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A Bridge Principle for Harmonic Maps

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Abstract. We prove a bridge principle for harmonic maps between general manifolds.

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1. Introduction

Given two minimal surfaces Σ_0, Σ_1 with nonempty boundaries and a thin strip connecting their boundaries, the bridge principle for minimal surfaces says that if the strip is thin enough, then the new boundary should span a minimal surface which is close to Σ_0 and Σ_1 joined by this thin strip. This problem has been studied by many mathematicians. Among them, Meeks and Yau [2] proved the bridge principle for orientable stable minimal surfaces in R^3; Smale [4] proved the bridge principle for minimal submanifolds (stable or unstable, but satisfying certain nondegenerate conditions) and constant mean curvature submanifolds in R^n of arbitrary dimension and codimension by the PDE technique; White [5, 6], using the geometric measure theory approach, proved Smale's theorems and extended them to the singular case.

On the other hand, Mou [3] by using a similar scheme and the results of [4], justified a bridge principle for harmonic maps with singularities from R^n into S^{n-1}, for $n \geq 3$, and then proved a prescribed singularities theorem for harmonic maps from domains in R^n into S^{n-1}, for $n \geq 3$.

By applying the scheme of [4], in this paper we provide a bridge principle for harmonic maps between general manifolds. More precisely, we have the following theorem:

THEOREM 1. *Let M_0, and M_1 be smooth, compact submanifolds with boundary, of a Riemannian manifold N,* $\dim M_0 = \dim M_1 \geq 2$, $\dim N \geq 3$, *and let f_ι: $M_\iota \to \tilde{N}$, be harmonic maps, $\iota = 0, 1$, where $\tilde{N}$ is a Riemannian manifold,* $\dim \tilde{N} \geq 3$. *Assume that the Jacobi operators of the tension fields corresponding to f_0, f_1 are nondegenerate.* Suppose that γ is an arc in N connecting ∂M_0 with ∂M_1.

Then, in any sufficiently small ε-neighborhood of γ, one can connect M_0 and M_1 at their boundaries by a bridge T_ε, and find a harmonic map F_ε from the bridged manifold $M_0\sharp_{T_\varepsilon}M_1$ to $\tilde{N}$. Furthermore, as the width of T_ε shrinks to zero (i.e., $\varepsilon \to 0$), $\{F_\varepsilon\}$ converges to f_ι (in C^k-topology, $k > 2$) on each compact subset of $M_\iota\backslash\gamma$, for $\iota = 0, 1$.

The Jacobi operators of the tension fields corresponding to f_0, f_1 are nondegenerate, if $\mathrm{dist}(\mathrm{spec}(L^\iota), 0) > 0$, with $L^\iota = \Delta_{M_\iota} - \sum g^{ij}_{M_\iota} R^{\tilde{N}}(\nabla_i f_\iota, \cdot)\nabla_j f_\iota$. Throughout this paper, we use the convention that two like indices in a term indicate a summation, $g^{ij}_{M_\iota}$ the metric tensor of M_ι, and $R^{\tilde{N}}(\cdot,\cdot)$ the curvature tensor of $\tilde{N}$. By a bridge (between M_0 and M_1), we mean an image of a diffeomorphic map $T\colon B^{n-1}\times [0, 1] \to N$, such that the intersection of the image with M_ι is equal to $T(B^{n-1}, \iota)$, where B^{n-1} is the unit ball of R^{n-1}. Moreover, $M_0\sharp_T M_1$ is a smooth manifold consisting of the union of M_0, M_1 and the bridge T.

2. Reformulation of Theorem 1

Theorem 1 will be proved by the following scheme: in each ε-tubular neighborhood of $M_0\cup\gamma\cup M_1$, we first construct a bridged manifold $M_0\sharp_{T_\varepsilon}M_1$ and an approximate harmonic map G_ε from $M_0\sharp_{T_\varepsilon}M_1$ into $\tilde{N}$ by gluing together f_0 and f_1 via a map on the bridge T_ε. Then we solve the harmonic map $F_\varepsilon\colon M_0\sharp_{T_\varepsilon}M_1 \to \tilde{N}$ via a perturbation of G_ε and prove sharp enough estimates on this perturbation.

More precisely, let $C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, \tilde{N})$ be the manifold of $C^{k,\alpha}$-maps from $M_0\sharp_{T_\varepsilon}M_1$ into $\tilde{N}$ and $T_{G_\varepsilon}C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, \tilde{N}) = C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$ be the tangent space of $C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, \tilde{N})$ at G_ε. For $G_\varepsilon \in C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, \tilde{N})$ and $\zeta \in C^{k,\alpha}_0(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$, the map $\exp_{G_\varepsilon}(\zeta) \in C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, \tilde{N})$ will be a perturbation of G_ε.

The tension fields τG_ε of G_ε is

$$\tau^\alpha G_\varepsilon(x) = \sum g^{ij}\left(\frac{\partial^2 G^\alpha_\varepsilon}{\partial x^i\partial x^j} - {}^{M_0\sharp_{T_\varepsilon}M_1}\,\Gamma^k_{ij}\frac{\partial G^\alpha_\varepsilon}{\partial x^k} + {}^{\tilde{N}}\,\Gamma^\alpha_{\beta\mu}(G_\varepsilon)\frac{\partial G^\beta_\varepsilon}{\partial x^i}\frac{\partial G^\mu_\varepsilon}{\partial x^j}\right)(x),$$

which belongs to $C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$ [1]. Define $\mathscr{P}$ to be the map which sends $\tau\exp_{G_\varepsilon}(\zeta)(x)$ to its parallel transport at $G_\varepsilon(x)$ along the geodesic $\exp_{G_\varepsilon(x)}(t\zeta)$, $0 \le t \le 1$. So $\mathscr{P}\tau\exp_{G_\varepsilon}(\zeta) \in C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$, and $\mathscr{P}\tau\exp_{G_\varepsilon}(\zeta) = 0$ if and only if $\tau\exp_{G_\varepsilon}(\zeta) = 0$.

To solve the solution, we linearize the tension field operator and obtain

$$\begin{aligned}
&\mathscr{P}\tau\exp_{G_\varepsilon}(\zeta)\\
&\quad = \mathscr{P}\tau\exp_{G_\varepsilon}(t\zeta)|_{t=0} + \left.\frac{\mathrm{d}(\mathscr{P}\tau\exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0} +
\end{aligned}$$

$$+\int_0^1 (1-t)\left(\frac{\mathrm{d}^2(\mathcal{P}\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t^2}\right)\mathrm{d}t$$

$$=\tau G_\varepsilon + \left.\frac{\mathrm{d}(\mathcal{P}\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0} + E(\zeta). \tag{2.1}$$

Denote $\{g_{ij}\}$, $\{h_{\alpha\beta}\}$ as the metrics of $M_0\sharp_{T_\varepsilon}M_1$, $\tilde{N}$ with respect to the coordinate systems $\{x^i\}$, $\{\tilde{x}^\beta\}$, and ${}^{M_0\sharp_{T_\varepsilon}M_1}\Gamma^k_{ij}$, ${}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}$ as the Christoffel symbols of g, h, respectively. By a direct calculation, it follows that

$$\left.\frac{\mathrm{d}(\mathcal{P}\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0} = \mathrm{d}\mathcal{P}|_{G_\varepsilon}\cdot\left.\frac{\mathrm{d}(\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}(\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0},$$

and

$$\left.\frac{\mathrm{d}(\tau^\alpha \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0}$$
$$=\sum g^{ij}\left(\frac{\partial^2\zeta^\alpha}{\partial x^i\partial x^j} - {}^{M_0\sharp_{T_\varepsilon}M_1}\Gamma^k_{ij}\frac{\partial\zeta^\alpha}{\partial x^k} + {}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma,\delta}\zeta^\delta\frac{\partial G_\varepsilon{}^\beta}{\partial x^i}\frac{\partial G_\varepsilon{}^\gamma}{\partial x^j}+\right.$$
$$\left.+\,{}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}\frac{\partial\zeta^\beta}{\partial x^i}\frac{\partial G_\varepsilon{}^\gamma}{\partial x^j} + {}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}\frac{\partial G_\varepsilon{}^\beta}{\partial x^i}\frac{\partial\zeta^\gamma}{\partial x^j}\right).$$

Applying the formula $\Delta\zeta = \sum g^{ij}\left(\nabla_i\nabla_j\zeta - \Gamma^k_{ij}\nabla_k\zeta\right)$, we then get

$$\left.\frac{\mathrm{d}(\mathcal{P}\tau \exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t}\right|_{t=0}$$
$$=\left\{\Delta\zeta + \sum g^{ij}\left({}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma,\delta}\zeta^\delta\frac{\partial G_\varepsilon{}^\beta}{\partial x^i}\frac{\partial G_\varepsilon{}^\gamma}{\partial x^j}\frac{\partial}{\partial\tilde{x}^\alpha}-\right.\right.$$
$$-\,{}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma,\delta}\zeta^\beta\frac{\partial G_\varepsilon{}^\gamma}{\partial x^i}\frac{\partial G_\varepsilon{}^\delta}{\partial x^j}\frac{\partial}{\partial\tilde{x}^\alpha}+$$
$$+\,{}^{M_0\sharp_{T_\varepsilon}M_1}\Gamma^k_{ij}\,{}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}\zeta^\beta\frac{\partial G_\varepsilon{}^\gamma}{\partial x^k}\frac{\partial}{\partial\tilde{x}^\alpha} - {}^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}\zeta^\beta\frac{\partial^2 G_\varepsilon{}^\gamma}{\partial x^i\partial x^j}\frac{\partial}{\partial\tilde{x}^\alpha}-$$
$$\left.\left.-\,{}^{\tilde{N}}\Gamma^\alpha_{\gamma\delta}\,{}^{\tilde{N}}\Gamma^\gamma_{\lambda\beta}\zeta^\lambda\frac{\partial G_\varepsilon{}^\beta}{\partial x^i}\frac{\partial G_\varepsilon{}^\delta}{\partial x^j}\frac{\partial}{\partial\tilde{x}^\alpha}\right)\right\} + \sum^{\tilde{N}}\Gamma^\alpha_{\beta\gamma}\tau^\gamma G_\varepsilon\cdot\zeta^\beta\frac{\partial}{\partial\tilde{x}^\alpha}$$
$$=\Delta\zeta - \sum g^{ij}\,{}^{\tilde{N}}R(\nabla_i G_\varepsilon,\zeta)\nabla_j G_\varepsilon$$
$$=L\zeta, \tag{2.2}$$

where the curvature tensor R is given by

$$R\left(\frac{\partial}{\partial\tilde{x}^\alpha},\frac{\partial}{\partial\tilde{x}^\beta}\right)\frac{\partial}{\partial\tilde{x}^\gamma}=\Gamma^\alpha_{\gamma\beta,\delta}\frac{\partial}{\partial\tilde{x}^\alpha}+\Gamma^\lambda_{\gamma\beta}\Gamma^\alpha_{\lambda\delta}\frac{\partial}{\partial\tilde{x}^\alpha}-\Gamma^\alpha_{\gamma\delta,\beta}\frac{\partial}{\partial\tilde{x}^\alpha}-\Gamma^\lambda_{\beta\delta}\Gamma^\alpha_{\lambda\gamma}\frac{\partial}{\partial\tilde{x}^\alpha}.$$

If L is invertible and $L^{-1}\colon C^{k-2,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N}) \to C_0^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$ is continuous, then we can reformulate Theorem 1 as a fixed point problem. This is because

$$\begin{aligned}\tau\exp_{G_\varepsilon}(\zeta)=0 \quad &\text{iff}\ \ \tau G_\varepsilon+L\zeta+E\zeta=0,\\ &\text{iff}\ \ \zeta=-L^{-1}\tau G_\varepsilon-L^{-1}E\zeta,\\ &\text{iff}\ \ \mathcal{T}\zeta=\zeta,\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}\mathcal{T}\zeta &= -L^{-1}\tau G_\varepsilon-L^{-1}E\zeta,\\ L\zeta &= \Delta\zeta-\sum g^{ij}\,{}^{\tilde{N}}\!R(\nabla_i G_\varepsilon,\zeta)\nabla_j G_\varepsilon,\\ E\zeta &= \int_0^1(1-t)\left(\frac{\mathrm{d}^2(\mathcal{P}\tau\exp_{G_\varepsilon}(t\zeta))}{\mathrm{d}t^2}\right)\mathrm{d}t.\end{aligned}\tag{2.4}$$

The approximate harmonic maps will be constructed in Section 3. We then prove in Section 4 that L is invertible and find a convex compact set $\mathcal{K}$ of the Banach space $C_0^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$, such that $\mathcal{T}\colon\mathcal{K}\to\mathcal{K}$ is continuous. By the Schauder fixed point theorem, the operator $\mathcal{T}$ has a fixed point. The solution is also shown to be a small perturbation of our approximate harmonic map when the width of the bridge shrinks to zero. This proves Theorem 1. The paper ends with an application which proves the bridge principle for strictly stable harmonic maps.

3. Construction of the Approximate Harmonic Maps

Case 1. $\dim M_0=\dim M_1=n\geq 3$:
First of all, we perturb γ slightly so that γ is tangent to $M_0\cup M_1$ and transversal to $\partial\{M_0\cup M_1\}$. We then construct a smooth bridged manifold $M_0\sharp_T M_1$, where T is a bridge in the c-tubular neighborhood of γ. Parametrize T by a smooth diffeomorphic map $\Phi\colon T\to B^{n-1}\times[0,1]$, so that $\Phi^{-1}(0,x^n)$, $0\leq x^n\leq 1$, is γ. Furthermore, we extend Φ to be defined on an open neighborhood of T in $M_0\sharp_T M_1$ onto $B^{n-1}\times(-a,1+a)$.

Similarly, let $\tilde{\gamma}$ be any arc connecting $f_0(\gamma(0))$ with $f_1(\gamma(1))$ and $\tilde{T}$ be a small open neighborhood of $\tilde{\gamma}$. Parametrize $\tilde{T}$ by $\tilde{\Phi}\colon\tilde{T}\to B^{m-1}\times(-b,1+b)$, such that $\tilde{\gamma}$ is parametrized by $(0,\tilde{x}^m)$, $0\leq\tilde{x}^m\leq 1$. Since f_ι are smooth up to the boundary, and γ is tangent to $M_0\cup M_1$ at its end points, without loss of generality we can assume that $f_0\colon B^{n-1}\times(-a,a)\to\tilde{T}$, and $f_1\colon B^{n-1}\times(1-a,1+a)\to\tilde{T}$ are smooth.

Let ϕ: $[-1,1] \to [0,1]$ be a smooth function satisfying $\phi(r) = 1$, for $|r| \leq (1/8)a$, and $\phi(r) = 0$, for $|r| \geq (1/4)a$. Let Π: $\Phi(T) \to \tilde{\Phi}(\tilde{T})$ be any fixed smooth map which maps γ into $\tilde{\gamma}$. Then we can find a bridge T', $\gamma \subset T' \subset T$, where we can define a map G: $M_0\sharp_{T'}M_1 \to \tilde{N}$ by

$$G(x) = \begin{cases} f_0(x), \quad x \in M_0, \\ \tilde{\Phi}^{-1}\{\phi(x^n)\tilde{\Phi}(f_0(x)) + (1-\phi(x^n))\Pi(x)\}, \quad x^n \in [0, \frac{1}{4}a], \\ \tilde{\Phi}^{-1}\{\Pi(x)\big\}, \quad x^n \in [\frac{1}{4}a, 1-\frac{1}{4}a], \\ \tilde{\Phi}^{-1}\{\phi(x^n-1)\tilde{\Phi}(f_1(x)) + (1-\phi(x^n-1))\Pi(x)\}, \\ \qquad x^n \in [1-\frac{1}{4}a, 1], \\ f_1(x), \quad x \in M_1. \end{cases} \tag{3.1}$$

For any $\varepsilon > 0$, we can find two positive constants δ_ε, $\tilde{\lambda}_\varepsilon$, such that δ_ε, $\tilde{\lambda}_\varepsilon < \varepsilon$ and G defined by Equation (3.1) maps the δ_ε-neighborhood of γ into the $\tilde{\lambda}_\varepsilon$-neighborhood of $\tilde{\gamma}$. Let f be a smooth function tangent to the y-axis ($y = f(r)$) and satisfying $f(0) = 1$, $f(r) = 1/2$, for $r > 1/4$ and $f'(r) < 0$, for $0 < r < 1/4$. Define f_ε by

$$f_\varepsilon(r) = \begin{cases} \varepsilon f(\frac{r}{\varepsilon}), & r \in [0, \frac{\varepsilon}{4}], \\ \frac{\varepsilon}{2}, & r \in [\frac{\varepsilon}{4}, 1-\frac{\varepsilon}{4}], \\ \varepsilon f(\frac{1-r}{\varepsilon}), & r \in [1-\frac{\varepsilon}{4}, 1]. \end{cases} \tag{3.2}$$

The bridge T_ε is defined by restricting T' to $\Phi^{-1}(\xi, x^n)$, with $0 \leq x^n \leq 1$, with the distance of $\Phi^{-1}(\xi, x^n)$ and $\Phi^{-1}(0, x^n)$ being less than $f_{\delta_\varepsilon}(x_n)$. The bridged manifold is defined by $M_0\sharp_{T_\varepsilon}M_1$ and the approximate harmonic map G_ε is defined by restricting G on $M_0\sharp_{T_\varepsilon}M_1$.

If $\mathcal{U}^\varepsilon = (\Omega_j^\varepsilon, \Psi_j^\varepsilon)$ is a parametrization of $M_0\sharp_{T_\varepsilon}M_1$, with $\Omega_1^\varepsilon = \big(M_0\sharp_{T_\varepsilon}M_1\big) \cap (B^{n-1} \times (-a, 1+a))$, $\Psi_1^\varepsilon = \Phi|_{\Omega_1^\varepsilon}$, then the metric tensors of $M_0\sharp_{T_\varepsilon}M_1$, denoted by g_{ij}, i, $j \in \{1, \ldots, n\}$, satisfy

$$\begin{aligned} g_{ij} &= \left\langle \Psi_*^\varepsilon\left(\frac{\partial}{\partial x^i}\right), \Psi_*^\varepsilon\left(\frac{\partial}{\partial x^j}\right)\right\rangle_N, \\ c_0^{-1} &< g_{ij}, \quad g^{ij} < c_0, \\ \sum_{ij}&(|g_{ij}|_{C^{k,\alpha}} + |g^{ij}|_{C^{k,\alpha}}) < c_k, \end{aligned} \tag{3.3}$$

for constant c_k which depends on k, M_0, M_1, γ, N, and is independent of ε.

By Equations (3.1) and (3.3), we obtain that the tension field τG_ε satisfies

$$\left(\int_{M_0\sharp_{T_\varepsilon}M_1} |\tau G_\varepsilon|^p \,\mathrm{d}v\right)^{1/p} \leq c(\mathrm{vol}\,(T_\varepsilon))^{1/p} \leq C\varepsilon^{(n-1)/p}, \tag{3.4}$$

where the constant C depends on f_0, f_1, M_0, M_1, γ, $\tilde{N}$, and is independent of ε.

Case 2. $\dim M_0 = \dim M_1 = n = 2$:
For simplicity and without loss of generality, we can assume that the length of γ and $\tilde{\gamma}$ is 1. Perturb γ and $\tilde{\gamma}$ so that γ is tangent to $M_0 \cup M_1$ and perpendicular to $\partial M_0 \cup \partial M_1$ at its endpoints, and the tangent vector of $\tilde{\gamma}$ is parallel to $\partial f_\iota/\partial \nu_\iota$ at $\tilde{\gamma}(\iota)$, with ν_ι being the outward normal of ∂M_ι at $\gamma(\iota)$. Then find a bridge T in the c-tubular neighborhood of γ, and a tubular neighborhood $\tilde{T}$ in the $\tilde{c}$-tubular neighborhood of $\tilde{\gamma}$. Let Φ: $T \to [0,1] \times [0,1]$, $\tilde{\Phi}$: $\tilde{T} \to B^{m-1} \times [0,1]$ be the parametrizations satisfying $\Phi(\gamma) \subset (0,[0,1])$, $\tilde{\Phi}(\tilde{\gamma}) \subset (0,[0,1])$, where Φ, $\tilde{\Phi}$ are normal coordinates of T and $\tilde{T}$ along γ and $\tilde{\gamma}$ respectively. Since f_ι are smooth up to the boundary, we can extend the parametrization of Φ and $\tilde{\Phi}$, and assume that f_0: $[0,1] \times (-a,a) \to \tilde{T}$, and f_1: $[0,1] \times (1-a, 1+a) \to \tilde{T}$ are smooth, for some a.

Now we find a monotone curve ϱ: $[0,1] \to B^{m-1} \times [0,1] \subset R^m$, satisfying $\tilde{\Phi}(\tilde{\gamma}) = \varrho$,

$$\frac{\mathrm{d}\varrho^\alpha}{\mathrm{d}t}(0) = \frac{\partial f_0^\alpha}{\partial x^2}(0,0) \quad \text{and} \quad \frac{\mathrm{d}\varrho^\alpha}{\mathrm{d}t}(1) = \frac{\partial f_1^\alpha}{\partial x^2}(0,1), \quad \text{for } \alpha \in \{1,\ldots,m\}.$$

Besides, let v: $[0,1] \to R^m$ be a smooth vector field, such that

$$v^\alpha(0) = \frac{\partial f_0^\alpha}{\partial x^1}(0,0) \quad \text{and} \quad v^\alpha(1) = \frac{\partial f_1^\alpha}{\partial x^1}(0,1), \quad \text{for } \alpha \in \{1,\ldots,m\}.$$

Define Π: $(x^1,x^2) \in [0,1] \times [0,1] \to R^m$ by

$$\Pi(x^1,x^2) = \varrho(x^2) + v(x^2)x^1 - \left.\frac{\mathrm{d}^2\varrho}{\mathrm{d}(x^2)^2}\right|_{(0,x^2)} (x^1)^2. \tag{3.5}$$

Note that, for $\alpha \in \{1,\ldots,m\}$,

$$\begin{aligned}
&\Pi^\alpha(0,\iota) = f_\iota^\alpha(\gamma(\iota)),\\
&\left.\frac{\mathrm{d}\Pi^\alpha(x^1,\iota)}{\mathrm{d}x^1}\right|_{x^1=0} = \frac{\partial f_\iota^\alpha}{\partial x^1}(\gamma(\iota)),\\
&\left.\frac{\mathrm{d}\Pi^\alpha(0,x^2)}{\mathrm{d}x^2}\right|_{x^2=\iota} = \frac{\partial f_0^\alpha}{\partial x^2}(\gamma(\iota)),\\
&\frac{\mathrm{d}^2\Pi^\alpha}{\mathrm{d}(x^1)^2}(0,x^2) = -\frac{\mathrm{d}^2\Pi^\alpha}{\mathrm{d}(x^2)^2}(0,x^2), \quad \text{for } x^2 \in [0,1].
\end{aligned} \tag{3.6}$$

For any $\varepsilon > 0$, we can find two positive constants δ_ε, $\tilde{\lambda}_\varepsilon < (1/2)\varepsilon$, such that Π defined by Equation (3.5), and f_ι, $\iota = 0, 1$, map the δ_ε-neighborhood of γ into the $\tilde{\lambda}_\varepsilon$-neighborhood of $\tilde{\gamma}$. As in Case 1, we construct $M_0 \sharp_{T_\varepsilon} M_1$ by defining T_ε to be the restriction of T to $\Phi^{-1}(x^1,x^2)$, with $0 \le x^2 \le 1$, and the distance of $\Phi^{-1}(x^1, x^2)$ and $\Phi^{-1}(0, x^2)$ less than $f_{\delta_\varepsilon}(x^2)$. The approximate harmonic map $G_\varepsilon : M_0 \sharp_{T_\varepsilon} M_1 \to \tilde{N}$ is defined by

$$G_\varepsilon(x) = \begin{cases} f_0(x), \quad x \in M_0, \\ \tilde{\Phi}^{-1}\{\phi_\varepsilon(x^2)\tilde{\Phi}(f_0(x)) + (1 - \phi_\varepsilon(x^2))\Pi(x)\}, \quad x^2 \in [0, \varepsilon], \\ \tilde{\Phi}^{-1}\{\Pi(x)\}, \quad x^2 \in [\varepsilon, 1 - \varepsilon], \\ \tilde{\Phi}^{-1}\{\phi_\varepsilon(x^2 - 1)\tilde{\Phi}(f_1(x)) + (1 - \phi_\varepsilon(x^2 - 1))\Pi(x)\}, \\ \qquad x^2 \in [1 - \varepsilon, 1], \\ f_1(x), \quad x \in M_1, \end{cases} \tag{3.7}$$

with

$$\begin{aligned} &0 \le \phi_\varepsilon(t) \le 1, \\ &\phi_\varepsilon(t) = 1, \quad |t| \le \frac{1}{4}\varepsilon, \\ &\phi_\varepsilon(t) = 0, \quad |t| \ge \frac{1}{2}\varepsilon. \end{aligned} \tag{3.8}$$

Using the same notation as in Case 1, we have g_{ij} satisfying Equation (3.3). Furthermore, by the normality of Φ and $\tilde{\Phi}$ along γ and $\tilde{\gamma}$, the Christoffel symbols of g, h satisfy

$$|{}^{M_0 \sharp_{T_\varepsilon} M_1}\Gamma^k_{ij}|(x) < c'\varepsilon, \quad |{}^{\tilde{N}}\Gamma^\alpha_{\beta\mu}|(G_\varepsilon(x)) < c'\varepsilon \tag{3.9}$$

on T_ε, with c' a uniform constant. Therefore, Equations (3.3), (3.6–3.9) imply that there exists a uniform constant c_1', such that $|\tau G_\varepsilon(x)| \le c_1'\varepsilon$, for (x^1, x^2) with $x^2 \in [\varepsilon, 1 - \varepsilon]$, and $\Phi^{-1}(x^1, x^2) \in T_\varepsilon$, and $|\tau G_\varepsilon(x)| \le c_1'$, for (x^1, x^2), with $x^2 \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$, and $\Phi^{-1}(x^1, x^2) \in T_\varepsilon$. Hence, there exists a uniform constant C, such that

$$\begin{aligned} &\left(\int_{M_0 \sharp_{T_\varepsilon} M_1} |\tau G_\varepsilon|^p \, \mathrm{d}v\right)^{1/p} \\ &\quad = \left(\int_0^\varepsilon \int_{S_{x^2}} |\tau G_\varepsilon|^p \, \mathrm{d}v + \int_\varepsilon^{1-\varepsilon} \int_{S_{x^2}} |\tau G_\varepsilon|^p \, \mathrm{d}v + \int_{1-\varepsilon}^1 \int_{S_{x^2}} |\tau G_\varepsilon|^p \, \mathrm{d}v\right)^{1/p} \\ &\quad \le c\Big\{\mathrm{vol}\,(T_\varepsilon \cap \Phi^{-1}([0, 1] \times [0, \varepsilon])) + \\ &\qquad + \varepsilon \cdot \mathrm{vol}\,(T_\varepsilon \cap \Phi^{-1}([0, 1] \times [\varepsilon, 1 - \varepsilon])) + \\ &\qquad + \mathrm{vol}\,(T_\varepsilon \cap \Phi^{-1}([0, 1] \times [1 - \varepsilon, 1]))\Big\}^{1/p} \\ &\quad \le C\varepsilon^{2/p}. \end{aligned} \tag{3.10}$$

4. Proof of Theorem 1

We note that $M_0 \sharp_{T_\varepsilon} M_1$, L, G_ε satisfy the following assumptions:

(a) There exists a finite collection of $C^{k,\alpha}$ parametrizations $\mathcal{U}^\varepsilon = (\Omega_j^\varepsilon, \Psi_j^\varepsilon)$, $\Psi_j^\varepsilon \colon \Omega_j^\varepsilon \to R^n$, such that $M_0 \sharp_{T_\varepsilon} M_1 = \cup_j \Omega_j^\varepsilon$.

(b) For $\zeta \in C^{k,\alpha}(M_0 \sharp_{T_\varepsilon} M_1, T_{G_\varepsilon} \tilde{N})$, define the norms $|\zeta|_{L^p}$, $|\zeta|_{H^k}$ by

$$|\zeta|_{L^p} = \left(\int |\zeta|^p \,\mathrm{d}v \right)^{1/p}; \quad |\zeta|_{H^k} = \sum_{j=0}^{k} \left(\int |\nabla^j \zeta|^2 \,\mathrm{d}v \right)^{1/2}$$

Moreover, by immersing $\tilde{N}$ into a Euclidean space R^r, and using the parametrization $\mathcal{U}^\varepsilon = (\Omega_j^\varepsilon, \Psi_j^\varepsilon)$, we can also define the norm $|\zeta|_{C^{k,\alpha}}$ as in [4].

(c) For each $x \in \Omega_j^\varepsilon$, either there is a closed ball of radius ε_1, $B_{\varepsilon_1}(x)$ with $x \in B_{\varepsilon_1}(x) \subset \Omega_j^\varepsilon$, or there is a closed half ball of radius ε_1, $B_{\varepsilon_1}^+(x)$ with $x \in B_{\varepsilon_1}^+(x) \subset \Omega_j^\varepsilon$. Here $\varepsilon_1 = C_0^{-1}\varepsilon$ with C_0 a constant independent of ε, and $B_{\varepsilon_1}^+(x)$ is the inverse image of $\Upsilon_j^\varepsilon \Psi_j^\varepsilon$ on a Euclidean closed half ball of radius ε_1, with Υ_j^ε being a diffeomorphism and having a uniform (independent of ε, j) C^1-norm.

(d) The metric on Ω_j^ε satisfies Equation (3.3).

(e) On each Ω_j^ε,

$$\left\{ \frac{\partial}{\partial \tilde{x}^1}, \dots, \frac{\partial}{\partial \tilde{x}^\beta}, \dots, \frac{\partial}{\partial \tilde{x}^m} \right\}$$

along $G_\varepsilon(x)$ generate $T_{G_\varepsilon} C^{k,\alpha}(\Omega_j^\varepsilon, \tilde{N})$ and satisfy

$$\left| \frac{\partial}{\partial \tilde{x}^\beta} \right|_{C^0}, \quad \left| \frac{\partial}{\partial \tilde{x}^\beta} \right|_{C^1} < C_0,$$

$$\sum_{\beta=1}^{m} \left| \frac{\partial}{\partial \tilde{x}^\beta} \right|_{C^k} \le C_k \varepsilon^{2-k}, \quad \text{for } k \ge 2,$$

$$\sum_{\beta=1}^{m} \left| \frac{\partial}{\partial \tilde{x}^\beta} \right|_{C^{k,\alpha}} \le C_k \varepsilon^{2-k-\alpha}, \quad \text{for } k \ge 2,$$

with C_k constants independent of ε.

(f) For $x_0 \in \Omega_j^\varepsilon \cap \partial(M_0 \sharp_{T_\varepsilon} M_1)$, there exists a $C^{k,\alpha}$-reparametrization Φ_s, straightening out of the boundary, with $\Phi_s(x_0) = 0$ and

$$\Phi_s \colon B_{\varepsilon/4}(x_0) \cap \Omega_j^\varepsilon \to R_+^n = \{x \in R^n : x_n \ge 0\},$$

$$\Phi_s\,(B_{\frac{\varepsilon}{4}}(x_0) \cap \partial(M_0 \sharp_{T_\varepsilon} M_1)) \subset R^{n-1} \times \{0\},$$

$$|\Phi_s|_{C^k} \le C_k \varepsilon^{1-k}, \quad \text{for } k \ge 1,$$

$$|\Phi_s|_{C^{k,\alpha}} \le C_k \varepsilon^{1-k-\alpha}, \quad \text{for } k \ge 1,$$

$$|\nabla \Phi_s| \ge C_1^{-1}.$$

(g) In coordinates $\{x^i\}$ and $\{\partial/\partial\tilde{x}^\beta|_{G_\varepsilon(x)}\}$ on Ω_j^ε and $T_{G_\varepsilon}C^{k,\alpha}(\Omega_j^\varepsilon, \tilde{N})$,

$$L\zeta = \sum \left(a_{ij}\frac{\partial^2\zeta^\beta}{\partial x^i \partial x^j} + b_i^{\beta\mu}\frac{\partial\zeta^\mu}{\partial x^i} + c^{\beta\mu}\zeta^\mu \right)\frac{\partial}{\partial\tilde{x}^\beta},$$

with

$$c_0^{-1}I \le (a_{ij}) \le c_0 I,$$

$$\sum_{i,j}(|a_{ij}|_{C^0} + \varepsilon^\alpha |a_{ij}|_{C^{0,\alpha}}) \le C_0,$$

$$\sum_{i,\mu,\beta}(\varepsilon|b_i^{\beta\mu}|_{C^0} + \varepsilon^{1+\alpha}|b_i^{\beta\mu}|_{C^{0,\alpha}}) \le C_0\varepsilon,$$

$$\sum_{\mu,\beta}(\varepsilon^2|c^{\beta\mu}|_{C^0} + \varepsilon^{2+\alpha}|c^{\beta\mu}|_{C^{0,\alpha}}) \le C_0\varepsilon^2.$$

We remark that condition (f) comes from Equation (3.2) and conditions (e) and (g) are from Equations (3.3), (3.6–3.8). Consequently, with minor modifications to the proofs of Lemmas 1–4, and Corollary 1 in [4], we can derive

LEMMA 1 (C^0 Estimate). *For* $\zeta \in C_0^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$, *we have*

$$|\zeta|_{C^0} \le c_p|\Delta\zeta|_{L^p}^{1-e_p}$$

for any $p > n/2$, *where* $e_p > 0$, $\lim_{p\to n/2} e_p = 0$, *and* c_p *depends on* $|\zeta|_{L^1}$, $\mathrm{vol}\,(M_0\sharp_{T_\varepsilon}M_1)$, *and* p.

LEMMA 2 (Interpolation inequalities). *For* $\zeta \in C^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$, $\sigma > 0$, *we have*

$$|\zeta|_{C^2} \le \sigma\varepsilon^\alpha|\zeta|_{C^{2,\alpha}} + C_\sigma\varepsilon^{-2}|\zeta|_{C^0},$$

$$|\zeta|_{C^{1,\alpha}} \le \sigma\varepsilon|\zeta|_{C^{2,\alpha}} + C_\sigma\varepsilon^{-1-\alpha}|\zeta|_{C^0},$$

$$|\zeta|_{C^1} \le \sigma\varepsilon^{1+\alpha}|\zeta|_{C^{2,\alpha}} + C_\sigma\varepsilon^{-1}|\zeta|_{C^0},$$

where C_σ *is a constant depending on* σ *and* C_0, C_1, C_2.

LEMMA 3 (Schauder estimates). *If* $\zeta \in C_0^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde{N})$ *satisfies* $L\zeta = F$, *then*

$$|\zeta|_{C^{2,\alpha}} \le C_\alpha\{\varepsilon^{-\alpha}|F|_{C^0} + |F|_{C^\alpha} + \varepsilon^{-2-\alpha}|\zeta|_{C^0}\},$$

where C_α *depends on* α, C_0, C_1, C_2, c_0.

LEMMA 4 (Eigenvalue estimate). *Suppose that* $\mathrm{dist}(\mathrm{spec}(L^\iota), 0) > d_\iota$, *with* $L^\iota = \Delta_{M_\iota} - \sum g_{M_\iota}^{ij} R^{\tilde{N}}(\nabla_i f_\iota, \cdot)\nabla_j f_\iota$. *Then positive constants* ε_1 *and* d *exist such* *that for all* $\varepsilon < \varepsilon_1$, $\operatorname{dist}(\operatorname{spec}(L), 0) > d$, *where* ε_1, d *depend on* $\min\{d_0, d_1\}$, c_0, C_0, C_1.

LEMMA 5 (L^2 estimate). *If* $\operatorname{dist}(\operatorname{spec}(L^\iota), 0) > d_\iota$, $\varepsilon < \varepsilon_1$ *with* ε_1 *as in Lemma* 4, *and* $\zeta \in C_0^{k,\alpha}(M_0 \sharp_{T_\varepsilon} M_1, T_{G_\varepsilon}\tilde{N})$, *then*

(a) $$|\zeta|_{L^2}^2 \leq d^{-2}(|\zeta^+|_{C^0} + |\zeta^-|_{C^0})|L\zeta|_{L^1},$$

(b) $$|\zeta|_{H^1} \leq C'|L\zeta|_{L^2},$$

where d *is as in Lemma* 4, ζ^+ *(respectively,* ζ^-*) is the orthogonal projection of* ζ *to the direct sum of eigenspaces of* L *with positive (negative) eigenvalues, and* C' *is a constant depending on* d_0 *and* c_0, C_0, C_1.

To find the convex set $\mathcal{K}$, the following observations are crucial:

LEMMA 6. *Let* $\zeta(x) \in C^{k,\alpha}(M_0 \sharp_{T_\varepsilon} M_1, T_{G_\varepsilon}\tilde{N})$, *with* $|\zeta|_{C^1}$ *bounded by a sufficiently small constant* c_2', *and* $\chi(x,t) = \exp_{G_\varepsilon}(t\zeta)$, *for* $t \in [0,1]$. *Then*

(a) $$\left|\frac{\partial \chi^\beta}{\partial t}\right| < C''|\zeta|,$$

(b) $$\left|\frac{\partial^2 \chi^\beta}{\partial x^i \partial t}\right| < C''(|\nabla\zeta| + |\zeta|^2),$$

(c) $$\left|\frac{\partial \chi^\beta}{\partial x^i}\right| < C''(1 + |\nabla\zeta| + |\zeta|^2),$$

(d) $$\left|\frac{\partial^2 \chi^\beta}{\partial t^2}\right| < C''|\zeta|^2,$$

(e) $$\left|\frac{\partial^3 \chi^\beta}{\partial x^i \partial t^2}\right| < C''(|\nabla\zeta|^2 + |\zeta|^2),$$

(f) $$\left|\frac{\partial^3 \chi^\beta}{\partial x^i \partial x^j \partial t}\right| < C''(|\nabla^2\zeta| + |\nabla\zeta|^2 + |\zeta|^2),$$

(g) $$\left|\frac{\partial^2 \chi^\beta}{\partial x^i \partial x^j}\right| < C''(1 + |\nabla^2\zeta| + |\nabla\zeta|^2 + |\zeta|^2),$$

(h) $$\left|\frac{\partial^4 \chi^\beta}{\partial x^i \partial x^j \partial t^2}\right| < C''(|\nabla^2\zeta|\,|\zeta| + |\nabla\zeta|^2 + |\zeta|^2),$$

where C'' *depends on* c_2', $\tilde{N}$, C_0, C_1 *and* C_2.

Proof. Define $\bar{\chi}(x,s,t,\zeta) = \exp_{G_\varepsilon}(st\zeta)$. Applying Taylor's expansion to χ and the geodesic equation

$$\frac{\partial^2 \bar{\chi}^\beta}{\partial s^2} + \sum \Gamma^\beta_{\mu\nu} \frac{\partial \bar{\chi}^\mu}{\partial s} \frac{\partial \bar{\chi}^\nu}{\partial s} = 0,$$

$$\left.\frac{\partial \bar{\chi}}{\partial s}\right|_{s=0} = t\zeta, \quad \bar{\chi}|_{s=0} = G_\varepsilon(x), \tag{4.1}$$

for $\beta, \mu, \nu \in 1, \dots m$, and $\Gamma^\beta_{\mu\nu} =^{\tilde{N}} \Gamma^\beta_{\mu\nu}$, we have

$$\begin{aligned}\chi^\beta(x,t) &= \exp^\beta_{G_\varepsilon}(t\zeta)\\ &= \exp^\beta_{G_\varepsilon}(st\zeta)|_{s=0} + \left.\frac{\mathrm{d}\exp^\beta_{G_\varepsilon}(st\zeta)}{\mathrm{d}s}\right|_{s=0} + \\ &\quad + \int_0^1 (1-s)\frac{\mathrm{d}^2\exp^\beta_{G_\varepsilon}(st\zeta)}{\mathrm{d}s^2}\,\mathrm{d}s\\ &= G^\beta_\varepsilon + t\zeta^\beta + \int_0^1 (1-s)\Gamma^\beta_{\mu\nu}(\bar{\chi}(x,s,t,\zeta))\frac{\partial\bar{\chi}^\mu}{\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}\,\mathrm{d}s. \end{aligned}\tag{4.2}$$

Directly from Equation (4.1), we have

$$\begin{aligned}&\frac{\partial\bar{\chi}^\beta}{\partial t}(x,1,0,\zeta) = \zeta^\beta, \quad \frac{\partial^2\bar{\chi}^\beta}{\partial x^i\partial t}(x,1,0,\zeta) = \frac{\partial\zeta^\beta}{\partial x^i},\\ &\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t}(x,1,0,\zeta) = \frac{\partial^2\zeta^\beta}{\partial x^i\partial x^j}, \quad \frac{\partial^2\bar{\chi}^\beta}{\partial t\partial s}(x,0,0,\zeta) = \zeta^\beta,\\ &\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial t\partial s}(x,0,0,\zeta) = \frac{\partial\zeta^\beta}{\partial x^i}, \quad \frac{\partial^4\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t\partial s}(x,0,0,\zeta) = \frac{\partial^2\zeta^\beta}{\partial x^i\partial x^j}.\end{aligned}$$

Therefore, by the compactness property and Taylor's expansion, we can find a small positive constant c_2 and a large constant C'', such that for $|\zeta| = 1$, $s \le 1$, $t < c'_2$,

$$\left|\frac{\partial\bar{\chi}^\beta}{\partial t}\right|(x,1,t,\zeta) \le |\zeta| + C''t, \tag{4.3a}$$

$$\left|\frac{\partial^2\bar{\chi}^\beta}{\partial x^i\partial t}\right|(x,1,t,\zeta) \le |\nabla\zeta| + C''(1+|\nabla\zeta|)t, \tag{4.3b}$$

$$\left|\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t}\right|(x,1,t,\zeta) \le |\nabla^2\zeta| + C''(1+|\nabla^2\zeta|+|\nabla\zeta|^2)t, \tag{4.3c}$$

$$\left|\frac{\partial^2\bar{\chi}^\beta}{\partial t\partial s}\right|(x,s,t,\zeta) \le |\zeta| + C''t, \tag{4.3d}$$

$$\left|\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial t\partial s}\right|(x,s,t,\zeta) \le |\nabla\zeta| + C''(1+|\nabla\zeta|)t, \tag{4.3e}$$

$$\left|\frac{\partial^4\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t\partial s}\right|(x,s,t,\zeta) \le |\nabla^2\zeta| + C''(1+|\nabla^2\zeta|+|\nabla\zeta|^2)t, \tag{4.3f}$$

where C'' depends on the local geometry of $\tilde{N}$, C_0, C_1 and C_2. Furthermore, by rescaling, we obtain that

$$\left|\frac{\partial\bar{\chi}^\beta}{\partial t}\right|(x,s,t,\zeta) \le C''|\zeta|, \tag{4.4a}$$

$$\left|\frac{\partial^2\bar{\chi}^\beta}{\partial x^i\partial t}\right|(x,s,t,\zeta) \le C''(|\nabla\zeta|+|\zeta|^2), \tag{4.4b}$$

$$\left|\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t}\right|(x,s,t,\zeta) \le C''(|\nabla^2\zeta|+|\nabla\zeta|^2|\zeta|^{-1}+|\zeta|^2), \tag{4.4c}$$

$$\left|\frac{\partial^2\bar{\chi}^\beta}{\partial t\partial s}\right|(x,s,t,\zeta) \le C''|\zeta|, \tag{4.4d}$$

$$\left|\frac{\partial^3\bar{\chi}^\beta}{\partial x^i\partial t\partial s}\right|(x,s,t,\zeta) \le C''(|\nabla\zeta|+|\zeta|^2), \tag{4.4e}$$

$$\left|\frac{\partial^4\bar{\chi}^\beta}{\partial x^i\partial x^j\partial t\partial s}\right|(x,s,t,\zeta) \le C''(|\nabla^2\zeta|+|\nabla\zeta|^2\,|\zeta|^{-1}+|\zeta|^2), \tag{4.4f}$$

for $t, s \in [0,1]$, $|\zeta|_{C^1} < c_2'$, where C'' is a uniform constant which may differ from the previous one. Thus (a) and (b) are directly from Equations (4.4a) and (4.4b) by setting $s=1$. And (c–e) can be derived successively by Taylor's expansion, (a), (b) and Equation (4.1). Besides, from Equation (4.4c), we have that for $t, s \in [0,1]$, $|\zeta| \le c_2'$,

$$\left|\frac{\partial^2\bar{\chi}^\beta}{\partial x^i\partial x^j}\right| \le C''(1+|\nabla^2\zeta|+|\nabla\zeta|^2|\zeta|^{-1}+|\zeta|^2). \tag{4.4g}$$

To prove (f), we can take derivatives of both sides of Equation (4.2) and get

$$\begin{aligned}\frac{\partial^3\chi^\beta}{\partial x^i\partial x^j\partial t} &= \frac{\partial^2\zeta^\beta}{\partial x^i\partial x^j} + \int_0^1 (1-s)\sum\Bigg\{\Gamma^\beta_{\mu\nu,\delta\lambda\omega}\frac{\partial\bar{\chi}^\omega}{\partial x^i}\frac{\partial\bar{\chi}^\lambda}{\partial x^j}\frac{\partial\bar{\chi}^\delta}{\partial t}\frac{\partial\bar{\chi}^\mu}{\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\\ &\quad+\Gamma^\beta_{\mu\nu,\delta\lambda}\frac{\partial^2\bar{\chi}^\lambda}{\partial x^i\partial x^j}\frac{\partial\bar{\chi}^\delta}{\partial t}\frac{\partial\bar{\chi}^\mu}{\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\\ &\quad+\Gamma^\beta_{\mu\nu,\delta\lambda}\frac{\partial\bar{\chi}^\lambda}{\partial x^j}\frac{\partial^2\bar{\chi}^\delta}{\partial x^i\partial t}\frac{\partial\bar{\chi}^\mu}{\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\\ &\quad+\Gamma^\beta_{\mu\nu,\delta}\frac{\partial^3\bar{\chi}^\delta}{\partial x^i\partial x^j\partial t}\frac{\partial\bar{\chi}^\mu}{\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\Gamma^\beta_{\mu\nu,\delta}\frac{\partial^2\bar{\chi}^\delta}{\partial x^j\partial t}\frac{\partial^2\bar{\chi}^\mu}{\partial x^i\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\\ &\quad+\Gamma^\beta_{\mu\nu,\delta}\frac{\partial\bar{\chi}^\delta}{\partial t}\frac{\partial^3\bar{\chi}^\mu}{\partial x^i\partial x^j\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\Gamma^\beta_{\mu\nu,\delta\lambda}\frac{\partial\bar{\chi}^\lambda}{\partial x^i}\frac{\partial\bar{\chi}^\delta}{\partial x^j}\frac{\partial^2\bar{\chi}^\mu}{\partial t\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\\ &\quad+\Gamma^\beta_{\mu\nu,\delta}\frac{\partial^2\bar{\chi}^\delta}{\partial x^i\partial x^j}\frac{\partial^2\bar{\chi}^\mu}{\partial t\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\Gamma^\beta_{\mu\nu,\delta}\frac{\partial\bar{\chi}^\delta}{\partial x^j}\frac{\partial^3\bar{\chi}^\mu}{\partial x^i\partial t\partial s}\frac{\partial\bar{\chi}^\nu}{\partial s}+\end{aligned}$$

$$
\begin{aligned}
&+ \Gamma^{\beta}_{\mu\nu,\delta} \frac{\partial \bar{\chi}^{\delta}}{\partial x^j} \frac{\partial^2 \bar{\chi}^{\mu}}{\partial t \partial s} \frac{\partial^2 \bar{\chi}^{\nu}}{\partial x^i \partial s} + \Gamma^{\beta}_{\mu\nu} \frac{\partial^4 \bar{\chi}^{\mu}}{\partial x^i \partial x^j \partial t \partial s} \frac{\partial \bar{\chi}^{\nu}}{\partial s} + \\
&+ \Gamma^{\beta}_{\mu\nu} \frac{\partial^3 \bar{\chi}^{\mu}}{\partial x^j \partial t \partial s} \frac{\partial^2 \bar{\chi}^{\nu}}{\partial x^i \partial s} + \Gamma^{\beta}_{\mu\nu,\delta} \frac{\partial \bar{\chi}^{\delta}}{\partial t} \frac{\partial^2 \bar{\chi}^{\mu}}{\partial x^i \partial s} \frac{\partial^2 \bar{\chi}^{\nu}}{\partial x^j \partial s} + \\
&+ \Gamma^{\beta}_{\mu\nu} \frac{\partial^2 \bar{\chi}^{\mu}}{\partial t \partial s} \frac{\partial^3 \bar{\chi}^{\nu}}{\partial x^i \partial x^j \partial s} + \cdots \Bigg\} \, \mathrm{d}s,
\end{aligned}
$$

where $\cdots$ represents similar terms. Plugging conditions (a–e), (4.4c–4.4g) to the right-hand side of the above inequality, we then obtain (f). The inequality (g) is derived by integration (f), and the inequality (h) is from Equation (4.1) and the estimates (a–g). □

LEMMA 7. *For* $\zeta \in C^{k,\alpha}(M_0 \sharp_{T_\varepsilon} M_1, T_{G_\varepsilon}\tilde{N})$, $|\zeta|_{C^1} \leq c_2'$ *with* c_2' *as in Lemma 6,* $E(\zeta)$ *defined by (2.4), we have*

$$
|E(\zeta)| \leq C'''(|\nabla^2 \zeta| \, |\zeta| + |\nabla \zeta|^2 + |\zeta|^2),
$$

with C''' *a uniform constant depending only on* C'', c_0, c_1, *and* $\tilde{N}$.

Proof. We follow the notation as in Lemma 6. It is sufficient to prove this lemma, if we can bound $(\partial^2(\mathcal{P}\tau\chi))/\partial t^2$ by $C(|\nabla^2\zeta|\,|\zeta| + |\nabla\zeta|^2 + |\zeta|^2)$. One has that

$$
\begin{aligned}
\frac{\partial^2(\mathcal{P}\tau\chi)}{\partial t^2} &= \frac{\partial}{\partial t}\left(D\mathcal{P} \cdot \frac{\partial \tau\chi}{\partial t}\right) \\
&= D^2\mathcal{P} \cdot \frac{\partial^2 \tau\chi}{\partial t^2},
\end{aligned}
$$

where $D\mathcal{P}$, $D^2\mathcal{P}$ denote the first and second differential of $\mathcal{P}$, respectively. Note that they just depend on the local geometry of $\tilde{N}$. Thus, we only need to estimate $(\partial^2\tau\chi)/\partial t^2$. A direct computation gives

$$
\begin{aligned}
\frac{\partial^2 \tau\chi}{\partial t^2} &= \sum \Bigg\{ \frac{\partial^2 \tau^{\beta}\chi}{\partial t^2} + 2\frac{\partial \tau^{\mu}\chi}{\partial t} \Gamma^{\beta}_{\mu\nu} \frac{\partial \chi^{\nu}}{\partial t} + \tau^{\mu}\chi \Gamma^{\beta}_{\mu\nu,\delta} \frac{\partial \chi^{\delta}}{\partial t} \frac{\partial \chi^{\nu}}{\partial t} + \\
&\quad + \tau^{\mu}\chi \Gamma^{\beta}_{\mu\nu} \frac{\partial^2 \chi^{\nu}}{\partial t^2} + \tau^{\mu}\chi \Gamma^{\delta}_{\mu\nu} \Gamma^{\beta}_{\delta\lambda} \frac{\partial \chi^{\nu}}{\partial t} \frac{\partial \chi^{\lambda}}{\partial t} \Bigg\} \frac{\partial}{\partial \tilde{x}^{\beta}}, \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau^{\beta}\chi &= \sum g^{ij} \left\{ \frac{\partial^2 \chi^{\beta}}{\partial x^i \partial x^j} - {}^{M_0\sharp_{T_\varepsilon} M_1}\Gamma^{k}_{ij} \frac{\partial \chi^{\beta}}{\partial x^k} + {}^{\tilde{N}}\Gamma^{\beta}_{\mu\nu} \frac{\partial \chi^{\mu}}{\partial x^i} \frac{\partial \chi^{\nu}}{\partial x^j} \right\}, \\
\frac{\partial}{\partial t}\tau^{\beta}\chi &= \sum g^{ij} \Bigg\{ \frac{\partial^3 \chi^{\beta}}{\partial x^i \partial x^j \partial t} - {}^{M_0\sharp_{T_\varepsilon} M_1}\Gamma^{k}_{ij} \frac{\partial^2 \chi^{\beta}}{\partial x^k \partial t} + \\
&\quad + {}^{\tilde{N}}\Gamma^{\beta}_{\mu\nu,\delta} \frac{\partial \chi^{\delta}}{\partial t} \frac{\partial \chi^{\mu}}{\partial x^i} \frac{\partial \chi^{\nu}}{\partial x^j} + {}^{\tilde{N}}\Gamma^{\beta}_{\mu\nu} \frac{\partial^2 \chi^{\mu}}{\partial x^i \partial t} \frac{\partial \chi^{\nu}}{\partial x^j} +
\end{aligned}
$$

$$+\,{}^{\tilde N}\Gamma^{\beta}_{\mu\nu}\frac{\partial\chi^{\mu}}{\partial x^{i}}\frac{\partial^{2}\chi^{\nu}}{\partial x^{j}\partial t}\Bigg\},$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\tau^{\beta}\chi \;=\; \sum g^{ij}\Bigg\{&\frac{\partial^4\chi^{\beta}}{\partial x^i\partial x^j\partial t^2} - {}^{M_0\sharp_{T_\varepsilon}M_1}\Gamma^{k}_{ij}\frac{\partial^3\chi^{\beta}}{\partial x^k\partial t^2}+\\ &+{}^{\tilde N}\Gamma^{\beta}_{\mu\nu,\delta\lambda}\frac{\partial\chi^{\lambda}}{\partial t}\frac{\partial\chi^{\delta}}{\partial t}\frac{\partial\chi^{\mu}}{\partial x^i}\frac{\partial\chi^{\nu}}{\partial x^j}+{}^{\tilde N}\Gamma^{\beta}_{\mu\nu,\delta}\frac{\partial^2\chi^{\delta}}{\partial t^2}\frac{\partial\chi^{\mu}}{\partial x^i}\frac{\partial\chi^{\nu}}{\partial x^j}+\\ &+{}^{\tilde N}\Gamma^{\beta}_{\mu\nu,\delta}\frac{\partial\chi^{\delta}}{\partial t}\frac{\partial^2\chi^{\mu}}{\partial x^i\partial t}\frac{\partial\chi^{\nu}}{\partial x^j}+{}^{\tilde N}\Gamma^{\beta}_{\mu\nu}\frac{\partial^2\chi^{\mu}}{\partial x^i\partial t}\frac{\partial^2\chi^{\nu}}{\partial x^i\partial t}+\\ &+{}^{\tilde N}\Gamma^{\beta}_{\mu\nu}\frac{\partial^3\chi^{\mu}}{\partial x^i\partial t^2}\frac{\partial\chi^{\nu}}{\partial x^j}+\dots\Bigg\}.\end{aligned}$$

Plugging the estimates in Lemma 6 into the right-hand side of the above formula, we obtain

$$\begin{aligned}&|\tau^{\beta}\chi| \le C'''(1+|\nabla^2\zeta|+|\nabla\zeta|^2+|\zeta|^2),\\ &\left|\frac{\partial}{\partial t}\tau^{\beta}\chi\right| \le C'''(|\nabla^2\zeta|+|\nabla\zeta|+|\zeta|),\\ &\left|\frac{\partial^2}{\partial t^2}\tau^{\beta}\chi\right| \le C'''(|\nabla^2\zeta|\,|\zeta|+|\nabla\zeta|^2+|\zeta|^2).\end{aligned}$$

The lemma then follows from the above estimates, Lemma 6, and Equation (4.5). □

Proof of Theorem 1. Lemma 4 implies that $\mathcal{T}$ is a continuous map from $C_0^{2,\alpha'}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde N)$ into itself, for $\forall\alpha' > 0$, $\varepsilon < \varepsilon_1$. Thus, by the Schauder fixed point theorem, to find the harmonic map in Theorem 1, we only need to find a convex compact set $\mathcal{K}$ of $C_0^{2,\alpha'}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde N)$, such that $\mathcal{T}$ maps $\mathcal{K}$ into itself. Define the convex compact sets $\mathcal{K}(\varepsilon,\sigma,\alpha)$ in $C_0^{2,\alpha'}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde N)$ by: for $n\ge 3$, $\alpha>\alpha'$, $\sigma_n>\sigma>0$, $\varepsilon<\tilde\varepsilon$,

$$\mathcal{K}(\varepsilon,\sigma,\alpha)=\Bigg\{\zeta\in C_0^{k,\alpha}(M_0\sharp_{T_\varepsilon}M_1, T_{G_\varepsilon}\tilde N)\;:\;\begin{aligned}&|\zeta|_{C^0}\le\varepsilon^{2-(2/n)-\sigma}\\ &|\zeta|_{C^{2,\alpha}}\le\varepsilon^{-(2/n)-\sigma-\alpha}\\ &|\zeta|_{H^1}\le\varepsilon^{(n-1)/2-\sigma}\quad(n>3)\\ &|\zeta|_{H^1}\le\varepsilon^{(3/2)-\sigma}\quad(n=3)\\ &|\nabla^2\zeta|_{L^2}\le\varepsilon^{(n^2-n-4/2n)-\sigma}\end{aligned}\Bigg\},$$

and for $n=2$, $\alpha>\alpha'$, $\sigma_2>\sigma>0$, $\varepsilon<\tilde\varepsilon$,

$$\mathcal{K}(\varepsilon, \sigma, \alpha) = \left\{ \zeta \in C_0^{k,\alpha}(M_0 \sharp_{T_\varepsilon} M_1, T_{G_\varepsilon} \tilde{N}) \ : \ \begin{aligned} &|\zeta|_{C^0} \leq \varepsilon^{(3/2)-\sigma} \\ &|\zeta|_{C^{2,\alpha}} \leq \varepsilon^{-(1/2)-\sigma-\alpha} \\ &|\zeta|_{H^1} \leq \varepsilon^{(3/2)-\sigma} \\ &|\nabla^2 \zeta|_{L^2} \leq \varepsilon^{(1/4)-\sigma} \end{aligned} \right\}.$$

It suffices to determine the constants $\sigma_n, \tilde{\varepsilon}$ so that $\mathcal{K}(\varepsilon, \sigma, \alpha)$ become invariant sets of $\mathcal{T}$.

Denote $V = -L^{-1}\tau u$, $W = -L^{-1}E\zeta$. We claim that there exist σ_n and constants K_i , J_i, K_i' , J_i' which depend on α, d, c_0, c_2', C, C''', C_0, C_1 and C_2, such that the following inequalities hold:

Case $n \geq 3$

(V1) $|V|_{C^0} \leq K_1 \varepsilon^{2-(2/n)-\theta}$, (W1) $|W|_{C^0} \leq J_1 \varepsilon^{2-(2/n)-\theta}$,
(V2) $|V|_{C^{2,\alpha}} \leq K_2 \varepsilon^{-(2/n)-\theta-\alpha}$, (W2) $|W|_{C^{2,\alpha}} \leq J_2 \varepsilon^{-(2/n)-\theta-\alpha}$,
(V3) $|V|_{H^1} \leq K_3 \varepsilon^{(n-1)/2-\theta}$ $(n > 3)$, (W3) $|W|_{H^1} \leq J_3 \varepsilon^{(n-1)/2-\theta}$ $(n > 3)$,
(V4) $|V|_{H^1} \leq K_4 \varepsilon^{(3/2)-\theta}$ $(n = 3)$, (W4) $|W|_{H^1} \leq J_4 \varepsilon^{(3/2)-\theta}$ $(n = 3)$,
(V5) $|\nabla^2 V|_{L^2} \leq K_5 \varepsilon^{(n^2-n-4/2n)-\theta}$, (W5) $|\nabla^2 W|_{L^2} \leq J_5 \varepsilon^{(n^2-n-4/2n)-\theta}$,

Case $n = 2$

(V1′) $|V|_{C^0} \leq K_1' \varepsilon^{(3/2)-\theta}$, (W1′) $|W|_{C^0} \leq J_1' \varepsilon^{(3/2)-\theta}$,
(V2′) $|V|_{C^{2,\alpha}} \leq K_2' \varepsilon^{-(1/2)-\theta-\alpha}$, (W2′) $|W|_{C^{2,\alpha}} \leq J_2' \varepsilon^{-(1/2)-\theta-\alpha}$,
(V3′) $|V|_{H^1} \leq K_3' \varepsilon^{(3/2)-\theta}$, (W3′) $|W|_{H^1} \leq J_3' \varepsilon^{(3/2)-\theta}$,
(V4′) $|\nabla^2 V|_{L^2} \leq K_4' \varepsilon^{(1/4)-\theta}$, (W4′) $|\nabla^2 W|_{L^2} \leq J_4' \varepsilon^{(1/4)-\theta}$,

for $\forall \zeta \in \mathcal{K}(\varepsilon, \sigma, \alpha)$, $\sigma < \sigma_n$, $\alpha < \alpha'$, $\varepsilon < \tilde{\varepsilon}_1 = \min(c_2, \varepsilon_1)$ and arbitrarily small positive θ. Assuming this claim, we choose $\tilde{\varepsilon} < \tilde{\varepsilon}_1 = \min(c_2, \varepsilon_1)$, such that

$$\max_{i \in \{1,\dots,5\}} (K_i + J_i) \leq \tilde{\varepsilon}^{-\sigma/2}, \quad \text{for } n \geq 3,$$

$$\max_{i \in \{1,\dots,4\}} (K_i' + J_i') \leq \tilde{\varepsilon}^{-\sigma/2}, \quad \text{for } n = 2.$$

It is easy to show that for $\sigma < \sigma_n$, $\varepsilon < \tilde{\varepsilon}$, $\alpha < \alpha'$, if $\zeta \in \mathcal{K}(\varepsilon, \sigma, \alpha)$, then $V + W \in \mathcal{K}(\varepsilon, \sigma, \alpha)$; i.e., $\mathcal{K}(\varepsilon, \sigma, \alpha)$ are invarant sets of $\mathcal{T}$.

Note that V does not depend on ζ (or $\mathcal{K}(\varepsilon, \sigma, \alpha)$) and that our L, (3.4), and (3.10) are analogous to L, (2.4), and (2.7) in [4]. Consequently, by Lemmas 1–5, we can derive (V1–V5), (V1′–V4′) in a similar way as that in [4] for any σ_n and $\tilde{\varepsilon} \leq \tilde{\varepsilon}_1$.

From now on, we denote the constants which depend on α, d, c_0, c_2', C, C''', C_0, C_1 and C_2 all by $\tilde{C}$, which may vary at different places.

By (b) in Lemmas 5 and 7, we have

$$\begin{aligned} |W|_{H^1} &\le \tilde{C}|E(\zeta)|_{L^2} \\ &\le \tilde{C}\left(\int (|\nabla^2\zeta|^2\,|\zeta|^2 + |\nabla\zeta|^4 + |\zeta|^4)\,\mathrm{d}v\right)^{1/2} \\ &\le \tilde{C}(|\zeta|_{C^0}|\nabla^2\zeta|_{L^2} + |\zeta|_{C^1}|\zeta|_{H^1}). \end{aligned}$$

Thus, by Lemma 2, we have

$$|W|_{H^1} \le \tilde{C}\varepsilon^{((n^2+n-4)/(2n))-2\sigma}, \quad \text{for } n > 3, \tag{4.6a}$$

$$|W|_{H^1} \le \tilde{C}\varepsilon^{(10/6)-2\sigma}, \quad \text{for } n = 3, \tag{4.6b}$$

$$|W|_{H^1} \le \tilde{C}\varepsilon^{(7/4)-2\sigma}, \quad \text{for } n = 2, \tag{4.6c}$$

for $\zeta \in \mathcal{K}(\varepsilon, \sigma, \alpha)$ and $\varepsilon < \tilde{\varepsilon}_1$. Therefore (W3), (W4), and (W3′) come from Equations (4.6a), (4.6b) and (4.6c) by setting

$$\sigma_n \le \frac{1}{n}, \quad J_3 = \tilde{C}, \quad \text{for } n > 3, \tag{4.7a}$$

$$\sigma_3 \le \frac{1}{12}, \quad J_4 = \tilde{C}, \tag{4.7b}$$

$$\sigma_2 \le \frac{1}{8}, \quad J_3' = \tilde{C}. \tag{4.7c}$$

We apply Lemma 1 to get

$$\begin{aligned} |W|_{C^0} &\le \tilde{c}_p|\Delta W|_{L^p}^{1-e_p} \\ &\le \tilde{C}(|LW|_{L^p} + |W|_{L^p})^{1-e_p} \\ &\le \tilde{C}(|E(\zeta)|_{L^p} + |W|_{L^p})^{1-e_p}. \end{aligned}$$

By Lemmas 2 and 7 and letting $p \to n/2$, we obtain, for $\zeta \in \mathcal{K}(\varepsilon, \sigma, \alpha)$,

$$\begin{aligned} |E(\zeta)|_{L^p} &\le \begin{cases} \tilde{C}\left(\int (|\nabla^2\zeta|^p\,|\zeta|^p + |\nabla\zeta|^{2p} + |\zeta|^{2p})\,\mathrm{d}v\right)^{1/p}, & n > 3, \\ |E(\zeta)|_{L^2}, & n = 3, \\ |E(\zeta)|_{L^2}, & n = 2, \end{cases} \\ &\le \begin{cases} \tilde{C}\big(|\zeta|_{C^0}|\nabla^2\zeta|_{C^0}^{(p-2)/p}|\nabla^2\zeta|_{L^2}^{2/p} \\ \quad + |\nabla\zeta|_{C^0}^{(2p-2)/p}|\nabla\zeta|_{L^2}^{2/p} + |\zeta|_{C^0}^{(2p-2)/p}|\zeta|_{L^2}^{2/p}\big), & n > 3, \\ \tilde{C}(|\zeta|_{C^0}|\nabla^2\zeta|_{L^2} + |\zeta|_{C^1}|\zeta|_{H^1}), & n = 3, \\ \tilde{C}(|\zeta|_{C^0}|\nabla^2\zeta|_{L^2} + |\zeta|_{C^1}|\zeta|_{H^1}), & n = 2. \end{cases} \end{aligned}$$

Thus

$$|E(\zeta)|_{L^p} \leq \begin{cases} \tilde{C}\varepsilon^{\frac{4n^2-10n+8}{n^2}-2\sigma}, & n > 3, \\ \tilde{C}\varepsilon^{\frac{5}{3}-2\sigma}, & n = 3, \\ \tilde{C}\varepsilon^{\frac{7}{4}-2\sigma}, & n = 2. \end{cases}$$

Besides, as $p \to n/2$, we have

$$\begin{aligned} |W|_{L^p} &\leq \begin{cases} |W|_{C^0}^{(p-2)/p}|W|_{L^2}^{2/p}, & n > 3, \\ |W|_{L^2}, & n = 3, \\ |W|_{L^2}, & n = 2, \end{cases} \\ &\leq \begin{cases} \tilde{C}|W|_{C^0}^{(p-2)/p}\varepsilon^{(2n^2+2n-8)/(n^2)-(8\sigma)n}, & n > 3, \\ \tilde{C}\varepsilon^{(5/3)-2\sigma}, & n = 3, \\ \tilde{C}\varepsilon^{(7/4)-2\sigma}, & n = 2. \end{cases} \end{aligned}$$

Hence, as $p \to n/2$, it follows that

$$|W|_{C^0} \leq \begin{cases} \tilde{C}(\varepsilon^{(4n^2-10n+8)/n^2-2\sigma} + |W|_{C^0}^{(p-2)/p}\varepsilon^{(2n^2+2n-8)/n^2-(8\sigma)/n}), & n > 3, \\ \tilde{C}\varepsilon^{(5/3)-2\sigma}, & n = 3, \\ \tilde{C}\varepsilon^{(7/4)-2\sigma}, & n = 2. \end{cases}$$

Applying Young's inequality $ab \leq \delta a^q + C_\delta b^r$ with

$$\begin{aligned} \delta &= 1/(2\tilde{C}), \quad a = |W|_{C^0}^{(p-2)/p}, \\ b &= \varepsilon^{(2n^2+2n-8)/n^2-(8\sigma)/n}, \quad q = \frac{p}{p-2}, \quad \text{and} \quad r = \frac{p}{2}, \end{aligned}$$

to the second term on the right-hand side of the inequality for the case $n > 3$, we then have

$$|W|_{C^0} \leq \begin{cases} \tilde{C}\varepsilon^{(4n^2-10n+8)/n^2-2\sigma}, & n > 3, \\ \tilde{C}\varepsilon^{(5/3)-2\sigma}, & n = 3, \\ \tilde{C}\varepsilon^{(7/4)-2\sigma}, & n = 2. \end{cases}$$

Thus, (W1), and (W1′) follow by setting

$$\sigma_n \leq \frac{n^2-4n+4}{n^2}, \quad J_1 = \tilde{C}, \quad \text{for } n > 3, \tag{4.8a}$$

$$\sigma_3 \leq \frac{1}{6}, \quad J_1 = \tilde{C}, \tag{4.8b}$$

$$\sigma_2 \leq \frac{1}{8}, \quad J_1' = \tilde{C}. \tag{4.8c}$$

And (W2), (W2′) come from Lemmas 3, 7, 2, the definition of $\mathcal{K}(\varepsilon, \sigma, \alpha)$, and

$$\sigma_n \leq 1 - \frac{1}{n}, \quad J_2 = \tilde{C}, \quad \text{for } n > 3, \tag{4.9a}$$

$$\sigma_3 \leq 1 - \frac{1}{3}, \quad J_2 = \tilde{C}, \tag{4.9b}$$

$$\sigma_2 \leq \frac{3}{4}, \quad J_2' = \tilde{C}. \tag{4.9c}$$

By standard elliptic theory, we have

$$|\nabla^2 W|_{L^2} \leq \tilde{C}(|LW|_{L^2} + |W|_{L^2}).$$

Plugging Equations (4.6a–4.6c) into the right-hand side of the above inequality and setting

$$\sigma_n \leq \frac{1}{2}, \quad J_5 = \tilde{C}, \quad \text{for } n > 3, \tag{4.10a}$$

$$\sigma_3 \leq \frac{2}{3}, \quad J_5 = \tilde{C}, \tag{4.10b}$$

$$\sigma_2 \leq \frac{3}{4}, \quad J_4' = \tilde{C}, \tag{4.10c}$$

we get (W5) and (W4′). From Equations (4.6a–4.10a), (4.6b–4.10b), (4.6c–4.10c), the claim is proved by choosing

$$\sigma_n = \frac{1}{n}, \quad \text{for } n > 3, \quad \sigma_3 = \frac{1}{12}, \quad \sigma_2 = \frac{1}{8}.$$

In the above process, we not only find the harmonic maps $F_\varepsilon\colon M_0 \sharp_{T_\varepsilon} M_1 \to \tilde{N}$, but we also show that F_ε converges to f_ι in $C^{1,\alpha''}$ on each compact subset of $M_\iota \backslash \gamma(\iota)$, for some positive $\alpha'' < 1$, $\iota = 0, 1$. Since M_ι, $\tilde{N}$, f_ι are smooth (up to the boundary), the proof of Theorem 1 is then complete by the standard potential theory. □

Remark 1. Suppose that $\dim \mathrm{M}_\iota \geq 3$, $\Pi\colon N \to \tilde{N}$ maps γ into $\tilde{\gamma}$, and $a > 0$. By formula (3.4), and Theorem 1, we can find a bridged map F_ε prescribing the same boundary value as Π on $\partial T_\varepsilon \backslash (B_{a/4}(\gamma(0) \cup B_{a/4}(\gamma(1))$, for $\forall \varepsilon < \tilde{\varepsilon}$ ($\tilde{\varepsilon}$ depends on a).

Remark 2. If $\dim M_\iota \geq 3$, for any smooth (or C^2-) extension map $G\colon M_0 \sharp_T M_1 \to \tilde{M}_0 \sharp_{\tilde{T}} \tilde{M}_1$ of f_0 and f_1, we can find a constant $\tilde{\varepsilon}$, and bridged harmonic maps $F_\varepsilon\colon M_0 \sharp_{T_\varepsilon} M_1 \to \tilde{M}_0 \sharp_{\tilde{T}_\varepsilon} \tilde{M}_1$, such that $F_\varepsilon = G$ on $\partial(M_0 \sharp_{T_\varepsilon} M_1)$, for $\forall \varepsilon < \tilde{\varepsilon}$. For the case of $\dim M_\iota = 2$, in order to solve the prescribing boundary value problem, we need to assume that the extension map G satisfies Equation (3.10).

5. An Example

A harmonic map $f : M \to \tilde{N}$ is called strictly stable, if the second variation of the energy functional $\mathcal{E}$ at f is positively definite. Since

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}(\exp_f(t\zeta))|_{t=0} &= \int_M \Big\langle -\Delta\zeta + \sum g^{ij}\,{}^{\tilde{N}}R(\nabla_i f,\zeta)\nabla_j f,\zeta\Big\rangle \mathrm{d}v \\ &= \int_M \langle -L^f\zeta,\zeta\rangle\,\mathrm{d}v,\end{aligned}$$

it follows that f is a strictly stable harmonic map if and only if $-L^f$ is positively definite, with $L^f = \Delta\zeta - \sum g^{ij}\,{}^{\tilde{N}}R(\nabla_i f,\zeta)\nabla_j f$.

COROLLARY 1. *Let M_0, M_1 be smooth compact submanifolds with boundary, of a Riemannian manifold N,* $\dim M_0 = \dim M_1 \geq 2$, $\dim N \geq 3$, *and f_ι: $M_\iota \to \tilde{N}$, be strictly stable harmonic maps, $\iota = 0, 1$, where $\tilde{N}$ is a Riemannian manifold,* $\dim \tilde{N} \geq 3$. *Suppose that $\gamma \subset N$, $\tilde{\gamma} \subset \tilde{N}$ are Jordan arcs, satisfying $\gamma \cap M_\iota = \gamma \cap \partial M_\iota = \gamma(\iota)$, $\tilde{\gamma}(\iota) = f_\iota(\gamma(\iota))$, $\iota = 0, 1$. Then, in any sufficiently small ε-neighborhood of γ, one can connect M_0 and M_1 at their boundaries by a bridge T_ε, and find a strictly stable harmonic map F_ε from the bridged manifold $M_0\sharp_{T_\varepsilon}M_1$ to the ε-neighborhood of $f_0(M_0) \cup \tilde{\gamma} \cup f_1(M_1)$. Furthermore, as the width of T_ε shrinks to zero, $\{F_\varepsilon\}_{\varepsilon\to 0}$, converges to f_ι (in the C^k-topology) on each compact subset of $M_\iota \backslash \gamma(\iota)$, for $\iota = 0, 1$.*

Proof. By Theorem 1 and the above observation, we only need to show that $-L^{F_\varepsilon}$ is positively definite for ε sufficiently small. This comes from Theorem 1 and a similar argument as in the proof of Lemma 4. More precisely, suppose that there does not exist $\delta > 0$, such that $-L^{F_\varepsilon}$ is positively definite for $0 < \varepsilon < \delta$, we then can find $\varepsilon_i \to 0$, $\lambda_{\varepsilon_i} \geq 0$, and ζ_{ε_i}, satisfying $-L^{F_{\varepsilon_i}}\zeta_{\varepsilon_i} = -\lambda_{\varepsilon_i}\zeta_{\varepsilon_i}$, and $|\zeta_{\varepsilon_i}|_{L^2} = 1$. Note that

$$\begin{aligned}-\lambda_{\varepsilon_i} &= -\int_{M_0\sharp_{T_{\varepsilon_i}}M_1} \Big\langle \Delta\zeta_{\varepsilon_i} - \sum g^{ij}\,{}^{\tilde{N}}R(\nabla_i F_{\varepsilon_i},\zeta_{\varepsilon_i})\nabla_j F_{\varepsilon_i},\ \zeta_{\varepsilon_i}\Big\rangle\,\mathrm{d}v \\ &\geq \int_{M_0\sharp_{T_{\varepsilon_i}}M_1} (|\nabla\zeta_{\varepsilon_i}|^2 - C|\zeta_{\varepsilon_i}|^2)\,\mathrm{d}v \\ &\geq -C.\end{aligned}$$

Following a similar argument as in the proof of Theorem 1, one can show that $|\zeta_{\varepsilon_i}|_{C^0}$ and $|\zeta_{\varepsilon_i}|_{H^1}$ have a uniform upper bound. Consequently, the L^2-norm of the restriction of ζ_{ε_i} on M_0, denoted by $|\zeta_{\varepsilon_i}|_{L^2;M_0}$, is greater than 1/3.

For $\dim M_\iota = n \geq 3$, define ϕ_l to be the smooth functions satisfying

$$\phi_l(x) = \begin{cases} 1, & x \in M_0\backslash B_{1/l}(\gamma(0)), \\ 0, & x \in M_0\sharp_{T_{\varepsilon_i}}M_1\backslash\,(M_0\backslash B_{1/2l}), \end{cases}$$

$$|\nabla\phi_l| \leq 4l, \quad |\nabla^2\phi_l| \leq 4l^2. \tag{5.1}$$

Then

$$\int_{M_0\sharp_{T_{\varepsilon_i}} M_1} \phi_l \zeta_{\varepsilon_i}\, L^{F_{\varepsilon_i}} \zeta_{\varepsilon_i}\, \mathrm{d}v = \lambda_{\varepsilon_i} \int_{M_0\sharp_{T_{\varepsilon_i}} M_1} \phi_l \zeta_{\varepsilon_i}\, \zeta_{\varepsilon_i}\, \mathrm{d}v \geq 0. \tag{5.2}$$

On the other hand, by the self-adjointness of $L^{F_{\varepsilon_i}}$, Theorem 1, $|\zeta_{\varepsilon_i}|_{L^2;M_0} > 1/3$, and the stability of f_0, we have

$$\begin{aligned}
&\underline{\lim}_{i,l\to\infty} \int_{M_0\sharp_{T_{\varepsilon_i}} M_1} \phi_l \zeta_{\varepsilon_i}\, L^{F_{\varepsilon_i}} \zeta_{\varepsilon_i}\, \mathrm{d}v \\
&\quad = \underline{\lim}_{i,l\to\infty} \int_{M_0\sharp_{T_{\varepsilon_i}} M_1} L^{F_{\varepsilon_i}} (\phi_l \zeta_{\varepsilon_i})\, \zeta_{\varepsilon_i}\, \mathrm{d}v \\
&\quad = \underline{\lim}_{i,l\to\infty} \int_{M_0\sharp_{T_{\varepsilon_i}} M_1} L^{G_{\varepsilon_i}} (\phi_l \zeta_{\varepsilon_i})\, \zeta_{\varepsilon_i}\, \mathrm{d}v \\
&\quad = \underline{\lim}_{i,l\to\infty} \int_{M_0} L^{f_0} (\phi_l \zeta_{\varepsilon_i})\, \zeta_{\varepsilon_i}\, \mathrm{d}v \\
&\quad = \underline{\lim}_{i,l\to\infty} \int_{M_0} L^{f_0} (\phi_l \zeta_{\varepsilon_i})\, \phi_l \zeta_{\varepsilon_i}\, \mathrm{d}v + I \\
&\quad \leq -\frac{d_0}{9} + I,
\end{aligned} \tag{5.3}$$

where the first term of the last inequality is obtained by L^{f_0}-eigenspaces decomposition,

$$-d_0 = \sup(\operatorname{spec} L^{f_0}) < 0 \quad \text{and} \quad I = \lim_{i,l\to\infty} \int_{M_0} L^{f_0}(\phi_l \zeta_{\varepsilon_i})(1-\phi_l)\zeta_{\varepsilon_i}\, \mathrm{d}v.$$

Moreover, by Equation (5.1), we have

$$\begin{aligned}
|I| &= \lim_{i,l\to\infty} \int_{M_0} |L^{f_0}(\phi_l \zeta_{\varepsilon_i})(1-\phi_l)\zeta_{\varepsilon_i}|\, \mathrm{d}v \\
&= \lim_{i,l\to\infty} \int_{M_0} |L^{G_{\varepsilon_i}}(\phi_l \zeta_{\varepsilon_i})(1-\phi_l)\zeta_{\varepsilon_i}|\, \mathrm{d}v \\
&\leq \lim_{i,l\to\infty} \int_{M_0} |\phi_l (1-\phi_l) L^{F_{\varepsilon_i}} \zeta_{\varepsilon_i}\, \zeta_{\varepsilon_i}|\, \mathrm{d}v + \\
&\qquad + \lim_{i,l\to\infty} \int_{M_0} |\nabla \phi_l|\, |\nabla \zeta_{\varepsilon_i}|(1-\phi_l)|\zeta_{\varepsilon_i}|\, \mathrm{d}v + \\
&\qquad + \lim_{i,l\to\infty} \int_{M_0} |\Delta \phi_l|\, |\zeta_{\varepsilon_i}|^2 (1-\phi_l)\, \mathrm{d}v
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{i,l\to\infty}(\lambda_{\varepsilon_i}|\zeta_{\varepsilon_i}|^2_{C^0}\,\mathrm{vol}\,(B_{1/l}) + |\nabla\phi_l|_{C^0}|\zeta_{\varepsilon_i}|_{C^0}|\nabla\zeta_{\varepsilon_i}|_{L^2}\,\mathrm{vol}\,(B_{1/l})^{1/2}) + \\
&\quad + \lim_{i,l\to\infty}|\Delta\phi_l|_{C^0}|\zeta_{\varepsilon_i}|^2_{C^0}\,\mathrm{vol}\,(B_{1/l}) \\
&\leq \lim_{l\to\infty} C'(l^{-n} + l^{1-(n/2)} + l^{2-n}) \\
&= 0.
\end{aligned} \tag{5.4}$$

Combining Equations (5.2–5.4), we get a contradiction.

To show the case for dim $M_\iota = 2$, we replace Equation (5.1) by

$$\phi_l(x) = \begin{cases} 1, & x \in M_0\backslash B_{1/l}(\gamma(0)), \\ 0, & x \in M_0\sharp_{T_{\varepsilon_i}} M_1\backslash\,(M_0\backslash B_{1/l^2}(\gamma(0))), \\ \dfrac{\log r\cdot l^2}{\log l}, & \text{otherwise}, \end{cases} \tag{5.1'}$$

with $r(x)$ being the distance from $\gamma(0)$ to x. Note that $\phi_l \in H^1$ and $|\nabla\phi_l|_{L^2} \to 0$. Thus following Equation (5.4), and applying integration by parts to the $\Delta\phi_l$-term, we get

$$\begin{aligned}
|I| &\leq \lim_{i,l\to\infty}(\lambda_{\varepsilon_i}|\zeta_{\varepsilon_i}|^2_{C^0}\,\mathrm{vol}\,(B_{1/l}) + |\nabla\phi_l|_{L^2}|\nabla\zeta_{\varepsilon_i}|_{L^2}|\zeta_{\varepsilon_i}|_{C^0} + |\nabla\phi_l|_{L^2}|\zeta_{\varepsilon_i}|^2_{C^0}) \\
&= 0,
\end{aligned}$$

which will also give a contradiction. □

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