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On asymptotics of the q-exponential and q-gamma functions ☆

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ABSTRACT

In this work we present a derivation for the complete asymptotic expansions of Euler's q-exponential function and Jackson's q-gamma function via Mellin transform. These formulas are valid everywhere, uniformly on any compact subset of the complex plane.

1. Introduction

Euler's q-exponential function $(z;q)_\infty$ is a fundamental building block in the theory of the basic hypergeometric series, it appears explicitly in many summation formulas, integral representations, orthogonal measures of the Askey–Wilson polynomials. The basic hypergeometric series are q-analogues of the classical hypergeometric series, that is, the latter is the $q \to 1$ limit of the former under appropriate scalings. Therefore, it is very important to know the asymptotic behavior of Euler's q-exponential function.

In this work we derive complete asymptotic expansions for Euler's q-exponential function and its related function, Jackson's q-gamma function. Unlike the approaches in [2,5], we first derive a Mellin transform for the logarithm of Euler's q-exponential function in terms of Euler's gamma function, Riemann zeta function and Hurwitz zeta function, then we left shift the integration contour and compute the sum of the residues, each of them is a product of a Bernoulli number (from Riemann zeta function) and a Bernoulli polynomial (Hurwitz zeta function).

2. Notations

Recall that Euler's gamma function is defined by [1,6,7]

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx, \quad \Re(s) > 1, \tag{2.1}$$

it can be extended to the whole complex plane via the functional equation

$$\Gamma(s+1) = s\Gamma(s), \tag{2.2}$$

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the obtained meromorphic function has simple poles at all non-positive integers with residues

$$\text{Residue}\left\{\Gamma(s),\ s=-n\right\}=\frac{(-1)^n}{n!},\quad n\in\mathbb{N}\cup\{0\}. \tag{2.3}$$

The function $\Gamma(z)$ also has the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},\quad z\in\mathbb{C}\backslash\mathbb{Z}. \tag{2.4}$$

The Bernoulli polynomials $B_n(x)$ and Bernoulli numbers B_n are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad \frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!} \tag{2.5}$$

for $|t|<2\pi$, they satisfy

$$B_1=-\frac{1}{2},\qquad B_{2n+1}=0,\quad n\in\mathbb{N} \tag{2.6}$$

and

$$B_n(1-z)=(-1)^nB_n(z),\quad z\in\mathbb{C},\ n\in\mathbb{N}\cup\{0\}. \tag{2.7}$$

The Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,z)$ are defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \zeta(s,z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^s},\quad \Re(s)>1,\ \Re(z)>0, \tag{2.8}$$

respectively, they can be extended to the whole complex plane to meromorphic functions with only simple pole at 1 with residues

$$\text{Residue}\left\{\zeta(s),\ s=1\right\}=1,\qquad \text{Residue}\left\{\zeta(s,z),\ s=1\right\}=1. \tag{2.9}$$

It is known that [6],

$$\zeta(-n,z)=-\frac{B_{n+1}(z)}{n+1},\qquad \zeta(s)=\zeta(s,1) \tag{2.10}$$

for $n\in\mathbb{N}\cup\{0\}$.

Throughout this work we use the standard notations [1,2,6,3–5]

$$(z;q)_\infty=\prod_{n=0}^{\infty}\left(1-zq^n\right),\qquad (z;q)_n=\frac{(z;q)_\infty}{(zq^n;q)_\infty},\quad z\in\mathbb{C},\ n\in\mathbb{Z}, \tag{2.11}$$

$$(z_1,\ldots,z_m;q)_\infty=\prod_{k=1}^{m}(z_k;q)_\infty,\qquad (z_1,\ldots,z_m;q)_n=\prod_{k=1}^{m}(z_k;q)_n \tag{2.12}$$

and

$$\frac{1}{\Gamma_q(z)}=\frac{(q^z;q)_\infty}{(q;q)_\infty}(1-q)^{z-1},\quad z\in\mathbb{C}, \tag{2.13}$$

where $|q|<1$. We also need [1,6,7]

$$\theta_1(v|\tau)=\frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^nq^{(n+1/2)^2}e^{(2n+1)\pi iv} \tag{2.14}$$

and

$$\theta_4(v|\tau)=\left(q^2,qe^{2\pi iv},qe^{-2\pi iv};q^2\right)_\infty, \tag{2.15}$$

they satisfy

$$\theta_4\left(v+\frac{\tau}{2}\middle|\tau\right)=iq^{-1/4}e^{-\pi iv}\theta_1(v|\tau) \tag{2.16}$$

and

$$\theta_1(v|\tau)=i\sqrt{\frac{i}{\tau}}e^{-\pi iv^2/\tau}\theta_1\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right), \tag{2.17}$$

where $q=e^{\pi i\tau}$, $\Im(\tau)>0$ and $v\in\mathbb{C}$.

3. Main results

Lemma 1. *For $c>1$, $x>0$, $\Re(z)>0$, $\sigma>1$, $t\in\mathbb{R}$ and $s=\sigma+it$, we have*

$$-\int_0^\infty \log\bigl(e^{-xz};e^{-x}\bigr)_\infty x^{s-1}\,dx=\Gamma(s)\zeta(s+1)\zeta(s,z) \tag{3.1}$$

and

$$\log\bigl(e^{-xz};e^{-x}\bigr)_\infty=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s+1)\zeta(s,z)\frac{ds}{x^s} \tag{3.2}$$

or

$$\bigl(e^{-xz};e^{-x}\bigr)_\infty=\exp\left\{\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s+1)\zeta(s,z)\frac{ds}{x^s}\right\}. \tag{3.3}$$

Theorem 2. *For $x>0$, $z\in\mathbb{C}$ and $m\in\mathbb{N}$ we have*

$$\bigl(e^{-xz};e^{-x}\bigr)_\infty=\frac{\sqrt{2\pi}}{\Gamma(z)}\exp\left\{-\frac{\pi^2}{6x}-\left(z-\frac12\right)\log x+\frac{x}{4}B_2(z)-\sum_{k=1}^{m-1}\frac{B_{2k+1}(z)B_{2k}}{2k(2k+1)!}x^{2k}\right\}\times\bigl\{1+\mathcal{O}\bigl(x^{2m}\bigr)\bigr\}, \tag{3.4}$$

as $x\to0^+$, uniformly for z in any compact subset of $\mathbb{C}$.

For the q-gamma function we have:

Corollary 3. *Assume that $q=e^{-x}$ with $x>0$, for any $z\in\mathbb{C}$ and $m\in\mathbb{N}$ we have*

$$\frac{\Gamma(z)}{\Gamma_q(z)}=\exp\left\{\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k)!}\left(z-1-\frac{B_{2k+1}(z)}{2k+1}\right)+\frac{x(z-1)(z-2)}{4}\right\}\bigl\{1+\mathcal{O}\bigl(x^{2m}\bigr)\bigr\} \tag{3.5}$$

as $x\to0^+$, uniformly for z in any compact subset of $\mathbb{C}$.

4. Proofs

4.1. Preliminaries

Lemma 4. *Let $|x|<2\pi$ and $\Re(x)>0$, then,*

$$\log\bigl(1-e^{-x}\bigr)=\log x-\frac{x}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k)!}. \tag{4.1}$$

Proof. From

$$\frac{\sinh x}{x}=\prod_{j=1}^{\infty}\left(1+\frac{x^2}{j^2\pi^2}\right),\quad z\in\mathbb{C},$$

$$\log(1+x)=-\sum_{j=1}^{\infty}\frac{(-x)^j}{j},\quad |x|<1$$

and

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^k B_{2k}}{2(2k)!},\quad k\in\mathbb{N}$$

we get

$$\log\sinh x = \log x + \log\prod_{j=1}^{\infty}\left(1+\frac{x^2}{j^2\pi^2}\right)$$
$$= \log x + \sum_{j=1}^{\infty}\log\left(1+\frac{x^2}{j^2\pi^2}\right) = \log x - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{kj^{2k}\pi^{2k}}$$
$$= \log x - \sum_{k=1}^{\infty}\frac{(-x^2)^k}{k\pi^{2k}}\zeta(2k) = \log x + \sum_{k=1}^{\infty}\frac{B_{2k}(2x)^{2k}}{2k(2k)!}$$

and (4.1) follows. □

Lemma 5. *Assume that $N \in \mathbb{N}$, $\Re(z) > 0$ and $s = \sigma + it$ with $\sigma > -2N$ and $t \in \mathbb{R}$, then we have*

$$\Gamma(s) = \mathcal{O}\left(e^{-\pi|t|/2}|t|^{\sigma-1/2}\right) \tag{4.2}$$

and

$$\zeta(s,z) = \mathcal{O}\left(|t|^{2N+1}\right), \quad s \neq 1 \tag{4.3}$$

as $t \to \pm\infty$, uniformly for z in any compact subset of $\Re(z) > 0$.

Proof. The above equations follow from

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s-\frac{1}{2}\right)\log s - s + \sum_{j=1}^{N}\frac{B_{2j}s^{1-2j}}{2j(2j-1)} - \frac{1}{2N}\int_0^{\infty}\frac{B_{2N}(\{x\})\,dx}{(x+s)^{2N}}\,dx$$

and

$$\zeta(s,z) = \left\{\frac{1}{z^s} + \frac{1}{(1+z)^s}\left(\frac{1}{2}+\frac{1+z}{s-1}\right) + \sum_{k=1}^{N}\frac{(s)_{2k-1}B_{2k}}{2k(1+z)^{s+2k-1}(2k-1)!} - \frac{(s)_{2N+1}}{(2N+1)!}\int_1^{\infty}\frac{B_{2N+1}(\{x\})}{(x+z)^{s+2N+1}}\,dx\right\}$$

respectively. Eq. (4.2) and case $z = 1$ of (4.3) are in [7], the general case for (4.3) is proved similarly. □

Lemma 6. *Let $a, z \in \mathbb{C}$ and $q \in (0,1)$ such that $|a| < 1$ and $\Re(z) > 0$, then we have*

$$\sum_{n=1}^{\infty}\frac{a^n}{n(1-q^n)} = -\log(a;q)_\infty \tag{4.4}$$

and

$$\sum_{n=1}^{\infty}\frac{q^{nz}}{n(1-q^n)} = -\log\left(q^z;q\right)_\infty, \tag{4.5}$$

where the logarithmic function is taken such that $\log x \in \mathbb{R}$ for $x > 0$.

Proof. For $|a| < 1$ we have

$$\sum_{n=1}^{\infty}\frac{a^n}{n(1-q^n)} = \sum_{n=1}^{\infty}\frac{a^n}{n}\sum_{k=0}^{\infty}q^{nk} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(aq^k)^n}{n}$$
$$= -\sum_{k=0}^{\infty}\log\left(1-aq^k\right) = -\log(a;q)_\infty,$$

where the interchanging of order of summations is permitted because of $\sum_{n=1}^{\infty}\frac{|a|^n}{n^{\nu}}\sum_{k=0}^{\infty}q^{nk} < \infty$ and Fubini theorem, the second formula is obtained by taking $a = q^z$. □

Lemma 7. *For $n \in \mathbb{N} \cup \{0\}$, $v \in \mathbb{C}$ with $z = \cos v$, let $U_n(z) = \frac{\sin((n+1)v)}{\sin v}$ be the Chebyshev polynomials of the second kind, then*

$$\left|U_n(z)\right| \leqslant \frac{(1+\sqrt{5})^n}{\sqrt{5}} e^{n|\Im(v)|} \tag{4.6}$$

and

$$\frac{\theta_1(v|\tau)}{2q^{1/4}\sin v} = 1 + \mathcal{O}(q^2) \tag{4.7}$$

as $q \to 0$, uniformly for z in any compact subset of $\mathbb{C}$.

Proof. From

$$U_n(z) = (2z)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \left(-4z^2\right)^{-k}$$

we obtain

$$\begin{aligned}\left|U_n(z)\right| &\leqslant |2z|^n \sum_{k=0}^{n} \binom{n-k}{k} = |2z|^n F_{n+1} \\ &\leqslant \frac{(1+\sqrt{5})^n}{\sqrt{5}} |z|^n \leqslant \frac{(1+\sqrt{5})^n}{\sqrt{5}} e^{n|\Im(v)|},\end{aligned}$$

for $|z| \geqslant \frac{1}{2}$, while for $|z| < \frac{1}{2}$ we have

$$\left|U_n(z)\right| \leqslant |2z|^{n-2\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \leqslant F_{n+1} < \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

From

$$\theta_1(v|\tau) = 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)v$$

we get

$$\frac{\theta_1(v|\tau)}{2q^{1/4}\sin v} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} U_{2n}(v),$$

and

$$\left|\frac{\theta_1(v|\tau)}{2q^{1/4}\sin v} - 1\right| \leqslant \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} q^{n^2} \left\{(1+\sqrt{5}) q e^{|\Im(v)|}\right\}^n,$$

which proves (4.7). □

4.2. Proof of Lemma 1

Proof. For $\Re(z) > 0$, $s = \sigma + it$ with $t \in \mathbb{R}$ and $\sigma > 1$, from (4.5) we have

$$\begin{aligned}-\int_0^\infty \log\left(e^{-xz}; e^{-x}\right)_\infty x^{s-1}\,dx &= \int_0^\infty \left\{\sum_{n=1}^{\infty} \frac{e^{-nxz}}{n(1-e^{-nx})}\right\} x^{s-1}\,dx \\ &= \sum_{n=1}^{\infty} \int_0^\infty \left\{\frac{e^{-nxz}}{n(1-e^{-nx})}\right\} x^{s-1}\,dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{1+s}} \int_0^\infty \frac{e^{-yz} y^{s-1}\,dy}{1-e^{-y}} = \Gamma(s)\zeta(s+1)\zeta(s,z),\end{aligned}$$

the interchange of the order of summation and integration is valid because of

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma}} \int_0^{\infty} \frac{e^{-y\Re(z)} y^{\sigma-1}\, dy}{1-e^{-y}} < \infty$$

and the Fubini theorem. The asymptotics of $\Gamma(s)$, $\zeta(s)$ and $\zeta(s,z)$ for $\Re(s) > 1$ and $\Re(z) > 0$ allows us to invert the Mellin transformation to get (3.2) and (3.3). □

4.3. Proof of Theorem 2

Proof. Let us first assume that $\Re(z) > 0$, considering the rectangular contour with corners at $c - iM$, $c + iM$, $d + iM$ and $d - iM$, then letting $M \to \infty$, it is clear that we have

$$\begin{aligned} -\log\left(e^{-xz}; e^{-x}\right)_{\infty} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s+1)\zeta(s,z) \frac{ds}{x^s} \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s)\zeta(s+1)\zeta(s,z) \frac{ds}{x^s} + \sum_{k=-2m+1}^{1} \text{Residue}\left\{ \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=k \right\}, \end{aligned}$$

where $c > 1$ and $d > -2m$. From

$$\begin{aligned} \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s} &= \frac{1}{s^2} \Gamma(s+1) \cdot \bigl(s\zeta(s+1)\bigr) \cdot \zeta(s,z) \cdot x^{-s} \\ &= \frac{1}{s^2} \bigl\{1 - \gamma s + \mathcal{O}(s^2)\bigr\} \times \bigl\{1 + \gamma s + +\mathcal{O}(s^2)\bigr\} \\ &\quad \times \left\{ \frac{1}{2} - z + s \log \frac{\Gamma(z)}{\sqrt{2\pi}} + \mathcal{O}(s^2) \right\} \times \bigl\{1 - s\log x + \mathcal{O}(s^2)\bigr\} \\ &= \frac{\frac{1}{2} - z}{s^2} + \left(\log \frac{\Gamma(z)}{\sqrt{2\pi}} + \left(z - \frac{1}{2}\right) \log x \right) s^{-1} + \mathcal{O}(1) \end{aligned}$$

we get

$$\text{Residue}\left\{ \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=0 \right\} = \log \frac{\Gamma(z)}{\sqrt{2\pi}} + \left(z - \frac{1}{2}\right) \log x.$$

Similarly we have

$$\begin{aligned} \text{Residue}\left\{ \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=1 \right\} &= \frac{\pi^2}{6x}, \\ \text{Residue}\left\{ \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=-1 \right\} &= -\frac{x}{4} B_2(z) \end{aligned}$$

and

$$\text{Residue}\left\{ \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=-2k \right\} = \frac{B_{2k+1}(z) B_{2k}}{2k(2k+1)!} x^{2k},$$

thus we have

$$\begin{aligned} -\log\left(e^{-xz}; e^{-x}\right)_{\infty} &= \frac{\pi^2}{6x} + \left(z - \frac{1}{2}\right) \log x + \log \frac{\Gamma(z)}{\sqrt{2\pi}} - \frac{x}{4} B_2(z) \\ &\quad + \sum_{k=1}^{m-1} \frac{B_{2k+1}(z) B_{2k}}{2k(2k+1)!} x^{2k} + \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)\zeta(s+1)\zeta(s,z)\, ds}{2\pi i x^s}. \end{aligned}$$

Notice that for $-2m < \rho < -2m-1$,

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\Gamma(s)\zeta(s+1)\zeta(s,z)\frac{ds}{x^s}=\operatorname{Residue}\left\{\frac{\Gamma(s)\zeta(s+1)\zeta(s,z)}{x^s},\ s=-2m\right\}+\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\Gamma(s)\zeta(s)\zeta(s,z)\frac{ds}{x^s}$$

$$=\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\Gamma(s)\zeta(s)\zeta(s,z)\frac{ds}{x^s}=\mathcal{O}\left(x^{\rho}\right),$$

uniformly for z in any compact subset of $\Re(z)>0$, thus we get

$$-\log\left(e^{-xz};e^{-x}\right)_\infty=\frac{\pi^2}{6x}+\left(z-\frac{1}{2}\right)\log x+\log\frac{\Gamma(z)}{\sqrt{2\pi}}-\frac{x}{4}B_2(z)+\sum_{k=1}^{m-1}\frac{B_{2k+1}(z)B_{2k}}{2k(2k+1)!}x^{2k}+\mathcal{O}\left(x^{2m}\right)$$

or

$$\left(e^{-xz};e^{-x}\right)_\infty=\frac{\sqrt{2\pi}}{\Gamma(z)}\exp\left\{-\frac{\pi^2}{6x}-\left(z-\frac{1}{2}\right)\log x+\frac{x}{4}B_2(z)-\sum_{k=1}^{m-1}\frac{B_{2k+1}(z)B_{2k}}{2k(2k+1)!}x^{2k}\right\}\left\{1+\mathcal{O}\left(x^{2m}\right)\right\}$$

as $x\to 0^+$, uniformly for z in any compact subset set of $\Re(z)>0$.

From (2.15), (2.16) and (2.17) we get

$$\left(q^2,q^2e^{2\pi iv},e^{-2\pi iv};q^2\right)_\infty=iq^{-1/4}e^{-\pi iv}\theta_1(v|\tau)$$
$$=-\sqrt{\frac{i}{\tau}}\exp\left\{-\pi iv-\frac{\pi i\tau}{4}-\frac{\pi iv^2}{\tau}\right\}\theta_1\left(\frac{v}{\tau}\middle|-\frac{1}{\tau}\right),$$

where $q=e^{\pi i\tau}$ with $\Im(\tau)>0$. Let $\tau=\frac{xi}{2\pi}$ and $v=-\frac{xzi}{2\pi}$ in the above equation, by applying (2.14) and Lemma 7 we obtain

$$\left(e^{-x},e^{-x(1-z)},e^{-xz};e^{-x}\right)_\infty=\sqrt{\frac{2\pi}{x}}\exp\left\{\frac{x}{8}-\frac{xz}{2}+\frac{xz^2}{2}\right\}\theta_1\left(z\middle|\frac{2\pi i}{x}\right)$$
$$=\sqrt{\frac{2\pi}{x}}\exp\left\{\frac{x}{8}-\frac{xz}{2}+\frac{xz^2}{2}-\frac{\pi^2}{2x}\right\}\times 2\sin\pi z\left\{1+\mathcal{O}\left(e^{-4\pi^2/x}\right)\right\},$$

as $x\to 0^+$, uniformly for z in any compact subset of $\mathbb{C}$. Then for $\Re(z)<1$, by applying (2.4), (2.6) and (2.7) we have

$$\left(e^{-xz};e^{-x}\right)_\infty=\sqrt{\frac{2\pi}{x}}\exp\left\{\frac{x}{8}-\frac{xz}{2}+\frac{xz^2}{2}-\frac{\pi^2}{2x}\right\}\times 2\sin\pi z\left\{1+\mathcal{O}\left(e^{-4\pi^2/x}\right)\right\}\times\left(e^{-x};e^{-x}\right)_\infty^{-1}\left(e^{-x(1-z)};e^{-x}\right)_\infty^{-1}$$
$$=\frac{\sqrt{2\pi}}{\Gamma(z)}\exp\left\{-\frac{\pi^2}{6x}-\left(z-\frac{1}{2}\right)\log x+\frac{x}{4}B_2(z)-\sum_{k=1}^{m-1}\frac{B_{2k+1}(z)B_{2k}}{2k(2k+1)!}x^{2k}\right\}\times\left\{1+\mathcal{O}\left(x^{2m}\right)\right\},$$

as $x\to 0^+$, uniformly for z in any compact subset of the half-plane $\Re(z)<1$. The theorem follows from combining the two cases for $\Re(z)<1$ and $\Re(z)>0$. □

4.4. Proof of Corollary 3

Proof. From (2.13), (4.1) and Theorem 2 we get

$$\frac{\Gamma(z)}{\Gamma_q(z)}=\frac{(q^z;q)_\infty}{(q;q)_\infty}(1-q)^{z-1}\Gamma(z)$$
$$=\exp\left\{\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k)!}\left(z-1-\frac{B_{2k+1}(z)}{2k+1}\right)+\frac{x(z-1)(z-2)}{4}\right\}\left\{1+\mathcal{O}\left(x^{2m}\right)\right\}.\qquad □$$

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