

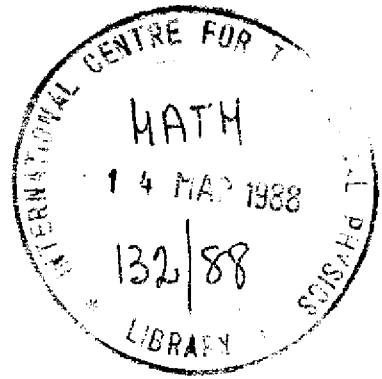
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STRICT TOPOLOGIES AND (DF)-SPACES *

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ABSTRACT

The topology β_p has been introduced and studied on $C_b(X, E)$, the space of all bounded continuous E -valued functions in [4]. In this paper we give a necessary and sufficient condition for the space $C_b(X, E)$ equipped with β_p to be a (DF)-space, and as a corollary we obtain directly a result of Mendoza for the space $C(X, E)$ equipped with τ_p to be a (DF)-space (cf. Theorem I.V.9.2 of [2]).

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1. Notation and terminology . Let X be a completely regular and Hausdorff space and let E be a locally convex Hausdorff space. We denote by $C_b(X, E)$ the space of all bounded continuous functions with values in E , and when E coincides with \mathbb{R} or \mathbb{C} , we denote $C_b(X, E)$ by $C_b(X)$. We designate by $cs(E)$ the set of all continuous semi-norms on E , by $\kappa(X)$ the set of all compact subsets of X and by \mathcal{P} a directed family of elements of $\kappa(X)$ which covers X . We denote by \mathcal{B} the set of all bounded functions $v: X \rightarrow \mathbb{R}^+$ which \mathcal{P} -vanish at infinity (i.e. for every $\epsilon > 0$, there is $K \in \mathcal{P}$ such that $\|v\|_{X/K} < \epsilon$). The uniform topology on X (resp. on the elements of \mathcal{P}) on $C_b(X, E)$ is denoted by τ_u (resp. $\tau_{\mathcal{P}}$).

The strict topology $\beta_{\mathcal{P}}$ on $C_b(X, E)$ has been defined in [4] by the system of the semi-norms

$$\{ \| \cdot \|_{v,p} : v \in \mathcal{B}, p \in cs(E) \}$$

where

$$\| f \|_{v,p} = \sup \{ v(x)p(f(x)) : x \in X \}, f \in C_b(X, E).$$

It has been shown in [4] that the spaces $(C_b(X, E), \tau_u)$ and $(C_b(X, E), \beta_{\mathcal{P}})$ have the same bounded subsets and that on these bounded subsets, the topology $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide.

We use the following notation: If A is a subset of X and B is an absolutely convex subset of E , then

$$(i) C_b(A, B) = \{ f \in C_b(X, E) : f(A) \subseteq B \}$$

(ii) $P(X, B)$ is the set of elements of the type $\sum_{j=1}^J f_j a_j$ where $J \in \mathbb{N}$, $\{ f_1, \dots, f_J \}$ is a finite continuous partition of unity on X , and a_1, \dots, a_J belong to B .

2. $(C_b(X, E), \beta_{\mathcal{P}})$ spaces of the type (DF) . A locally convex E is a (DF) - space if E has a fundamental sequence of bounded sets and every bornivorous subset of E which is the intersection of a sequence of closed absolutely convex zero-neighbourhoods in E is a zero-neighbourhood in E .

PROPOSITION. If E has a fundamental sequence of bounded sets and V is a bornivorous barrel in $(C_b(X, E), \tau_{\mathcal{P}})$, then there is a bornivorous barrel set W in E such that $C_b(X, W) \subseteq V$.

Proof. Since $(C_b(X, E), \tau_{\mathcal{P}})$ is a subspace of $(C(X, E), \tau_{\mathcal{P}})$ and the equality (c) of the proposition I.6.4 in [2] is also valid in the case $(C_b(X, E), \tau_{\mathcal{P}})$ for an arbitrary space X , then the proof of the proposition I.6.5 in [2] can be adapted to this case.

THEOREM. $(C_b(X, E), \beta_{\mathcal{P}})$ is a (DF)-space if and only if E is a (DF)-space and every countable union elements of \mathcal{P} is a subset of an element of \mathcal{P} .

In order to prove this theorem, we establish the following lemma first:

LEMMA. For a compact subset K of X , any arbitrary absolutely convex subset B of E , and any closed absolutely convex zero-neighbourhood V in E , one has

$$C_b(K, \bar{B}) \subseteq P(K, B) + C_b(K, V).$$

Proof. By a lemma in [3], we have

$$C_b(K, \bar{B}) \subseteq P(K, \bar{B}) + C_b(K, \frac{1}{2}V)$$

and for any $e \in \bar{B}$, there is $e' \in B$ such that $e \in e' + \frac{1}{2}V$. Therefore

$$P(K, \bar{B}) \subseteq P(K, B + \frac{1}{2}V) \subseteq P(K, B) + P(K, \frac{1}{2}V),$$

consequently

$$\begin{aligned} C_b(K, \bar{B}) &\subseteq P(K, B) + P(K, \frac{1}{2}V) + C_b(K, \frac{1}{2}V) \\ &\subseteq P(K, B) + C_b(K, V). \end{aligned}$$

Proof of theorem. The condition is necessary. By lemma 3.5 of [4], $(C_b(X), \beta_{\mathcal{P}})$ is a (DF)-space. Now by a result of [1], $(C_b(X), \beta_{\mathcal{P}})$ is a (DF)-space if and only if the union of every sequence of elements of \mathcal{P} is a subset of an element of \mathcal{P} .

The condition is sufficient. By proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$. Now, let $(B_n)_{n \in \mathbb{N}}$ be a fundamental sequence of absolutely convex bounded subsets of E , then obviously $(C_b(X, B_n))_{n \in \mathbb{N}}$ is a fundamental sequence of

absolutely convex bounded subsets of $(C_b(X, E), \tau_U)$. Therefore by proposition 3.3 of [4] this sequence is also a fundamental family of bounded sets for $(C_b(X, E), \tau_{\mathcal{P}})$. Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of closed absolutely convex zero-neighbourhoods in $(C_b(X, E), \tau_{\mathcal{P}})$, such that $V = \bigcap_{n \in \mathbb{N}} V_n$ is a bornivorous in $(C_b(X, E), \tau_{\mathcal{P}})$. For each $n \in \mathbb{N}$, there is then a zero-neighbourhood U_n in E and $K_n \in \mathcal{P}$ such that $C_b(K_n, U_n) \subseteq V_n$. Since V is a bornivorous barrel set, by the above proposition there is a bornivorous barrel set W in E such that $C_b(X, W) \subseteq V$. Since E is a (DF)-space, $U := \bigcap_{n \in \mathbb{N}} [\frac{1}{2}(W + U_n)]$ is a zero-neighbourhood in E . Now, let $K \in \mathcal{P}$ such that $\bigcup_{n \in \mathbb{N}} K_n \subseteq K$. It follows from the above lemma that

$$\begin{aligned} C_b(K, U) &\subseteq \bigcap_{n \in \mathbb{N}} C_b(K, [\frac{1}{2}(W + U_n)]) \\ &\subseteq \bigcap_{n \in \mathbb{N}} [P(K, \frac{1}{2}(W + U_n)) + C_b(K, \frac{1}{2}U_n)] \\ &\quad \bigcap_{n \in \mathbb{N}} [P(K, \frac{1}{2}W) + P(K, \frac{1}{2}U_n) + C_b(K, \frac{1}{2}U_n)] \\ &\subseteq \bigcap_{n \in \mathbb{N}} [\frac{1}{2}C_b(K, W) + \frac{1}{2}C_b(K, U_n)] \\ &\subseteq \bigcap_{n \in \mathbb{N}} [\frac{1}{2}V + \frac{1}{2}V_n] \subseteq \bigcap_{n \in \mathbb{N}} V_n = V. \end{aligned}$$

This complete the proof.

Remarks (a). Since by proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$ if and only if every countable union of elements of \mathcal{P} is contained in an element of \mathcal{P} , (and this implies that X is pseudo-compact), in fact we have shown that $(C(X, E), \tau_{\mathcal{P}})$ is a (DF)-space (cf. theorem I.V.4.2 of [2]).

(b). When \mathcal{P} is the family $\kappa(X)$ (resp. $\mathcal{A}(X)$) of all compact (resp. finite) subsets of X , the $\beta_{\mathcal{P}}$ is the substrict (resp. weak-strict) topology β_0 (resp. δ) on $C_b(X, E)$ (cf. [4]). Then in these cases we have obtained a necessary and sufficient condition for the space $C_b(X, E)$ equipped with the topology β_0 (resp. δ) to be a (DF)-space.

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