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Article *in* Mathematatische Nachrichten · January 1995

Impact Factor: 0.68 · DOI: 10.1002/mana.19951740123

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Math. Nachr. 174 (1995) 317–322

Grothendieck Spaces of Compact Operators

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(Received February 2, 1994)
(Revised Version June 14, 1994)

Abstract. In this paper we study the Grothendieck spaces among the operator spaces $L_e(E'_c, F)$. Conditions under which $L_e(E'_c, F)$ contains complemented copy of c_0 are given. We apply these results to spaces of the type $C_b(X; F)$ endowed with strict topologies.

1. Introduction

Let E and F be two locally convex spaces. The ε-product $E\varepsilon F$ is the space $L_e(E'_c, F)$ of all $\sigma(E', E) - \sigma(F, F')$ continuous linear operators from E' into F which transform equicontinuous subsets of E' into relatively compact subsets of F, endowed with the topology of uniform convergence on the equicontinuous subsets of E' [20]. It is well known that when E and F are complete locally convex spaces, then $E \widehat{\otimes}_\varepsilon F \hookrightarrow L_e(E'_c, F)$. Recently various properties of $L_e(E'_c, F)$ have been widely studied, see for example [3, 4, 6, 7, 12, 17], and the reader is referred to the survey article [17] for the basic information about this important space of operators. FRENICHE [11], has extended the definition of Grothendieck space to the locally convex setting. A locally convex space E is said to be Grothendieck if the $\sigma(E', E)$ and $\sigma(E', E'')$ sequential convergences coincide on equicontinuous subsets of E'.

We now recall a few definitions. If X is a Hausdorff completely regular space, then $C(X; F)$ denotes the vector space of the continuous functions on X with values in F, and $C_b(X; F)$ denotes the vector subspace of bounded continuous functions. If F is the scalar field, we simply write $C(X)$ and $C_b(X)$, respectively. Let $\mathscr{P}$ be a directed family of compact subsets of X which covers X. Then we denote by $\mathscr{B}$ the set of all bounded functions $v: X \to R^+$ which $\mathscr{P}$-vanish at infinity (i.e., for every $\varepsilon > 0$, there is a $K \in \mathscr{P}$ such that $\|v\|_{X \setminus K} \leq \varepsilon$). The notation τ_u(resp. $\tau_{\mathscr{P}}$) stands for the locally convex topology on $C_b(X; F)$ of uniform convergence on X(resp. on the elements of $\mathscr{P}$). The strict topology $\beta_{\mathscr{P}}$ is defined in [21] as the locally convex topology on $C_b(X; F)$ described by the system of semi-norms $\{\|\cdot\|_{v,p}: p \in cs(F), v \in \mathscr{B}\}$, where

$$\|f\|_{v,p} = \sup\{v(x)\, p(f(x)): x \in X\}$$

for each f in $C_b(X; F)$.

It is well known that the space $(C_b(X; F), \tau_u)$ and the space $(C_b(X; F), \beta_{\mathscr{P}})$ have the same bounded subsets and that on the bounded sets, the topologies $\tau_{\mathscr{P}}$ and $\beta_{\mathscr{P}}$ coincide. If A is a subset of X and B is an absolutely convex subset of F, then

$$C(A, B) := \{f \in C(X; F) : f(A) \subset B\}.$$

A topological space X is angelic [9], if the following two conditions are satisfied by any relatively countably compact subset K of X: (1) $\overline{K}$ is compact; (2) every $x \in \overline{K}$ is the limit of some sequence in K. A locally convex space E is c_0-barrelled if every $\sigma(E', E)$-null sequence in E' is equicontinuous, and E is a Mazur space if sequentially continuous elements in the algebraic dual E^* of E are continuous, i.e., in E'.

2. Main results

The following characterization of the Grothendieck spaces which are c_0-barrelled in terms of linear continuous operators and its proof were inspired by Theorem 3.1 in [8].

Theorem 2.1. *For a c_0-barrelled space E, the following are equivalent:*

(i) *E is a Grothendieck space.*

(ii) *For each locally convex space F for which $(F', \sigma(F', F))$ is a Mazur space, every continuous linear operator $T: E \to F$ carries bounded sets into weakly relatively compact sets.*

(iii) *For each complete locally convex space F with a countable family $\{K_n : n \in \mathbb{N}\}$ of relatively $\sigma(F, F')$-countably compact sets such that $\bigcup_{n=1}^{\infty} K_n$ is $\sigma(F, F')$-dense in F, every continuous linear operator $T: E \to F$ carries bounded sets into weakly relatively compact sets.*

(iv) *Every continuous linear operator $T: E \to c_0$ carries bounded sets into weakly relatively compact sets.*

Proof. (i) $\Rightarrow$ (ii): By the result 42.2.1 of [15], it is enough to show that $T''(E'') \subset F$. For every sequence (f'_n) in F' which $\sigma(F', F)$ converges to zero, $(T'(f'_n))$ is $\sigma(E', E)$-convergent to zero. Since E is c_0-barrelled, it follows that $\{T'(f'_n)\}$ is an equicontinuous subset of E' and consequently $(T'(f'_n))$ is $\sigma(E', E'')$-convergent to zero. For each $e'' \in E''$ we have

$$(T''(e''))(f'_n) = e''(T'(f'_n)) \to 0.$$

Thus, $T''(e'')$ is $\sigma(F', F)$ sequentially continuous and our assumption on F implies that $T''(e'') \in F$.

(ii) $\Rightarrow$ (iii): By the Eberlein-Smulian Theorem 3.10.5 of [9], one can show that $(F', \sigma(F', F))$ in an angelic space. In particular, every equicontinuous subset of F' is relatively $\sigma(F', F)$-sequentially compact. Therefore by Theorem 1.5 of [8] it follows that $(F', \sigma(F', F))$ is a Mazur space.

(iii) $\Rightarrow$ (iv): The proof of it is trivial.

(iv) $\Rightarrow$ (i): If (e'_n) is a $\sigma(E', E)$-null sequence, then by our hypothesis on E, $\{e'_n\}$ is equicontinuous and $T: E \to c_0$ defined by $T(e) = (e'_n(e))$ is a continuous linear operator. It follows from 42.2.1 of [15] that $T''(E'') \subset c_0$. This implies that $T''(e'') = (e''(e'_n)) \in c_0$ for each $e'' \in E''$. Thus the sequence (e'_n), $\sigma(E', E'')$-converges to zero, and E is a Grothendieck space. □

We denote by Φ_0, the subspace of c_0 whose elements have only finitely many non-zero coordinates. Let E_1 be a subspace of E so that there exists a topological isomorphism $T: E_1 \to \Phi_0$. We identify l^1 with the dual space Φ_0' and define

$$e_n = T^{-1}(u_n), \quad e_n' = T'(u_n'),$$

where (u_n) and (u_n') are the standard Schauder bases of c_0 and l^1 respectively. If B is the closed unit ball of Φ_0, then there exists an absolutely convex zero-neighbourhood U_1 in E such that $U_1 \cap E_1 = T^{-1}(B)$. If p is the gauge of U_1, then $|e_n'(e)| \leq p(e)$ for every $e \in E$ and every $n \in \mathbf{N}$. The following theorem is related to a result of FRENICHE [10]. Here we give a characterization of the $\sigma(F', F)$-null sequences in F' when $L_e(E_c', F)$ is a Grothendieck space.

Theorem 2.2. *Let E and F be two locally convex spaces and suppose that E contains a subspace topologically isomorphic to Φ_0. If $L_e(E_c', F)$ is a Grothendieck space, then the $\sigma(F', F)$ and $\beta(F', F)$ sequential convergence coincide in the equicontinuous subsets of F'.*

Proof. We preserve the notation introduced in the above paragraph and proceed by contradiction. If this is not the case, there exist a zero-neighborhood V_1 in F and a sequence (f_n') in V_1^0 such that (f_n') is $\sigma(F', F)$-null but it is not $\beta(F', F)$-null. Thus there exist $\alpha > 0$ and a bounded sequence (f_n) in F such that $|f_n'(f_n)| \geq \alpha$ for every n (by passing to a subsequence if necessary). Let $H = L_e(E_c', F)$ and let $h_n' = e_n' \otimes f_n' \in H'$, where $h_n'(h) = \langle h(e_n'), f_n' \rangle$ for every $h \in H, n \geq 1$; we now show that the sequence (h_n') is $\sigma(H', H)$-null. But (e_n') is in U_1^0 and therefore $h(e_n')$ is a relatively compact subset of F. Thus $h_n'(h)$ tends to zero for each $h \in H$; $\{h_n'\}$ is an equicontinuous subset of H' [Proposition 2.1 of [3]]. Since Σe_n is a weakly unconditionally Cauchy series and (f_n) is a bounded sequence in F, by Theorem 2.4 of [3] the series $\Sigma e_n \otimes f_n$ is weakly unconditionally Cauchy in H. Thus it converges to an element $h'' \in H''$. Since

$$|h''(h_n')| = |h_n'(\Sigma e_n \otimes f_n)| = |f_n'(f_n)| \geq \alpha$$

for every n, the sequence (h_n') is not $\sigma(H', H'')$-null, and this is a contradiction. □

Remarks.

(a). The above theorem is also valid for $L_e(F_c', E)$. In fact, the map $h \to h'$ is a topological isomorphism of $L_e(E_c', F)$ onto $L_e(F_c', E)$.

(b). If F is a Fréchet space, then by the Josefson-Nissenzweig theorem for Fréchet spaces [BONET-LINDSTRÖM-VALDIVIA [1]], $\sigma(F', F)$ and $\beta(F', F)$ converging sequences coincide if and only if F is a Montel space, and we have the following corollary.

Corollary 2.1. *Let E be a locally convex space containing a subspace isomorphic to c_0 and let F be a Fréchet space. If $L_e(E_c', F)$ is a Grothendieck space, then F is a Montel space.*

(c). When F is an infinite dimensional Banach space, the above corollary shows that $L_e(E_c', F)$ is not a Grothendieck space.

CEMBRANOS [2] has shown that if K is an infinite compact Hausdorff space and if E is an infinite dimensional Banach space, then $C(K; E)$ contains a complemented copy of c_0. This result has been generalized to the injective tensor products by SAAB and SAAB [18]: If

E and F are two infinite dimensional Banach spaces and E contains a subspace isomorphic to c_0, then $E \widehat{\otimes}_\varepsilon F$ contains a complemented subspace isomorphic to c_0. FRENICHE [10] has independently proven that if a locally convex space E has an isomorphic subspace to Φ_0 and F is an infinite dimensional normed space, then $E \widehat{\otimes}_\varepsilon F$ contains a complemented subspace isomorphic to c_0. Recently RYAN [16] has obtained a simple proof of SAAB and SAAB's result and has shown that a c_0 copy in $E \widehat{\otimes}_\varepsilon F$ is in fact complemented in $K(E', F)$, where $K(E', F)$ is the space of all compact operator from E' into F. The following theorem is an analogous result for the space $L_e(E'_c, F)$.

Theorem 2.3 *If E is a complete locally convex Hausdorff space containing a copy E_1 of c_0 and if F is an infinite dimensional Banach space, then $E \widehat{\otimes}_\varepsilon F$ has a copy of c_0 which is complemented in $L_e(E'_c, F)$.*

Proof. By the Josefson-Nissenzweig theorem [5, page 219], there exists a sequence (f'_n) in F' such that (f'_n) is $\sigma(F', F)$-null and $\|f'_n\| = 1$ for $n \geq 1$. Let $(f_n) \subset F$ be such that $1/2 \leq \|f_n\| \leq 2$ and $f'_n(f_n) = 1$ for $n \geq 1$. Suppose that the sequence (e_n) is equivalent to the unit vector basis of c_0 in E_1, and suppose that (e'_n) is the corresponding orthogonal sequence in E'_1. We will show that the closed linear span $H := [e_n \otimes f_n]$ in $L_e(E'_c, F)$ is isomorphic to c_0. We first show that H in the induced topology is isomorphic to a Banach space. To this end, as in Theorem 2.2 and its preceeding paragraph, we let U_1 be an absolutely convex zero-neighborhood in E such that $U_1 \cap E_1 = T^{-1}(B)$ and let V_1 be the unit ball of F. Since $U_1 \cap E_1$ is bounded, for any absolutely convex zero-neighborhood U in E there exist $\alpha > 0$ such that the restriction of the zero-neighborhood $N\left(U_1^0, \frac{1}{\alpha} V_1\right)$ in $L_e(E'_c, F)$ to H is contained in the zero-neighborhood $N(U^0, V_1)$ in $L_e(E'_c, F)$. Recall that $N(U^0, V)$ denotes the set of all $h \in L_e(E'_c, F)$ such that $h(U^0) \subset V$. Thus H is isomorphic to a Banach space.

Moreover, as in the proof of Theorem 2.2, $\Sigma e_n \otimes f_n$ is weakly unconditionally Cauchy series and in $\inf_n \|e_n \otimes f_n\| > 0$, therefore an appeal to Corollary 7 of [5, page 45] shows that the sequence $(e_n \otimes f_n)$ is equivalent to a c_0-basis in $E \widehat{\otimes}_\varepsilon F$ and consequently in $L_e(E'_c, F)$. We define now the map

$$p : L_e(E'_c, F) \mapsto H$$

by

$$p(h) = \Sigma \langle h(e'_n), f'_n \rangle \, e_n \otimes f_n \,, \qquad h \in L_e(E'_c, F) \,.$$

Since $\{e'_n\}$ is an equicontinuous subset of E', $\langle h(e'_n), f'_n \rangle \to 0$. One can now easily show that p is in fact a continuous linear projection from $L_e(E'_c, F)$ onto H. □

By $K_b^b(E, F)$ we mean the space of all weakly continuous linear operators which transform bounded sets into relatively compact subsets of F, endowed with the topology of uniform convergence on bounded sets in E. The following corollary offers a refinement of a result in [16].

Corollary 2.2. *If E is a locally convex metric space such that E'_b contains a copy of c_0 and if F is an infinite dimensional Banach space, then $K_b^b(E, F)$ contains a complemented subspace isomorphic to c_0.*

Proof. If E is metrizable, then by the result 21.6.4 of [14], E'_b is a complete space. The example 0.2 of [3] shows that $K_b^b(E, F)$ is topologically isomorphic to $L_e((E'_b)'_c, F)$, which completes the proof. □

3. Applications

FRENICHE [11] has shown that $C(X; F)$ equipped with the compact-open topology is a Grothendieck space if and only if $C(K; F)$ is a Grothendieck space for every compact subset K of X. The following result is analogous to one direction of FRENICHE's result.

Theorem 3.1. *Let X be a completely regular Hausdorff space and let F be a locally convex space. If $(C_b(X; F), \beta_{\mathscr{P}})$ is a Grothendieck space, then for any $K \in \mathscr{P}$, $(C(K; F), \tau_u)$ is a Grothendieck space.*

Proof. Let $K \in \mathscr{P}$ and let T be the restriction operator from $(C_b(X; F), \beta_{\mathscr{P}})$ into $(C(K; F), \tau_u)$. For each absolutely convex bounded subset B of F one has

$$C(K, B) \subseteq C(X, B) + C(K, V)$$

for every closed and absolutely convex zero-neighborhood V in F [19]. It is a well known fact that $C(X, B)$ is a $\beta_{\mathscr{P}}$-bounded subset of $C_b(X; F)$. Thus the closure of $T(C(X, B))$ contains the bounded set $C(K, B)$ of $(C(K, F), \tau_u)$ and by Proposition 2.3(b) of [11], $(C(K, F), \tau_u)$ is a Grothendieck space. □

KHURANA and VIELMA [13] have shown that if X has an infinite compact subset and F is a Banach space, then $(C_b(X; F), \beta_0)$ is a Grothendieck space if and only if F is finite dimensional and $(C_b(X), \beta_0)$ is a Grothendieck space. Let us recall that if $\mathscr{P}$ is the family of all compact subsets of X, then $\beta_{\mathscr{P}}$ is the strict topology β_0 on $C_b(X; F)$. The following corollary offers a refinement of one direction of this result.

Corollary 3.1. *If $(C_b(X; F), \beta_{\mathscr{P}})$ is a Grothendieck space, F is a quasicomplete locally convex space and X has an infinite compact subset $K \in \mathscr{P}$, then the $\sigma(F', F)$ and $\beta(F', F)$ sequential convergence coincide in the equicontinuous subsets of F'.*

Proof. By Theorem 3.1, $(C(K; F), \tau_u)$ is a Grothendieck space, and Example 0–5 of [3] shows that $(C(K; F), \tau_u)$ is topologically isomorphic to $L_e(C(K)', F)$. By theorem 2.2. the proof is ocmpleted. □

Remarks. (a) In the case that F is a Fréchet space, by a result in [1], F must be a Montel space, thus we have a refinement of the Corollary 3.3. of [11].

(b) When F is a Banach space, we obtain Theorem 3 of [13].

Acknowledgement. This work has been supported by the University of Isfahan, Iran, while the author was visiting the Department of Mathematics and Computer Sciences at Kent State University, Kent, Ohio, U.S.A. The author is very much indebted to Professor J. DIESTEL for several useful discussions and for his very valuable suggestions. The author is also grateful to the referee for making many useful comments and remarks.

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