

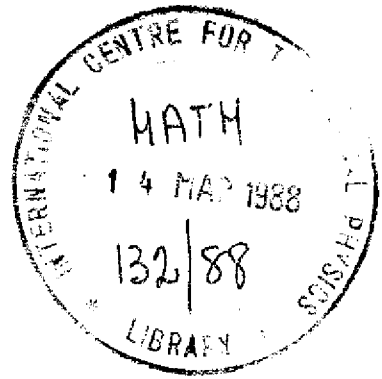
See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/280769322>

df space strict

Dataset · August 2015

READS

14



INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

STRICT TOPOLOGIES AND (DF) -SPACES

J. Zafarani



**INTERNATIONAL
ATOMIC ENERGY
AGENCY**



**UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION**



International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

STRICT TOPOLOGIES AND (DF)-SPACES *

J. Zafarani **

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The topology β_p has been introduced and studied on $C_b(X, E)$, the space of all bounded continuous E -valued functions in [4]. In this paper we give a necessary and sufficient condition for the space $C_b(X, E)$ equipped with β_p to be a (DF)-space, and as a corollary we obtain directly a result of Mendoza for the space $C(X, E)$ equipped with τ_p to be a (DF)-space (cf. Theorem I.V.9.2 of [2]).

MIRAMARE - TRIESTE

February 1988

* Submitted for publication.

** Permanent address: Department of Mathematics, University of Isfahan, Isfahan, Iran.

1. Notation and terminology. Let X be a completely regular and Hausdorff space and let E be a locally convex Hausdorff space. We denote by $C_b(X, E)$ the space of all bounded continuous functions with values in E , and when E coincides with \mathbb{R} or \mathbb{C} , we denote $C_b(X, E)$ by $C_b(X)$. We designate by $cs(E)$ the set of all continuous semi-norms on E , by $\kappa(X)$ the set of all compact subsets of X and by \mathcal{P} a directed family of elements of $\kappa(X)$ which covers X . We denote by \mathcal{B} the set of all bounded functions $v: X \rightarrow \mathbb{R}^+$ which \mathcal{P} -vanish at infinity (i.e. for every $\epsilon > 0$, there is $K \in \mathcal{P}$ such that $\|v\|_{X/K} < \epsilon$). The uniform topology on X (resp. on the elements of \mathcal{P}) on $C_b(X, E)$ is denoted by τ_u (resp. $\tau_{\mathcal{P}}$).

The strict topology $\beta_{\mathcal{P}}$ on $C_b(X, E)$ has been defined in [4] by the system of the semi-norms

$$\{ \| \cdot \|_{v,p} : v \in \mathcal{B}, p \in cs(E) \}$$

where

$$\| f \|_{v,p} = \sup \{ v(x)p(f(x)) : x \in X \}, f \in C_b(X, E).$$

It has been shown in [4] that the spaces $(C_b(X, E), \tau_u)$ and $(C_b(X, E), \beta_{\mathcal{P}})$ have the same bounded subsets and that on these bounded subsets, the topology $\tau_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ coincide.

We use the following notation: If A is a subset of X and B is an absolutely convex subset of E , then

$$(i) C_b(A, B) = \{ f \in C_b(X, E) : f(A) \subseteq B \}$$

(ii) $P(X, B)$ is the set of elements of the type $\sum_{j=1}^J f_j a_j$ where $J \in \mathbb{N}$, $\{ f_1, \dots, f_J \}$ is a finite continuous partition of unity on X , and a_1, \dots, a_J belong to B .

2. $(C_b(X, E), \beta_{\mathcal{P}})$ spaces of the type (DF). A locally convex E is a (DF)-space if E has a fundamental sequence of bounded sets and every bornivorous subset of E which is the intersection of a sequence of closed absolutely convex zero-neighbourhoods in E is a zero-neighbourhood in E .

PROPOSITION. If E has a fundamental sequence of bounded sets and V is a bornivorous barrel in $(C_b(X, E), \tau_{\mathcal{P}})$, then there is a bornivorous barrel set W in E such that $C_b(X, W) \subseteq V$.

Proof. Since $(C_b(X, E), \tau_{\mathcal{P}})$ is a subspace of $(C(X, E), \tau_{\mathcal{P}})$ and the equality (c) of the proposition I.6.4 in [2] is also valid in the case $(C_b(X, E), \tau_{\mathcal{P}})$ for an arbitrary space X , then the proof of the proposition I.6.5 in [2] can be adapted to this case.

THEOREM. $(C_b(X, E), \beta_{\mathcal{P}})$ is a (DF)-space if and only if E is a (DF)-space and every countable union elements of \mathcal{P} is a subset of an element of \mathcal{P} .

In order to prove this theorem, we establish the following lemma first:

LEMMA. For a compact subset K of X , any arbitrary absolutely convex subset B of E , and any closed absolutely convex zero-neighbourhood V in E , one has

$$C_b(K, \bar{B}) \subseteq P(K, B) + C_b(K, V).$$

Proof. By a lemma in [3], we have

$$C_b(K, \bar{B}) \subseteq P(K, \bar{B}) + C_b(K, \frac{1}{2}V)$$

and for any $e \in \bar{B}$, there is $e' \in B$ such that $e \in e' + \frac{1}{2}V$. Therefore

$$P(K, \bar{B}) \subseteq P(K, B + \frac{1}{2}V) \subseteq P(K, B) + P(K, \frac{1}{2}V),$$

consequently

$$\begin{aligned} C_b(K, \bar{B}) &\subseteq P(K, B) + P(K, \frac{1}{2}V) + C_b(K, \frac{1}{2}V) \\ &\subseteq P(K, B) + C_b(K, V). \end{aligned}$$

Proof of theorem. The condition is necessary. By lemma 3.5 of [4], $(C_b(X), \beta_{\mathcal{P}})$ is a (DF)-space. Now by a result of [1], $(C_b(X), \beta_{\mathcal{P}})$ is a (DF)-space if and only if the union of every sequence of elements of \mathcal{P} is a subset of an element of \mathcal{P} .

The condition is sufficient. By proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$. Now, let $(B_n)_{n \in \mathbb{N}}$ be a fundamental sequence of absolutely convex bounded subsets of E , then obviously $(C_b(X, B_n))_{n \in \mathbb{N}}$ is a fundamental sequence of

absolutely convex bounded subsets of $(C_b(X, E), \tau_U)$. Therefore by proposition 3.3 of [4] this sequence is also a fundamental family of bounded sets for $(C_b(X, E), \tau_{\mathcal{P}})$. Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of closed absolutely convex zero-neighbourhoods in $(C_b(X, E), \tau_{\mathcal{P}})$, such that $V = \bigcap_{n \in \mathbb{N}} V_n$ is a bornivorous in $(C_b(X, E), \tau_{\mathcal{P}})$. For each $n \in \mathbb{N}$, there is then a zero-neighbourhood U_n in E and $K_n \in \mathcal{P}$ such that $C_b(K_n, U_n) \subseteq V_n$. Since V is a bornivorous barrel set, by the above proposition there is a bornivorous barrel set W in E such that $C_b(X, W) \subseteq V$. Since E is a (DF)-space, $U := \bigcap_{n \in \mathbb{N}} [\frac{1}{2}(W + U_n)]$ is a zero-neighbourhood in E . Now, let $K \in \mathcal{P}$ such that $\bigcup_{n \in \mathbb{N}} K_n \subseteq K$. It follows from the above lemma that

$$\begin{aligned} C_b(K, U) &\subseteq \bigcap_{n \in \mathbb{N}} C_b(K, [\frac{1}{2}(W + U_n)]) \\ &\subseteq \bigcap_{n \in \mathbb{N}} [P(K, \frac{1}{2}(W + U_n)) + C_b(K, \frac{1}{2}U_n)] \\ &\quad \bigcap_{n \in \mathbb{N}} [P(K, \frac{1}{2}W) + P(K, \frac{1}{2}U_n) + C_b(K, \frac{1}{2}U_n)] \\ &\subseteq \bigcap_{n \in \mathbb{N}} [\frac{1}{2}C_b(K, W) + \frac{1}{2}C_b(K, U_n)] \\ &\subseteq \bigcap_{n \in \mathbb{N}} [\frac{1}{2}V + \frac{1}{2}V_n] \subseteq \bigcap_{n \in \mathbb{N}} V_n = V. \end{aligned}$$

This complete the proof.

Remarks (a). Since by proposition 3.1 of [4], $\beta_{\mathcal{P}}$ coincides with $\tau_{\mathcal{P}}$ if and only if every countable union of elements of \mathcal{P} is contained in an element of \mathcal{P} , (and this implies that X is pseudo-compact), in fact we have shown that $(C(X, E), \tau_{\mathcal{P}})$ is a (DF)-space (cf. theorem I.V.4.2 of [2]).

(b). When \mathcal{P} is the family $\kappa(X)$ (resp. $\mathcal{A}(X)$) of all compact (resp. finite) subsets of X , the $\beta_{\mathcal{P}}$ is the substrict (resp. weak-strict) topology β_0 (resp. δ) on $C_b(X, E)$ (cf. [4]). Then in these cases we have obtained a necessary and sufficient condition for the space $C_b(X, E)$ equipped with the topology β_0 (resp. δ) to be a (DF)-space.

ACKNOWLEDGEMENTS

The author is grateful to Professor J. Schmets for useful suggestions. He would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

- [1] K.Noureddin ,Note sur les espaces D_b ,Math. Ann. 219(1976),97-103.
- [2] J.Schmets , Space of vector-valued continuous functions,L.N.M. 1003, (1983),Springer Verlag.
- [3] J.Schmets and J.Zafarani, Strict topologies and (gDF)-spaces,Arch.Math. 49(1987), 227-231.
- [4] J.Zafarani , Locally convex topologies on spaces of vector-valued continuous functions ,Bull.Soc.Roy.Liege,55(1986), 353-362.

