambda > \omega$, $\lambda \in \rho(A)$ and $\|(\lambda - A)^{-1}\| \leqslant C|\lambda|^\gamma$. Then A generates a β-times integrated semigroup on X, where $\beta > \gamma + 1$.*

Assume that the operator $P(D)$ has maximal domain in the distributional sense on $L^p(\mathbf{R}^n)$ ($1 \leqslant p < \infty$), and thus it is closed and densely defined. It is known that $D(P(D)) = W^{m,p}(\mathbf{R}^n)$ for $1 < p < \infty$. The following Lemma 4.3(a) can be found in [9,17], and Lemma 4.3(b) follows from Lemma 4.3(a) and Definition 4.1, immediately.

Lemma 4.3. *Let $1 \leqslant p < \infty$ and $\beta > n_p := n|\frac{1}{2} - \frac{1}{p}|$.*
  (a) $iP(D)$ generates a β-times integrated group $T(t)$ ($t \in \mathbf{R}$) on $L^p(\mathbf{R}^n)$, and

$$\|T(t)\|_{L^p - L^p} \leqslant C|t|^\beta \quad \text{for } t \in \mathbf{R}.$$

  (b) $\{\lambda \in \mathbf{C} | \operatorname{Re} \lambda \neq 0\} \subset \rho(iP(D))$ on $L^p(\mathbf{R}^n)$, and

$$\|(\lambda - iP(D))^{-1}\|_{L^p - L^p} \leqslant C|\lambda|^\beta / |\operatorname{Re} \lambda|^{\beta + 1} \quad \text{for } \operatorname{Re} \lambda \neq 0.$$

Let V be a measurable function defined on $\mathbf{R}^n$. We consider V as a multiplication operator on $L^p(\mathbf{R}^n)$ with $D(V) := \{f \in L^p(\mathbf{R}^n) | Vf \in L^p(\mathbf{R}^n)\}$. The domain of $iP(D) + V$ is $D(P(D)) \cap D(V)$. Denote by I'_p ($1 \leqslant p \leqslant 2$), the following subset of $[1, \infty]$:

$$I'_p = \begin{cases} \left[p, \dfrac{\tau' p}{2 - p} \right) & \text{if } 1 \leqslant p < \tau', \\[2ex] \left(\dfrac{p(2 - \tau')}{2 - p}, \dfrac{\tau' p}{2 - p} \right) & \text{if } \tau' \leqslant p < 2, \\[2ex] \{\infty\} & \text{if } p = 2, \end{cases}$$

where $\tau = n/h(m,n,k)$ and $h(m,n,k)$ is defined as in Section 2.

Theorem 4.4. *Let $V = V_1 + V_2$ with $V_j \in L^{s_j}(\mathbf{R}^n)$ for some $s_j \in (\frac{n}{m}, \infty]$ ($j = 1, 2$).*
  (a) If $s_j \in I'_p$ for some $p \in [1, 2]$, then $iP(D) + V$ generates a β-times integrated group on $L^p(\mathbf{R}^n)$, where $\beta > n_p + 1$.
  (b) If $s_j \in I'_{p'}$ for some $p \in (2, \infty)$, then an extension of $iP(D) + V$, i.e. $(-iP(D) + \bar{V})^*$ generates a β-times integrated group on $L^p(\mathbf{R}^n)$, where $\beta > n_p + 1$.

Proof. Since $iP(D) + V$ and $-(iP(D) + V)$ satisfy the same assumptions, it suffices to show that $iP(D) + V$ generates a β-times integrated semigroup on $L^p(\mathbf{R}^n)$.

We consider first the case $1 \leqslant p \leqslant 2$. Let $\frac{1}{q_j} = \frac{1}{p} - \frac{1}{s_j}$ ($j = 1, 2$). Then $s_j \in I'_p$ implies $q_j \in I_p$ (I_p is defined in Section 2). So we obtain by Theorem 2.5 and Hölder's inequality that

$$\|V(\lambda - iP(D))^{-1}\|_{L^p - L^p} \leqslant \sum_{j=1,2} \|V\|_{L^{q_j} - L^p} \|(\lambda - iP(D))^{-1}\|_{L^p - L^{q_j}}$$

$$\leqslant C \sum_{j=1,2} \|V\|_{L^{s_j}} |\operatorname{Re} \lambda|^{\frac{n}{ms_j} - 1}.$$

In view of $\frac{n}{ms_j} - 1 < 0$, there exists $\omega \geqslant 1$ such that

$$\|V(\lambda - iP(D))^{-1}\|_{L^p - L^p} \leqslant 1/2 \quad \text{for } \operatorname{Re} \lambda > \omega.$$

Consequently, $\lambda \in \rho(iP(D) + V)$ and

$$(\lambda - iP(D) - V)^{-1} = (\lambda - iP(D))^{-1} \sum_{j=0}^{\infty} \left(V(\lambda - iP(D))^{-1} \right)^j.$$

This implies by Lemma 4.3(b) that

$$\|(\lambda - iP(D) - V)^{-1}\|_{L^p - L^p} \leqslant 2\|(\lambda - iP(D))^{-1}\|_{L^p - L^p}$$

$$\leqslant C|\lambda|^{n_p + \varepsilon} \quad \text{for } \operatorname{Re} \lambda > \omega,$$

where $\varepsilon \in (0, \beta - n_{p-1})$. It follows now from Lemma 4.2 that $iP(D) + V$ generates a β-times integrated semigroup on $L^p(\mathbf{R}^n)$.

Next, we consider the case $2 < p < \infty$. From the proof of (a) one sees that $-iP(D) + \bar{V}$ is densely defined on $L^{p'}(\mathbf{R}^n)$, and thus $(-iP(D) + \bar{V})^*$ exists and is densely defined on $L^p(\mathbf{R}^n)$. It is easy to check $iP(D) + V \subset (-iP(D) + \bar{V})^*$. Also, an adjointness argument implies

$$\|(\lambda - (-iP(D) + \bar{V})^*)^{-1}\|_{L^p - L^p} = \|(\bar{\lambda} - (-iP(D) + \bar{V}))^{-1}\|_{L^{p'} - L^{p'}}.$$

Since $s_j \in I'_{p'}$ and $n_{p'} = n_p$, this leads to the same estimate as in the case $1 < p < 2$. It follows therefore from Lemma 4.2 that $(-iP(D) + \bar{V})^*$ generates a β-times integrated semigroup on $L^p(\mathbf{R}^n)$. $\quad\square$

When $2 < p < \infty$, we rewrite $I'_{p'}$ as

$$
I'_{p'} = \begin{cases}
\left(\dfrac{p(2 - \tau')}{p - 2}, \dfrac{\tau' p}{p - 2} \right) & \text{if } 2 < p \leqslant \tau, \\[2ex]
\left[\dfrac{p}{p - 1}, \dfrac{\tau' p}{p - 2} \right) & \text{if } \tau < p < \infty.
\end{cases}
$$

From this and the subsequent proposition one sees that it is not always true that $iP(D) + V = (-iP(D) + \bar{V})^*$ in Theorem 4.3(b), so is such situation in [4, Theorem 6]. This means that the operator $iP(D) + V$ in [4, Theorem 6] should be replaced by $(-iP(D) + \bar{V})^*$ for $p > 2$.

Proposition 4.5. *Let* $1 < p < \infty$ *and* $\frac{n}{m} < s < \infty$.

(a) *If* $V \in L^s(\mathbf{R}^n)$ *and* $s \geqslant p$, *then* $W^{m,p}(\mathbf{R}^n) \subset D(V)$ *and* V *is a compact operator from* $W^{m,p}(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$.

(b) *If* $V \in L^s(\mathbf{R}^n)$ *and* $s \geqslant \max\{p, p'\}$, *then* $iP(D) + V = (-iP(D) + \bar{V})^*$ *on* $L^p(\mathbf{R}^n)$.

(c) *If* $1 \leqslant s < p$, *then there exists* $V \in L^s(\mathbf{R}^n)$ *such that* $W^{m,p}(\mathbf{R}^n) \cap D(V) = \{0\}$.

Proof. (a) is a direct consequence of Theorem 10.2 in [15, p. 147], in which condition (5) is satisfied. Since $\rho(P(D)) \neq \emptyset$ on $L^{p'}(\mathbf{R}^n)$, (b) follows from (a) and Theorem 6.1 in [15, p. 94]. To show (c), we will modify slightly an example in [14, p. 60]. Let $f(x) = |x|^{-n/p}$ for $|x| \leqslant 1$ and $= 0$ for $|x| > 1$. Define

$$
V(x) = \sum_{j=1}^{\infty} 2^{-j} f(x - \alpha_j) \quad \text{for } x \in \mathbf{R}^n,
$$

where $\{\alpha_j\}_{j=1}^{\infty} = \mathbf{Q}^n$ ($\mathbf{Q}$ denotes the set of all rational numbers). Then

$$
\|V\|_{L^s} \leqslant \sum_{j=1}^{\infty} 2^{-j} \|f(x - \alpha_j)\|_{L^s} = \|f\|_{L^s} < \infty.
$$

If $0 \neq g \in W^{m,p}(\mathbf{R}^n) \cap D(V)$, one has $g \in C(\mathbf{R}^n)$ by Sobolev's embedding theorem. Thus, $|g(x)| \geqslant c > 0$ in some open subset $\Omega \subset \mathbf{R}^n$. Taking $\alpha_j \in \Omega$ yields

$$
\int_{\mathbf{R}^n} |V(x) g(x)|^p \, dx \geqslant 2^{-np} c^p \int_{|x - \alpha_j| \leqslant \delta} |x - \alpha_j|^{-n} \, dx = \infty \quad \text{for small } \delta > 0,
$$

which contradicts $g \in D(V)$. $\square$

Set $I'_p = I'_{p'} \cap [p, \infty)$ for $p \in (2, \infty)$. Then $I'_p = \emptyset$ for $p \geqslant 2 + \tau'$ and

$$I'_p = \begin{cases} \left(\dfrac{p(2 - \tau')}{p - 2}, \dfrac{\tau'p}{p - 2} \right) & \text{if } 2 < p \leqslant 4 - \tau', \\[3mm] \left[p, \dfrac{\tau'p}{p - 2} \right) & \text{if } 4 - \tau' < p < 2 + \tau'. \end{cases}$$

Combining Theorem 4.4 and Proposition 4.5(b) leads to

Theorem 4.6. *Let* $1 \leqslant p < 2 + \tau'$ *and* $V = V_1 + V_2$ *with* $V_j \in L^{s_j}(\mathbf{R}^n)$ *for some* $s_j \in I'_p \cap \left(\frac{n}{m}, \infty \right]$ $(j = 1, 2)$. *Then* $iP(D) + V$ *generates a* β-*times integrated group on* $L^p(\mathbf{R}^n)$, *where* $\beta > n_p + 1$.

Corresponding to Corollary 2.6 we have

Corollary 4.7. *Let* $1 \leqslant p \leqslant 3$, $n_p < m/2$ *and* $V \in L^{\frac{p}{|p-2|}}(\mathbf{R}^n)$. *Then* $iP(D) + V$ *generates a* β-*times integrated group on* $L^p(\mathbf{R}^n)$, *where* $\beta > n_p + 1$.

In order to give L^p estimates of the solution for Schrödinger equations we need Straub's fractional powers (cf. [13]). If a densely defined operator A is the generator of a β-times integrated group $T(t)$ $(t \in \mathbf{R})$ satisfying $\|T(t)\| \leqslant Ce^{\omega|t|}$ $(t \in \mathbf{R})$, then for $\delta, \varepsilon > 0$ the fractional powers $(\omega + \delta \pm A)^{\beta + \varepsilon}$ are well-defined and their domains are independent of $\delta > 0$. We note that $D((\omega + \delta + A)^{\beta + \varepsilon}) \cap D((\omega + \delta - A)^{\beta + \varepsilon})$ for small $\varepsilon > 0$ contains the dense subspace $D(A^{[\beta]+1})$. The following result is a consequence of Theorem 4.6 and Theorem 1.1 in [13].

Theorem 4.8. *Suppose* p, V *and* β *satisfy the assumptions of* Theorem 4.6. *Then, there exist constants* $C, \omega > 0$ *such that for every data* $u_0 \in D((\omega + iP(D) + V)^{\beta}) \cap D((\omega - iP(D) - V)^{\beta})$, *the Cauchy problem* (∗) *has a unique solution* $u \in C(\mathbf{R}, L^p(\mathbf{R}^n))$ *and*

$$\|u(t, \cdot)\|_{L^p} \leqslant Ce^{\omega|t|} \|(\omega \pm iP(D) \pm V)^{\beta} u_0\|_{L^p} \quad \text{for } t \in \mathbf{R},$$

where we choose + (*resp.* −) *if* $t \geqslant 0$ (*resp.* < 0).

When P is nondegenerate (i.e. $\det\left(\frac{\partial^2 P(\xi)}{\partial \xi_i \partial \xi_j} \right)_{n \times n} \neq 0$ for $\xi \in \mathbf{R}^n \backslash \{0\}$), the Gaussian curvature of Σ is nonzero everywhere (cf. [11]). In this case $k = 2$, and thus $h(m, n, k) = \frac{n(m-2)}{2(m-1)}$. In order to compare our results with those in [4] for homogeneous polynomial P, we denote by $I_p(\tau)$ (resp. $I'_p(\tau)$) the set I_p (resp. I'_p) defined in Section 2 (resp. 4). As a direct consequence of Theorems 2.4 and 4.4(a) we have

Corollary 4.9. *Suppose P is nondegenerate. Let $1 \leqslant p \leqslant 2$ and $\tau_0 = \frac{2(m-1)}{m-2}$. If $q \in I_p(\tau_0)$ (resp. $s_j \in I_p'(\tau_0) \cap (\frac{n}{m}, \infty])$), then the conclusion of* Theorem 2.4 *(resp. 4.4(a)) is true.*

We first note that for homogeneous polynomial P, the hypothesis (H2) in [4] is equivalent to that P is nondegenerate. Thus, Corollary 4.9 improves the corresponding Theorem 2′ and 6 in [4] in several respects:

(1) For fixed $p \in [1, 2)$, the interval $I_p(\tau_0)$ is replaced by smaller $(q(\tau_1, p), p']$ in Theorem 2′, where $\frac{1}{q(\tau,p)} = \frac{1}{\tau p} + \frac{1}{\tau' p'}$ and $\tau_1 = \frac{2n(m-1)}{mn-2n-3m+2}$. In fact, it is clear that $q(\tau, p)$ $(\tau > 2)$ is strictly increasing. Since $p' \in I_p(\tau_0)$ and $2 < \tau_0 < \tau_1$, $I_p(\tau_0)$ contains properly $(q(\tau_1, p), p']$. Similarly, the interval $I_p'(\tau_0)$ is replaced by smaller $[\frac{p}{2-p}, \frac{\tau_1' p}{2-p})$ in Theorem 6.

(2) The hypothesis (H3′), i.e. $n > 3 + \frac{4}{m-2}$ is required in Theorems 2′ and 6, but not in Corollary 4.9. For example, when $n = 2, 3$ and $m \geqslant 4$, one has $n \leqslant 3 + \frac{4}{m-2}$, and thus Theorems 2′ and 6 cannot deal with such a case. However, in this case, $I_p'(\tau_0) \cap (\frac{n}{m}, \infty] = I_p'(\tau_0)$, which means that for every $p \in [1, 2]$, we can choose q and s_j's values such that the conclusion of Corollary 4.9 holds.

(3) We first note that $p \neq 1$ in Theorems 2′ and 6, but it is admitted that $p = 1$ in Corollary 4.9. Furthermore, it is required in Theorem 6 that $p > \frac{2c}{c+1}$, where c is an integer with $c > \frac{n}{m-1}$. However, the restriction of p in Corollary 4.9 is only caused by $I_p'(\tau_0) \cap (\frac{n}{m}, \infty] \neq \emptyset$, which is equivalent to $p > \frac{2n}{n+2m-2}$. It is easy to see that this is naturally an improvement of the corresponding condition in Theorem 6.

(4) The conclusion in Theorem 6 (see its remark for homogeneous P) is that $iP(D) + V$ generates a smooth distribution group on $L^p(\mathbf{R}^n)$ of order β, which is equivalent to a β-times integrated group on $L^p(\mathbf{R}^n)$, where β is an integer with $\beta > n_p + 2$. Our conclusion in Corollary 4.9 however admits that β is a real number with $\beta > n_p + 1$.

Acknowledgments

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