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STRICT TOPOLOGIES AND (DF)-SPACES *

J. Zafarani **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The topology β_ρ has been introduced and studied on $C_b(X,E)$, the space of all bounded continuous E-valued functions in [4]. In this paper we give a necessary and sufficient condition for the space $C_b(X,E)$ equipped with β_ρ to be a (DF)-space, and as a corollary we obtain directly a result of Mendoza for the space $C(X,E)$ equipped with τ_ρ to be a (DF)-space (cf. Theorem I.V.9.2 of [2]).

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** Permanent address: Department of Mathematics, University of Isfahan, Isfahan, Iran.

1. Notation and terminology . Let X be a completely regular and Hausdorff space and let E be a locally convex Hausdorff space. We denote by $C_b(X,E)$ the space of all bounded continuous functions with values in E ,and when E coincides with $\mathbb{R}$ or $\mathbb{C}$, we denote $C_b(X,E)$ by $C_b(X)$. We designate by cs(E) the set of all continuous semi-norms on E , by $\kappa(X)$ the set of all compact subsets of X and by $\mathcal{P}$ a directed family of elements of $\kappa(X)$ which covers X. We denote by $\mathcal{B}$ the set of all bounded functions $v:X \longrightarrow \mathbb{R}^+$ which $\mathcal{P}$-vanish at infinity (i.e. for every $\varepsilon>0$,there is $K\in\mathcal{P}$ such that $\|v\|_{X/K}<\varepsilon$). The uniform topology on X (resp. on the elements of $\mathcal{P}$) on $C_b(X,E)$ is denoted by τ_u(resp. $\tau_{\mathcal{P}}$).

The strict topology $\beta_{\mathcal{P}}$ on $C_b(X,E)$ has been defined in [4] by the system of the semi-norms

$$\{ \|\cdot\|_{v,p} : v\in\mathcal{B},\ p\in cs(E) \}$$

where

$$\|f\|_{v,p} = \sup(\ v(x)p(f(x)) : x\in X\ )\ ,\ f\in C_b(X,E)\ .$$

It has been shown in [4] that the spaces $(C_b(X,E),\tau_u)$ and $(C_b(X,E),\beta_{\mathcal{P}})$ have the same bounded subsets and that on these bounded subsets ,the toplogy $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide .

We use the following notation : If A is a subset of X and B is an absolutely convex subset of E ,then

(i) $C_b(A,B) := \{ f\in C_b(X,E) : f(A)\subseteq B \}$

(ii) $P(X,B)$ is the set of elements of the type $\sum_{j=1}^{J} f_j \otimes e_j$ where $J\in\mathbb{N}$, $\{ f_1,\dots,f_J\}$ is a finite continuous partition of unity on X ,and $e_1,\dots,e_J$ belong to B.

2. $(C_b(X,E),\ \beta_{\mathcal{P}})$ spaces of the type (DF) . A locally convex E is a (DF) -space if E has a fundamental sequence of bounded sets and every bornivorous subset of E which is the intersection of a sequence of closed absolutely convex zero-neighbourhoods in E is a zero-neighbourhood in E .

PROPOSITION. If E has a fundamental sequence of bounded sets and V is a bornivorous barrel in $(C_b(X,E), \tau_{\mathcal{P}})$, then there is a bornivorous barrel set W in E such that $C_b(X,W) \subseteq V$.

Proof. Since $(C_b(X,E), \tau_{\mathcal{P}})$ is a subspace of $(C(X,E), \tau_{\mathcal{P}})$ and the equality (c) of the proposition I.6.4 in [2] is also valid in the case $(C_b(X,E),\tau_{\mathcal{P}})$ for an arbitrary space X, then the proof of the proposition I.6.5 in [2] can be adapted to this case.

THEOREM. $(C_b(X,E),\beta_{\mathcal{P}})$ is a (DF)-space if and only if E is a (DF)-space and every countable union elements of $\mathcal{P}$ is a subset of an element of $\mathcal{P}$.

In order to prove this theorem, we establish the following lemma first:

LEMMA. For a compact subset K of X, any arbitrary absolutely convex subset B of E, and any closed absolutely convex zero-neighbourhood V in E, one has

$$C_b(K,\bar{B}) \subseteq P(K,B) + C_b(K,V).$$

Proof. By a lemma in [3], we have

$$C_b(K,\bar{B}) \subseteq P(K,\bar{B}) + C_b(K,\tfrac{1}{2}V)$$

and for any $e\in\bar{B}$, there is $e'\in B$ such that $e\in e'+\frac{1}{2}V$. Therefore

$$P(K,\bar{B}) \subseteq P(K,B+\tfrac{1}{2}V) \subseteq P(K,B)+P(K,\tfrac{1}{2}V),$$

consequently

$$C_b(K,\bar{B}) \subseteq P(K,B)+P(K,\tfrac{1}{2}V)+C_b(K,\tfrac{1}{2}V)$$

$$\subseteq P(K,B)+C_b(K,V).$$

Proof of theorem. The condition in necessary. By lemma 3.5 of [4], $(C_b(X),\beta_{\mathcal{P}})$ is a (DF)-space. Now by a result of [1], $(C_b(X),\beta_{\mathcal{P}})$ is a (DF)-space if and only if the union of every sequence of elements of $\mathcal{P}$ is a subset of an element of $\mathcal{P}$.

The condition is sufficient. By proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$. Now, let $(B_n)_{n\in\mathbb{N}}$ be a fundamental sequence of absolutely convex bounded subsets of E, then obviously $(C_b(X,B_n))_{n\in\mathbb{N}}$ is a fundamental sequence of

absolutely convex bounded subsets of $(C_b(X,E),\tau_u)$. Therefore by proposition 3.3 of [4] this sequence is also a fundamental family of bounded sets for $(C_b(X,E),\tau_{\mathcal{P}})$. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of closed absolutely convex zero-neighbourhoods in $(C_b(X,E),\tau_{\mathcal{P}})$, such that $V=\bigcap_{n\in\mathbb{N}} V_n$ is a bornivorous in $(C_b(X,E),\tau_{\mathcal{P}})$. For each $n\in\mathbb{N}$, there is then a zero-neighbourhood U_n in E and $K_n\in\mathcal{P}$ such that $C_b(K_n,U_n)\subseteq V_n$. Since V is a bornivorous barrel set, by the above proposition there is a bornivorous barrel set W in E such that $C_b(X,W)\subseteq V$. Since E is a (DF)-space, $U:=\bigcap_{n\in\mathbb{N}}[\frac{1}{4}(\overline{W+U_n})]$ is a zero-neighbourhood in E. Now, let $K\in\mathcal{P}$ such that $\bigcup_{n\in\mathbb{N}}K_n\subseteq K$. It follows from the above lemma that

$$C_b(K,U) \subseteq \bigcap_{n\in\mathbb{N}} C_b(K,[\tfrac{1}{4}(\overline{W+U_n})])$$

$$\subseteq \bigcap_{n\in\mathbb{N}} [P(K,\tfrac{1}{4}(W+U_n))+C_b(K,\tfrac{1}{4}U_n)]$$

$$\bigcap_{n\in\mathbb{N}} [P(K,\tfrac{1}{4}W)+P(K,\tfrac{1}{4}U_n)+C_b(K,\tfrac{1}{4}U_n)]$$

$$\subseteq \bigcap_{n\in\mathbb{N}} [\tfrac{1}{2}C_b(K,W)+\tfrac{1}{2}C_b(K,U_n)]$$

$$\subseteq \bigcap_{n\in\mathbb{N}} [\tfrac{1}{2}V+\tfrac{1}{2}V_n] \subseteq \bigcap_{n\in\mathbb{N}} V_n = V.$$

This complete the proof.

Remarks (a). Since by proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$ if and only if every countable union of elements of $\mathcal{P}$ is contained in an element of $\mathcal{P}$, (and this implies that X is pseudo-compact), in fact we have shown that $(C(X,E),\tau_{\mathcal{P}})$ is a (DF)-space. (cf. theorem I.V.4.2 of [2]).

(b). When $\mathcal{P}$ is the family $\kappa(X)$ (resp. $\mathcal{A}(X)$) of all compact (resp. finite) subsets of X, the $\beta_{\mathcal{P}}$ is the substrict (resp. weak-strict) topology β_0 (resp. δ) on $C_b(X,E)$ (cf. [4]). Then in these cases we have obtained a necessary and sufficient condition for the space $C_b(X,E)$ equipped with the topology β_0 (resp. δ) to be a (DF)-space.

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