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Number of zeros of interval polynomials

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ABSTRACT

In this paper, we develop a rigorous algorithm for counting the real interval zeros of polynomials with perturbed coefficients that lie within a given interval, without computing the roots of any polynomials. The result generalizes Sturm's Theorem for counting the roots of univariate polynomials to univariate interval polynomials.

1. Introduction

Polynomials with perturbed coefficients, which we will call interval polynomials, are commonly used in various areas of science and in engineering applications, because of floating-point computing or because of errors in inputting polynomials into the computer environment. Manipulation such polynomials, for example, studying their zeros, is a very important problem in practical applications. It is known that, one can obtain the zeros of an interval polynomial by computing the roots of some exact polynomials. However, because the roots of a polynomial are very sensitive to its coefficients, and it is hard to solve algebraic equations of high degree, the stability and the complexity of numerical methods present very challenging problems, whose solutions will require the introduction of novel techniques.

Several papers have considered the zeros of interval polynomials. In [1], Levkovich et al. proposed a method for checking whether a given interval polynomial has a robust root distribution with respect to a given symmetric sector of the complex plane. In [2], the maximal modulus of the zeros of interval polynomials were investigated. In [3], Ferreira et al. discussed the distribution of the complex zeros of interval polynomials of degrees two, three and four. They further explored the real zeros of a special class of multivariate interval polynomials in [4]. In [5], Fan et al. gave a bound on the number of interval zeros using the degrees of interval polynomials, and described the boundaries of the complex block zeros. They also provided a numerical algorithm to bound the interval zeros and complex block zeros. In this paper, we will mainly concentrate on the number of interval zeros in a given interval of interval polynomials. The main idea is as follows: firstly, we translate the problem to computing the intervals in which the product of two bound functions (see the definition in Section 2) is non-positive. Secondly, we classify the roots of the bound functions into three categories and study how to compute the number of elements in each category without computing the roots. Lastly, we determine the relationship between the number of interval zeros of the given interval polynomial and the number of roots of different kinds of the bound functions.

This paper is organized as follows. In Section 2, some basic definitions and propositions are reviewed. In Section 3, we discuss computation of the number of different kinds of roots of bound functions using Sturm sequences. In Section 4, we relate the number of interval zeros of an interval polynomial to the number of roots of its bound functions and give an algorithm to compute this number. Finally, an example is given to illustrate the algorithm.

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2. Interval polynomials and zero sets

In this section, we review some basic concepts about univariate interval polynomials [5].

An *interval polynomial* of degree n is a polynomial whose coefficients are intervals:

$$[f](x) := \sum_{i=0}^{n}[a_i, b_i]\,x^i := \left\{\sum_{i=0}^{n} f_i x^i : f_i \in [a_i, b_i],\ i = 0, 1, \ldots, n\right\},$$

where $[a_i, b_i]$, $i = 0, 1, \ldots, n$, are bounded closed intervals. $[a_n, b_n]$ is called the *leading coefficient* and n is called the *degree* of $[f](x)$.

The *upper/lower bound functions* of $[f](x)$ are defined by

$$\mathcal{U}f(x) := \begin{cases} \mathcal{U}f^{+}(x) = \sum_{i=0}^{n} b_i x^i & x \geqslant 0, \\ \mathcal{U}f^{-}(x) = \sum_{0 \leqslant 2i \leqslant n} b_{2i}x^{2i} + \sum_{0 \leqslant 2i+1 \leqslant n} a_{2i+1}x^{2i+1} & x < 0, \end{cases}$$

$$\mathcal{L}f(x) := \begin{cases} \mathcal{L}f^{+}(x) = \sum_{i=0}^{n} a_i x^i & x \geqslant 0, \\ \mathcal{L}f^{-}(x) = \sum_{0 \leqslant 2i \leqslant n} a_{2i}x^{2i} + \sum_{0 \leqslant 2i+1 \leqslant n} b_{2i+1}x^{2i+1} & x < 0. \end{cases}$$

Note that both the upper and lower bound functions of $[f](x)$ are piecewise polynomials with joints at $x = 0$. Naturally, we require that there exists at least one k such that $a_k \lneqq b_k$, that is, $[f](x)$ is not an ordinary polynomial. Then, $\mathcal{L}f(x) \leqslant \mathcal{U}f(x)$ for any $x \in [a, b]$ and the equality holds if and only if $a_0 = b_0$ and $x = 0$.

Let $[f](x)$ be a real interval polynomial. The graph of $[f](x)$ is denoted by $G([f])$ and is given by

$$G([f]) = \{(x, y) \in \mathcal{R}^2 : \text{ there exists an } h \in [f](x) \text{ such that } y = h(x)\}.$$

The following lemma shows that the graph of $[f](x)$ is a simply connected region bounded by $\mathcal{L}f(x)$ and $\mathcal{U}f(x)$ in the plane.

Lemma 1 (*Ferreira et al. [3]*)**.** *The graph of an interval polynomial* $[f](x)$ *is given by*

$$G([f]) = \{(x, y) \in \mathcal{R}^2 : \mathcal{L}f(x) \leqslant y \leqslant \mathcal{U}f(x)\}.$$

The real *zero* of an interval polynomial $[f](x)$ is defined as

$$Z([f]) \triangleq \{t_0 \in \mathcal{R} : \text{there exists an } h \in [f](x) \text{ such that } h(t_0) = 0\}.$$

By Lemma 1, $[f](t_0) = [\mathcal{L}f(t_0), \mathcal{U}f(t_0)]$, for all $t_0 \in \mathcal{R}$. Thus, we have

$$Z([f]) = \{t_0 \in \mathcal{R} : \mathcal{L}f(t_0) \cdot \mathcal{U}f(t_0) \leqslant 0\}.$$

In this case, the zero set of $[f](x)$ is in fact composed of several closed intervals (except for the intervals containing $+\infty$ or $-\infty$), and the endpoints (except $\pm\infty$) of the interval zeros must be roots of the upper bound function $\mathcal{L}f(x)$ or the lower bound function $\mathcal{U}f(x)$. We call each of these intervals an *interval zero* of $[f](x)$. The aim of this paper is to *count the interval zeros of* $[f](x)$ *without computing the roots of any polynomials.*

Remark 2. If $[a, a]$ is an interval zero of $[f](x)$, then the sum of the multiplicities of $\mathcal{U}f(x)$ and $\mathcal{L}f(x)$ at $x = a$ is at least two. We consider a as a "multiple endpoint" of $[a, a]$. Hence, every interval zero has two endpoints.

Remark 3. If $(-\infty, a]$ is an interval zero of $[f](x)$, then the leading interval coefficient of $[f](x)$ must contain 0. Then, there exists b such that $[b, +\infty)$ is also an interval zero of $[f](x)$. In this case, we count $(-\infty, a] \bigcup [b, +\infty)$ as one interval zero. In other words, if we require that the leading interval coefficient does not contain 0, then $[f](x)$ cannot have infinite interval zeros.

As an example, Fig. 1 shows the graph and zeros of the interval polynomial $[f](x) = x^2 + [-3, 3]x + [2, 6]$. Here, $[f](x)$ has two interval zeros: $[-2, -1]$ and $[1, 2]$.

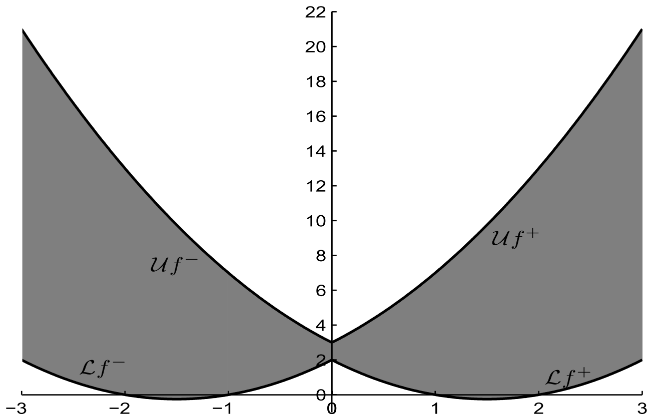

Fig. 1. Graph of $x^2 + [-3, 3]x + [2, 6]$.

3. Roots of bound functions

As we have seen in the previous section, the number of interval zeros of an interval polynomial $[f](x)$ is equal to the number of intervals in which $g(x) = \mathcal{L}f(x) \cdot \mathcal{U}f(x) \leqslant 0$. Therefore, we now consider the roots of $g(x)$. Note that g is a piecewise polynomial with a joint at 0. Studying the roots of g is, in some sense, equivalent to studying the roots of two polynomials in $(-\infty, 0)$ and $(\infty, 0)$. Thus, we begin with Sturm's Theorem, which provides powerful tools and efficient algorithms for counting the roots of univariate polynomials. The following definition and theorem are well known in algebra textbooks.

Definition 4. A Sturm chain or Sturm sequence is a finite sequence of polynomials $\{p_0, p_1, \ldots, p_m\}$ of decreasing degree, with these following properties:

(1) $p_0 = p$ is square free (no square factors);
(2) if $p(t_0) = 0$, then $\text{sign}(p_1(t_0)) = \text{sign}(p'(t_0))$;
(3) if $p_i(t_0) = 0$ for some $0 < i < m$, then $\text{sign}(p_{i-1}(t_0)) = -\text{sign}(p_{i+1}(t_0))$;
(4) p_m does not change its sign.

Theorem 5 (*Sturm [6]*). *Let $S = \{p_0 = p, p_1, \ldots, p_m\}$ be a Sturm sequence, where p is a square-free polynomial, and let $\sigma(S, x)$ denote the number of sign changes (zeros are not counted) in the sequence S. Then for two real numbers $a < b$, the number of zeros of p in the open interval (a, b) is $\sigma(S, a) - \sigma(S, b)$.*

Remark 6. Sturm himself proposed to choose the intermediate results when applying a Euclidean algorithm to a polynomial p and its derivative p' to obtain a Sturm sequence, known as the canonical Sturm sequence of p:

$$
\begin{aligned}
&p_0(x) := p(x),\\
&p_1(x) := p'(x),\\
&p_2(x) := -\text{rem}(p_0, p_1),\\
&\vdots\\
&p_m(x) := -\text{rem}(p_{m-2}, p_{m-1}),\\
&0 = -\text{rem}(p_{m-1}, p_m).
\end{aligned} \tag{1}
$$

Here, $\text{rem}(p_i, p_{i+1})$ denotes the remainder of p_i divided by p_{i+1}.

Remark 7. Ordinarily, we use the canonical Sturm sequence to determine the number of zeros of p. In this case, even if p is not square-free, $\sigma(S, a) - \sigma(S, b)$ is the number of distinct roots of p in (a, b) whenever $a < b$ are real numbers such that neither a nor b is a zero of p.

In 1853, Sylvester observed that Sturm Theorem can be extended to add an inequality as a side condition:

Theorem 8 (*Sylvester [7]*). *Let p, q be real univariate polynomials and $a < b$ be elements of $\mathcal{R}$ that are not zeros of p or q. Let T be the finite sequence of polynomials obtained from p and $p'q$ by successive Euclidean division with a negative remainder* *sequence.* $\sigma(T, x)$ *denotes the number of sign changes (zeros are not counted) in* T. *Then the difference of the number of solutions of*

$$(1)\ \begin{cases} a < t_0 < b, \\ p(t_0) = 0, \\ q(t_0) > 0 \end{cases}$$

and the number of solutions of

$$(2)\ \begin{cases} a < t_0 < b, \\ p(t_0) = 0, \\ q(t_0) < 0 \end{cases}$$

is $\sigma(T, a) - \sigma(T, b)$.

It is then straightforward to obtain the following corollary:

Corollary 9. *Let* p, q, a, b, T *be as in Theorem 8. Let* u^+, u^- *denote the number of solutions of* (1) *and* (2), *and let* S_1, S_2 *denote the canonical Sturm sequence of* p *and the greatest common divisor* $\gcd(p, q)$ *respectively. Then*

$$\begin{cases} u^+ = \dfrac{\sigma(S_1, a) - \sigma(S_1, b) - \sigma(S_2, a) + \sigma(S_2, b) + \sigma(T, a) - \sigma(T, b)}{2}, \\ u^- = \dfrac{\sigma(S_1, a) - \sigma(S_1, b) - \sigma(S_2, a) + \sigma(S_2, b) - \sigma(T, a) + \sigma(T, b)}{2}. \end{cases}$$

Proof. Note that solutions of $a < t_0 < b,\ p(t_0) = 0$ can be divided into three parts: solutions of (1), (2) in Theorem 8, and solutions of

$$\begin{cases} a < t_0 < b, \\ p(t_0) = 0, \\ q(t_0) = 0. \end{cases}$$

Thus, we have

$$u^+ + u^- = \sigma(S_1, a) - \sigma(S_1, b) - \sigma(S_2, a) + \sigma(S_2, b).$$

By Theorem 8, we have

$$u^+ - u^- = \sigma(T, a) - \sigma(T, b).$$

Thus, the result is obvious. □

Now, we consider the problem of counting the roots of a polynomial $p(x)$ which are local minima or maxima. Let t_0 be a root of p. The Taylor series at t_0 of $p(x)$ is

$$p(x) = p'(t_0)(x - t_0) + \frac{p''(t_0)}{2!}(x - t_0)^2 + \frac{p'''(t_0)}{3!}(x - t_0)^3 + \cdots + \frac{p^{(m)}(t_0)}{m!}(x - t_0)^m,$$

where m denotes the degree of p. By observing the behavior of $p(x)$ near t_0, we have:

Proposition 10. *Let* $t_0 \in \mathcal{R},\ p(x)$ *be as described above,* $p(t_0) = 0$. *Then,*

(1) t_0 *is a local minimum point if and only if there exists an odd integer* r *such that* $p'(t_0) = p''(t_0) = \cdots = p^{(r)}(t_0) = 0$ *and* $p^{(r+1)}(t_0) > 0$.
(2) t_0 *is a local maximum point if and only if there exists an odd integer* r *such that* $p'(t_0) = p''(t_0) = \cdots = p^{(r)}(t_0) = 0$ *and* $p^{(r+1)}(t_0) < 0$.

Proof. Note that when $p'(t_0) = p''(t_0) = \cdots = p^{(r)}(t_0) = 0$, p behaves approximately as $\bar{p} = \frac{p^{(r+1)}(t_0)}{(r+1)!}(x - t_0)^{r+1}$ in a sufficiently small neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon)$. $\bar{p}$ is a local minimum or a local maximum at t_0 if and only if r is odd, and then the sign of the coefficient $\frac{p^{(r+1)}(t_0)}{(r+1)!}$ implies the property of local minimality or local maximality. □

On the basis of Corollary 9 and Proposition 10, we give an algorithm to compute the number of roots of each kind of polynomial $p(x)$ in an interval.

Algorithm 1.

Input: A polynomial $p(x)$ and a real interval (a, b).

Output: Three integers $\{s, t, r\}$, where s, t, r denote the number of roots of $p(x)$ which are locally minimal, locally maximal, and neither locally minimal nor locally maximal, respectively.

Step 1. Compute the number w of all distinct roots of p in (a, b) using the canonical Sturm sequence of p. If $w = 0$, output $\{0, 0, 0\}$ and terminate the algorithm.

Step 2. Let $s := 0$, $t := 0$, and let m be the degree of p. For $1 \leqslant i \leqslant m$, perform the following two steps.

Step 3. Let $h_i(x)$ be the greatest common divisor of $p, p', \ldots, p^{(i)}$, and let u_i^+, u_i^- denote the number of solutions of

$$\begin{cases} a < t_0 < b, \\ h_i(t_0) = 0, \\ p^{(i+1)}(t_0) > 0 \end{cases}$$

and

$$\begin{cases} a < t_0 < b, \\ h_i(t_0) = 0, \\ p^{(i+1)}(t_0) < 0, \end{cases}$$

respectively. u_i^+, u_i^- can be computed from the formula in Corollary 9.

Step 4. $s := s + u_i^+$, $t := t + u_i^-$, $i := i + 2$, go to Step 3.

Step 5. $r := w - s - t$. Output $\{s, t, r\}$.

As $t_0 \in Z([f])$ if and only if $g(t_0) = \mathcal{L}f(t_0) \cdot \mathcal{U}f(t_0) \leqslant 0$ (see the notation introduced in page 3), the number of interval zeros of $[f](x)$ is equal to the number of intervals in which $g(x)$ is non-positive. This number is related to the numbers of roots of different kinds of $g(x)$. We describe the relationships between the roots of $g(x)$ and those of $\mathcal{U}f(x)$ and $\mathcal{L}f(x)$ as follows.

Lemma 11. *$g(x) = \mathcal{U}f(x) \cdot \mathcal{L}f(x)$, $0 \neq t_0 \in \mathcal{R}$ is a root of $g(x)$. Then*

(1) *The set of roots of $g(x)$ is the union of the sets of roots of $\mathcal{U}f(x)$ and $\mathcal{L}f(x)$.*

(2) *t_0 is a local minimum of $g(x)$ if and only if $\mathcal{L}f(t_0) = 0$ and t_0 is a local minimum of $\mathcal{L}f(x)$, or $\mathcal{U}f(t_0) = 0$ and t_0 is a local maximum of $\mathcal{U}f(x)$.*

(3) *t_0 is a local maximum of $g(x)$ if and only if $\mathcal{L}f(t_0) = 0$ and t_0 is a local maximum of $\mathcal{L}f(x)$, or $\mathcal{U}f(t_0) = 0$ and t_0 is a local minimum of $\mathcal{U}f(x)$.*

(4) *Any two roots of $g(x)$ such that one is a local minimum and the other is a local maximum cannot be adjoint, i.e., there must be some root that is neither a local minimum nor a local maximum lying between them.*

Proof. The claim in (1), (4) is obvious and (2) is dual to (3), thus, we need only prove (2). Note that $\mathcal{U}f(t_0) > \mathcal{L}f(t_0)$, and $\mathcal{L}f(t_0) = 0$ or $\mathcal{U}f(t_0) = 0$ by $g(t_0) = 0$. The function g has a local minimum at t_0 if and only if $g(x) > 0$ on a sufficiently small neighborhood D of t_0, that is, if and only if $\mathcal{L}f, \mathcal{U}f$ have the same sign in D which is equivalent to $\mathcal{L}f(t_0) = 0$, $\mathcal{U}f(x) > 0$, $\mathcal{L}f(x) > 0$ on D or $\mathcal{U}f(t_0) = 0$, $\mathcal{L}f(x) < 0$, $\mathcal{U}f(x) < 0$ on D. Then we conclude that g has a local minimum at t_0 if and only if $\mathcal{L}f(t_0) = 0$ and $\mathcal{L}f(x)$ has a local minimum at t_0, or $\mathcal{U}f(t_0) = 0$ and $\mathcal{U}f(x)$ is maximal at t_0. □

4. Number of interval zeros

Now we turn to the main problem. Given an interval polynomial $[f](x) = \sum_{i=0}^{n}[a_i, b_i]x^i$ and an interval $(a, b) \subseteq (-\infty, \infty)$, determine the number of interval zeros of $[f](x)$ in (a, b). The theorem below shows that we can obtain the number of interval zeros of $[f](x)$ by counting the roots of $\mathcal{U}f$ and $\mathcal{L}f$ that are local minima or maxima.

Theorem 12. *$[f](x)$ is an interval polynomial, and $\mathcal{U}f(x)$, $\mathcal{L}f(x)$ are the upper and lower bound functions of $[f](x)$, respectively. Let $a, b \in \mathcal{R}$, such that neither of a, b is a zero of $[f](x)$. Let $\{x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t, z_1, z_2, \ldots, z_r\}$ be the distinct roots of $g(x) = \mathcal{U}f(x) \cdot \mathcal{L}f(x)$ lying in (a, b), such that x_i's are local minima, y_i's are local maxima and z_i's are the roots that are neither local minima nor local maxima. Then, the number of interval zeros of $[f](x)$ in (a, b) is $s + r/2$.*

Proof. Note that $[\alpha, \beta]$ is an interval zero of $[f](x)$ if and only if $g(\alpha) = g(\beta) = 0$, $g(\alpha-) > 0$, $g(\beta+) > 0$ and $g(x) \leqslant 0$ for x in (α, β). The polynomial g has a local minimum at x_u provided that $g(x) > 0$ in a sufficiently small neighborhood of x_u. Then, each x_u corresponds to an interval zero $[x_u, x_u]$ of $[f](x)$. For any two adjoint z_u and z_v (i.e., there is no z_k between them) satisfying $g(z_u-) > 0$, if there exists $y_{k_1}, \ldots, y_{k_q}$ such that $z_u < y_{k_1} < \cdots < y_{k_q} < z_v$, we have $g(z_v+) > 0$ and $g(x) \leqslant 0$ holds in $[z_u, z_v]$. It follows that, $[z_u, z_v]$ is an interval zero of $[f](x)$. In other words, the roots of g that are local maxima have no contribution to the number of interval zeros of $[f](x)$. Considering the sign changes of $g(x)$ when x varies from a to b, we see that the sign of $g(x)$ changes once if and only if x passes through some z_k. As $g(a) > 0$ and $g(b) > 0$ have the same sign, r must be even. It follows that the number of interval zeros is equal to $s + r/2$. □

For a given function h, we classify its real roots in (a, b) into three categories: local minima, local maxima and the remainder(that are neither local minima nor local maxima), and we denote the number of elements in each category by $s(h), t(h), r(h)$ respectively. By Theorem 12, we need to compute $s(g), r(g)$.

Firstly, we consider the case where (a, b) is a positive interval, namely, $b > a > 0$. In this case, we need only consider the polynomial $g(x) = \mathcal{U}f^+(x) \cdot \mathcal{L}f^+(x)$. By Lemma 11, we have:

$$\begin{cases} s(g) = s(\mathcal{L}f^+) + t(\mathcal{U}f^+), \\ t(g) = t(\mathcal{L}f^+) + s(\mathcal{U}f^+), \\ r(g) = r(\mathcal{L}f^+) + r(\mathcal{U}f^+). \end{cases}$$

Each s, t, r for $\mathcal{L}f^+$ and $\mathcal{U}f^+$ can be obtained from Algorithm 1. The case of $a < b < 0$ can be analyzed similarly.

The case of (a, b) containing 0 is somewhat more complicated. In this case, g is not a polynomial over (a, b), and we must consider whether 0 is a root of g. If 0 is not a root of g, then the numbers of roots of g in (a, b) in the three categories can be computed in $(a, 0)$ and $(b, 0)$ as before. If 0 is a root of g, we divide (a, b) into three parts: $(a, -\tau)$, $(-\tau, \tau)$ and (τ, b), where τ is a positive number such that $g(\tau) \neq 0$ and $g(x)$ has no roots in $(-\tau, \tau)$ except 0. Such a τ can be obtained from the Cauchy bound.

Theorem 13 (*Cauchy [8]*). *Let h be defined by $h(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \in \mathbb{C}[x]$ and*

$$r_0 = \max\{|z_1|, \ldots, |z_m|\},$$

where $z_1, z_2, \ldots, z_n$ are the complex roots of $h(x)$. Then

$$r_0 \leqslant 1 + \max\{|a_0|, \ldots, |a_{m-1}|\}.$$

If z is a nonzero root of h given in Theorem 13, with $a_m \neq 0$, then z is a root of $a_m x^{m-k} + \cdots + a_{k+1}x + a_k$, where $k = \min\{i : a_i \neq 0\}$. So $1/z$ is a root of

$$x^{m-k} + \frac{a_{k+1}}{a_k} x^{m-1-k} + \cdots + \frac{a_m}{a_k}.$$

By Theorem 13,

$$\left|\frac{1}{z}\right| \leqslant 1 + \max\left\{\left|\frac{a_{k+1}}{a_k}\right|, \ldots, \left|\frac{a_m}{a_k}\right|\right\}.$$

Thus we have

$$|z| \geqslant \left(1 + \frac{1}{|a_k|} \max\{|a_{k+1}|, \ldots, |a_m|\}\right)^{-1}.$$

In other words, the nonzero roots of a polynomial have a lower bound given by the coefficients of the polynomial.

When $0 \in (a, b)$ and 0 is a root of g, we can choose τ smaller than the minimum of the lower bounds of nonzero roots of the polynomials: $\mathcal{U}f^+$, $\mathcal{U}f^-$, $\mathcal{L}f^+$ and $\mathcal{L}f^-$, then g has no root in $(-\tau, \tau)$ except 0. The numbers of roots of g in $(a, -\tau)$ and (τ, b) in the three categories can be obtained as before, and the remaining work is to determine whether the root 0 of g is a local minimum or maximum or neither. This is related to the left and right derivatives of $\mathcal{U}f$ and $\mathcal{L}f$ at 0, which can be described by the coefficients of $[f](x)$. For example, we consider the case of $0 = a_0 < b_0$, where $[a_0, b_0]$ is the "constant coefficient" of $[f](x)$(note that if 0 is a root of g, then $a_0 \cdot b_0 = 0$). Then g has a local maximum at zero if and only if $\mathcal{L}f^+(0+) < 0$ and $\mathcal{L}f^-(0-) < 0$. The last sentence occurs if and only if there exist odd integers r_1, r_2 such that

$$\begin{cases} (\mathcal{L}f^+)^{(1)}(0) = \cdots = (\mathcal{L}f^+)^{(r_1-1)}(0) = 0, \ (\mathcal{L}f^+)^{(r_1)}(0) < 0, \\ (\mathcal{L}f^-)^{(1)}(0) = \cdots = (\mathcal{L}f^-)^{(r_2-1)}(0) = 0, \ (\mathcal{L}f^-)^{(r_2)}(0) > 0, \end{cases}$$

which allow us to conclude the existence of odd integers r_1, r_2 such that

$$\begin{cases} a_0 = a_1 = \cdots = a_{r_1-1} = 0, \quad a_{r_1} < 0, \\ a_0 = b_1 = a_2 = b_3 = \cdots = a_{r_2-1} = 0, \quad b_{r_2} > 0. \end{cases}$$

We can judge which category of roots(local minima, local maxima or neither) 0 belong to, as in the cases of $a_0 < b_0 = 0$ and $a_0 = b_0 = 0$, by the corresponding conditions on the endpoints $a_i's$, $b_i's$. Now we translate the analysis above into the following algorithm.

Algorithm 2.

Input: $[f](x) := \sum_{i=0}^{n} [a_i, b_i] x^i$ and (a, b), neither a nor b is a zero of $[f](x)$.

Output: the number of interval zeros of $[f](x)$ in (a, b).

Step 1. If $a \geqslant 0$, execute Algorithm 1 twice with the inputs $\{\mathcal{L}f^+, (a, b)\}$ and $\{\mathcal{U}f^+, (a, b)\}$ respectively. Let the corresponding output be $\{s_1, t_1, r_1\}$ and $\{s_2, t_2, r_2\}$. Let $s = s_1 + t_2$, $t = t_1 + s_2$, $r = r_1 + r_2$. Go to Step 7.
Step 2. If $b \leqslant 0$, execute Algorithm 1 twice with the inputs $\{\mathcal{U}f^-, (a, b)\}$ and $\{\mathcal{L}f^-, (a, b)\}$ respectively. Let the corresponding output be $\{s_1, t_1, r_1\}$ and $\{s_2, t_2, r_2\}$. Let $s = s_1 + t_2$, $t = t_1 + s_2$, $r = r_1 + r_2$. Go to Step 7.
Step 3. If $0 \in (a, b)$ and $a_0 \cdot b_0 \neq 0$, execute Algorithm 1 four times with the inputs $\{\mathcal{L}f^+, (0, b)\}$, $\{\mathcal{U}f^+, (0, b)\}$, $\{\mathcal{L}f^-, (a, 0)\}$, and $\{\mathcal{U}f^-, (a, 0)\}$ respectively. Let the corresponding output be $\{s_i, t_i, r_i\}$ $(1 \leqslant i \leqslant 4)$. Let

$$s = s_1 + t_2 + s_3 + t_4, \qquad t = t_1 + s_2 + t_3 + s_4, \qquad r = r_1 + r_2 + r_3 + r_4.$$

Go to Step 7.
Step 4. Compute τ such that $\mathcal{U}f^+$, $\cdot\mathcal{U}f^-$, $\mathcal{L}f^+$, $\mathcal{L}f^-$ have no roots in $(-\tau, \tau)$ except 0.
Step 5. Execute Algorithm 1 four times with the inputs $\{\mathcal{U}f^+, (\tau, b)\}$, $\{\mathcal{L}f^+, (\tau, b)\}$, $\{\mathcal{U}f^-, (a, -\tau)\}$, and $\{\mathcal{L}f^-, (a, -\tau)\}$ respectively. Let the outputs be $\{s_i, t_i, r_i\}$ $(1 \leqslant i \leqslant 4)$ in turn. Let $s = s_1 + t_2 + s_3 + t_4$, $t = t_1 + s_2 + t_3 + s_4$, $r = r_1 + r_2 + r_3 + r_4$.
Step 6. If the root 0 is a local minimum, $s := s + 1$; if it is a local maximum, $t := t + 1$; otherwise, $r := r + 1$.
Step 7. Output the number of interval zeros $s + r/2$.

We illustrate the algorithm with an example.

Example 14. Let $[f](x)$ be the interval polynomial

$$[f](x) = \left[-2, -\frac{198}{100}\right]x^6 + \left[19, \frac{1901}{100}\right]x^5 + \left[-55, -\frac{5499}{100}\right]x^4 + \left[29, \frac{2901}{100}\right]x^3 + \left[69, \frac{6902}{100}\right]x^2 + \left[-36, -\frac{3599}{100}\right]x + \left[0, \frac{505}{114}\right].$$

We constructed this example by adding $\frac{1}{100}$ or $\frac{2}{100}$ randomly to all coefficients of the polynomial $(-x + 4)(x + 1)x(2x - 1)(x - 3)^2$ except the constant coefficient. Now we count the interval zeros of $[f](x)$ in $(-6, 6)$.

Following Algorithm 2, Steps 1, 2 and 3 are omitted. In Step 4, we have

$$\begin{cases} \mathcal{U}f^+(x) = -\frac{198}{100}x^6 + \frac{1901}{100}x^5 - \frac{5499}{100}x^4 + \frac{2901}{100}x^3 + \frac{6902}{100}x^2, -\frac{3599}{100}x + \frac{505}{114}, \\ \mathcal{L}f^+(x) = -2x^6 + 19x^5 - 55x^4 + 29x^3 + 69x^2 - 36x, \\ \mathcal{U}f^-(x) = -\frac{198}{100}x^6 + 19x^5 - \frac{5499}{100}x^4 + 29x^3 + \frac{6902}{100}x^2 - 36x + \frac{505}{114}, \\ \mathcal{L}f^-(x) = -2x^6 + \frac{1901}{100}x^5 - 55x^4 + \frac{2901}{100}x^3 + 69x^2 - \frac{3599}{100}x. \end{cases}$$

We select $\tau = \frac{1}{20}$ satisfying

$$\tau < \left(1 + \frac{\frac{6902}{100}}{4}\right)^{-1}.$$

In Step 5, we execute Algorithm 1 with input $\{\mathcal{U}f^+, (\frac{1}{20}, 6)\}$ and obtain the output $\{0, 0, 3\}$. This means that $\mathcal{U}f^+$ has three real roots in $(\frac{1}{20}, 6)$ and none of them is a local minimum or a local maximum. Similarly, we execute Algorithm 1 with inputs $\{\mathcal{L}f^+, (\frac{1}{20}, 6)\}$, $\{\mathcal{U}f^-, (-6, -\frac{1}{20})\}$, and $\{\mathcal{L}f^-, (-6, -\frac{1}{20})\}$, then we obtain the outputs $\{1, 0, 2\}$, $\{0, 0, 1\}$, and $\{0, 0, 1\}$, respectively. Thus, we have $s = 1, r = 7, t = 1$. In Step 5, we find that 0 is a root and neither a local minimum nor a maximum; therefore, we output $s = 1, r = 8, t = 1$. In Step 7, we obtain the number of interval zeros of $[f](x)$ is $s + \frac{r}{2} = 5$.

In fact, these five interval zeros are $[-1.0179, -1]$, $[0, 0.24333]$, $[0.24460, 0.5]$, $[3, 3]$, and $[4, 4.4521]$, as shown in Fig. 2.

To verify the algorithm, we need to generate a set of interval polynomials. For each interval polynomial $[f]$ in the set, a polynomial f with given degree n and random coefficients range $[-t, t]$ is generated, and a random variation value in $[-\delta, \delta]$ is added to each coefficient of f to create $[f]$. Then we compute the number of interval zeros of this interval polynomial in $(-a, a)$. For a group of specified values of n, t, δ and a, we test our algorithm with ten interval polynomials generated in the described fashion. In Table 1, "Average Number" denotes the average number of interval zeros of the ten interval polynomials and "Average Time" denotes the average running time of Algorithm 2 testing for one interval polynomial. The running times are collected in Maple 14 on a PC with a 3.2G CPU and 2G memory.

Our algorithm is complete for rational coefficients. Besides arithmetic operations, the algorithm only requires computation of some Sturm sequences and their sign changes. As a result, most work is concentrated on computing

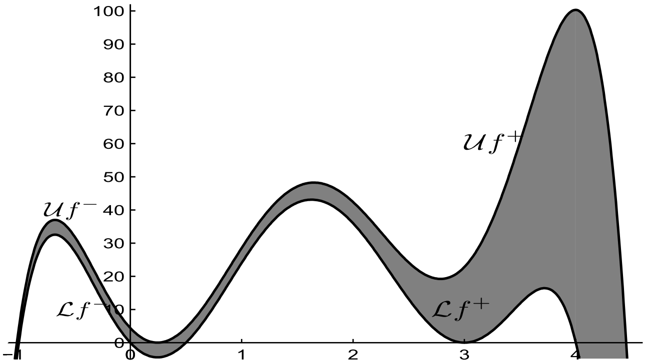

Fig. 2. Graph of the interval polynomial in Example 14.

Table 1
Computing number of interval zeros of interval polynomials.

n	t	δ	$a\,(\times 10^3)$	Average number	Average time (s)
50	100	$\frac{1}{10}$	1	3.6	0.2921
50	1000	1	10	3.4	0.2203
150	100	$\frac{1}{10}$	1	4.0	4.1844
150	1000	1	10	2.9	17.4719
300	100	$\frac{1}{10}$	10	4.7	44.0703
300	1000	1	100	4.4	232.6031

greatest common divisors. The expansion of large integers in the process may reduce the efficiency badly. When we adopt approximate computation, or in some cases, we consider interval polynomials with floating-point coefficients, an algorithm for computing approximate greatest common divisors can be used to promote the efficiency. There have been papers addressing this matter, for instance [9,10]. We also implement an analogous program for this case, citing the program of computing approximate greatest common divisors accomplished by Z. Yang. On average, the program can deal with interval floating-point polynomials with degrees less than 500 and coefficients in $(-10^3, 10^3)$ within 50 s. Computing approximate greatest common divisors usually takes more than half of the running time.

Remark 15. When we consider interval polynomials with floating-point coefficients, we cannot define the interval zeros to be absolutely correct and the precision must be introduced carefully. For example, in Example 14, when we preserve ten digits after the decimal point, we get five interval zeros, and if we preserve only two digits, we get four interval zeros, where the second and third interval zeros will be merged into one interval [0, 0.5]. This occurs because some of the end points of the interval zeros are too close to be distinguished by a few digits.

5. Conclusions

In this paper, we have given an algorithm for counting the interval zeros of univariate interval polynomials without computing the roots of any polynomials. The algorithm only requires computation of some Sturm sequences and their sign changes. We also implement the algorithm in Maple and give some experiment results. An interesting direction is to generalize the results in this paper to multivariate interval polynomials. Studying the intersection of interval polynomials and considering the corresponding Bezout theorem for interval polynomial curves is our task in the near future.

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References

[1] A. Levkovich, N. Cohen, E. Zezeb, A root-distribution criterion for an interval polynomial in a sector, Journal of Mathematical Control and Information 13 (1996) 321–333.
[2] D. Hertz, C.S. Adjiman, C.A. Floudas, Two results on bounding the roots of interval polynomials, Computers and Chemical Engineering 23 (1999) 1333–1339.
[3] J.A. Ferreira, F. Patrício, F. Oliveira, A priori estimates for the zeros of interval polynomials, Journal of Computational and Applied Mathematics 136 (2001) 271–281.

[4] J.A. Ferreira, F. Patrício, F. Oliveira, On the computation of solutions of systems of interval polynomial equations, Journal of Computational and Applied Mathematics 173 (2005) 295–302.
[5] X. Fan, J. Deng, F. Chen, Zeros of univariate interval polynomials, Journal of Computational and Applied Mathematics 216 (2) (2008) 563–573.
[6] C. Yap, Fundamental Problems in Algorithmic Algebra, Oxford University Press, 2000.
[7] J.J. Sylvester, On a theory of the syzegetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure, Philosophical Transactions of the Royal Society of London 143 (1853) 407–548.
[8] A.L. Cauchy, Exercices de mathématiques, IV Année de Bure Fréres, 1829.
[9] P. Chin, R.M. Corless, G.F. Corliss, Optimization strategies for the approximate GCD problem, in: Proceedings of the 1998 International Symposium on Symbolic and Algebraic Computation, 1998, pp. 228–235.
[10] R.M. Corless, S.M. Watt, L.H. Zhi, QR factoring to compute the GCD of univariate approximate polynomials, IEEE Transactions on Signal Processing 52 (12) (2004) 3394–3402.