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# UNIQUE CONTINUATION AND PERSISTENCE PROPERTIES OF SOLUTIONS OF THE 2-COMPONENT DEGASPERIS-PROCESI EQUATIONS*

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**Abstract** In this article, the unique continuation and persistence properties of solutions of the 2-component Degasperis-Procesi equations are discussed. It is shown that strong solutions of the 2-component Degasperis-Procesi equations, initially decaying exponentially together with its spacial derivative, must be identically equal to zero if they also decay exponentially at a later time.

**Key words** 2-component Degasperis-Procesi equation; unique continuation property; persistence property

**2000 MR Subject Classification** 35B30; 35G25; 35L05

## 1 Introduction

In this article, we consider the following initial value problem (IVP) of the 2-component generalization of the Degasperis-Procesi equations:

$$\begin{cases} u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx} + k_1\rho\rho_x, t > 0, & x \in \mathbb{R}, \\ \rho_t + k_2(\rho u)_x + k_3\rho u_x = 0, t > 0, & x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $k_1, k_2, k_3 \in \mathbb{R}$ are fixed constants and there are at least one of $k_2, k_3 \neq 0$. (1.1) was quite recently proposed by Popowicz in [15], the construction was based on the observation that the second Hamiltonian operator of the Degasperis-Procesi equation could be considered as the Dirac reduced Poisson tensor of the second Hamiltonian operator of the Boussinesq equation.

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For the 2-component Degasperis-Procesi equation (1.1), well-posedness and blow-up phenomena seem not to have been discussed. For $\rho \equiv 0$, equation (1.1) becomes the Degasperis-Procesi equation, modeling nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. Escher et al in [7] demonstrated that, given $u_0(x) \in H^s(\mathbb{R}), s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique solution to the Degasperis-Procesi equation, such that

$$u(t,x) \in C([0,T); H^s(\mathbb{R})) \times C^1([0,T); H^{s-1}(\mathbb{R})).$$

In [19], Zhou proved several blow-up results for the integrable Degasperis-Procesi equation.

Considering the well-posedness of the IVP of the 2-component Camassa-Holm equation

$$\begin{cases} u_t - u_{xxt} + ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} + \rho\rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $k \in \mathbb{R}$ is a fixed constant, was derived in [6], just following line by line what was done for (1.2), we can establish the following well-posedness theorem for (1.1) similarly.

**Theorem 1.1** Given $(u_0(x), \rho_0(x)) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s \geq 2$, then there exist a maximal $T > 0$ and a unique solution $(u(t,x), \rho(t,x))$ to (1.1), such that

$$(u(t,x), \rho(t,x)) \in C([0,T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})).$$

The proof follows from Kato's theory [11] directly, as it was applied to the 2-component Camassa-Holm equation in [6]. So, to be concise, we omit the detailed proof. For other properties of solutions to (1.2), we can refer to the reference [8].

Our objective here is to formulate decay conditions on a solution at two different times which guarantee that $u \equiv 0, \rho \equiv 0$ is the unique solution of (1.1). The idea of proving unique continuation results of nonlinear dispersive equations under decay assumptions of the solution at two different times was motivated by the recent works of Escauriaza et al in [4, 5] on the nonlinear Schrödinger and the k-generalized KdV equations respectively. More recently, Himonas et al in [9] proved the persistence and unique continuation properties of the Camassa-Holm equation.

In this article, we are concerned with the unique continuation property for (1.1). Unique continuation property (UCP) says, if a solution $u$ or vector $\mathbf{u}$ to certain evolution equation or system vanishes on some nonempty open subset $\mathcal{O}$ of $\Omega$, then, it vanishes in the horizontal component of $\mathcal{O}$, where $\Omega$ is the domain of the evolution operator under consideration.

A pioneer work in this direction is due to Carleman [2]. His method was based on the weighted estimates for the associated solutions. Later, Carleman's method was improved and extended to address the UCP for various nonlinear PDEs. Using Carleman type estimates, Saut and Scheurer [16] proved the UCP for a general class of dispersive equations in one space dimension. In particular, the KdV equation falls in the class considered in [16]. Also, Tataru [17] proved the UCP for Schrödinger equation by deriving some Carleman type estimates. Isakov [10] considered a large class of evolution equations with nonhomogeneous principal part and proved the UCP. B. Y. Zhang [18] used inverse scattering transform and the results from Hardy function theory to verify that the solution to the KdV equation cannot be supported in the horizontal half rays at two different moments unless it vanishes identically. He also proved the same result for the modified KdV equation by the Miura transformation. This slightly stronger result implies the UCP for the KdV equation obtained in [16]. Recently, Bourgain [1] applied a different approach and proved that, if a solution u to a dispersive equation has compact support in a nontrivial time interval $I = [t_1, t_2]$, then u vanishes identically, using complex variables technique along with Paley-Wiener theorem. More recently, Kenig [12] introduced a new method and proved that, if a sufficiently smooth solution $u$ to a generalized KdV equation is supported in a half ray at two different instants of time, then u vanishes identically. Exponential decay property of solution is essential in the arguments employed in [12]. The Bourgain's approach was used by Panthee et al in [3, 14] to verify the UCP for Zakharov-Kuznetsov equation of the higher spatial dimensions and a mixed equation of type KdV and Schrödinger. Also, there is a recent work due to Kenig [13] dealing with the UCP for Schrödinger equation.

Before presenting our main results, we give the following notations.

Notion: We shall say that for any $\alpha \in \mathbb{R}$,

$$|f(x)| \sim O(\mathrm{e}^{\alpha x}) \quad \text{as} \quad x \uparrow \infty \quad \text{if} \quad \lim_{x\to\infty} \frac{|f(x)|}{\mathrm{e}^{\alpha x}} = L \neq 0,$$

and

$$|f(x)| \sim o(\mathrm{e}^{\alpha x}) \quad \text{as} \quad x \uparrow \infty \quad \text{if} \quad \lim_{x\to\infty} \frac{|f(x)|}{\mathrm{e}^{\alpha x}} = 0.$$

For the 2-component Degasperis-Procesi equation (1.1), the main results of this work read as follows.

**Theorem 1.2** Assume that, for some $T > 0$ and $s \geq 2$,

$$(u(t,x), \rho(t,x)) \in C([0,T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$$

is a strong solution of the initial value problem (IVP) associated to the equation (1.1). If $u_0(x) = u(0,x), \rho_0(x) = \rho(0,x)$ satisfies that, for some $\theta \in (0,1)$ and any $k_1 \in \mathbb{R}$,

$$|u_0(x)|, |\partial_x u_0(x)|, |\rho_0(x)| \sim O(\mathrm{e}^{-\theta x}) \quad \text{as} \quad x \uparrow \infty,$$

then,

$$|u(t,x)|, |\partial_x u(t,x)|, |\rho(t,x)| \sim O(\mathrm{e}^{-\theta x}) \quad \text{as} \quad x \uparrow \infty,$$

uniformly in the time interval $[0,T]$.

This result is concerned with some persistence properties of the solution of (1.1) in $L^\infty$-spaces with exponential weights. Our main result on unique continuation property is based on the above persistence property, which reads as follows.

**Theorem 1.3** Assume that, for some $T > 0$ and $s \geq 2$,

$$(u(t,x), \rho(t,x)) \in C([0,T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$$

is a strong solution of the IVP associated to (1.1). If for some $\alpha \in (\frac{1}{2}, 1)$,

(1) For $k_1 < 0$,

$$|u_0(x)| \sim o(\mathrm{e}^{-x}), |\partial_x u_0(x)| \sim O(\mathrm{e}^{-\alpha x}), |\rho_0(x)| \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,$$

and there exists $t_1 \in (0, T]$ such that

$$|u(t_1, x)| \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty,$$

then,

$$u \equiv 0, \quad \rho \equiv 0.$$

(2) For $k_1 = 0$,

$$|u_0(x)| \sim o(\mathrm{e}^{-x}), |\partial_x u_0(x)| \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,$$

and there exists $t_1 \in (0, T]$ such that

$$|u(t_1, x)| \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty,$$

then,

$$u \equiv 0, \quad \rho(t, x) = \rho_0(x).$$

The following result establishes the optimality of Theorem 1.3. At the same time, a strong non-trivial solution $u(t, x)$ of (1.1), that is, the first flow of (1.1), corresponding to data with fast decay at infinity will immediately behave asymptotically in the $x$-variable at infinity, as the "peakon" solution.

**Theorem 1.4** Assume that, for some $T > 0$ and $s \geq 2$,

$$(u(t, x), \rho(t, x)) \in C([0, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$$

is a strong solution of the IVP associated to (1.1). If $u_0(x) = u(0, x), \rho_0(x) = \rho(0, x)$ satisfy that, for some $\alpha \in (\frac{1}{2}, 1)$ and any $k_1 \in \mathbb{R}$,

$$|u_0(x)| \sim O(\mathrm{e}^{-x}), |\partial_x u_0(x)| \sim O(\mathrm{e}^{-\alpha x}), |\rho_0(x)| \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,$$

then,

$$|u(t, x)| \sim O(\mathrm{e}^{-x}), |\rho(t, x)| \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty,$$

uniformly in the time interval $[0, T]$.

**Notation** In the following, we denote by $*$ the spatial convolution. Given a Lebesgue space $L^p(\mathbb{R})$, we denote its norm by $\|\cdot\|_p$. Because all spaces of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. In contrast, if there is no particular emphasis, all spatial variable of functions is $x$, for simplicity, we drop $x$ in our notations of functions if there is no ambiguity.

It is convenient to rewrite the first equation of (1.1) in its formally equivalent integral-differential form

$$u_t + uu_x + \partial_x G * \left(\frac{3}{2}u^2 - \frac{k_1}{2}\rho^2\right) = 0, \tag{1.3}$$

where $G(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. For any $s \in \mathbb{R}$, the integral operator

$$\mathcal{D} = (I - \partial_x^2)^{-1} : H^s \longrightarrow H^{s+2}$$

defines a bounded linear operator on the indicated Sobolev space. Moreover,

$$\mathcal{D}(f) = (G * f)(x) = \int_{\mathbb{R}} G(x - y) f(y) \mathrm{d}y, \quad \forall f(x) \in H^s.$$

The remainder of this article is organized as follows. In Section 2, we prove Theorem 1.2 with regard to the persistence property for (1.1). The proofs of Theorems 1.3 and 1.4 will be presented in Section 3.

## 2 Proofs of Theorem 1.2

Our proof is motivated by the idea of article [9].

**Proof** For simplicity, we introduce the following notations:

$$F(u,\rho)=\frac{3}{2}u^2-\frac{k_1}{2}\rho^2,$$
$$M=\sup_{t\in[0,T]}\|u(t)\|_{H^s},$$

and

$$K=|k_2|+|k_3|>0.$$

First, multiplying the second equation in (1.1) by $\rho^{2p-1}$ ($p\in\mathbb{Z}^+$), we get

$$\int_{\mathbb{R}}\rho^{2p-1}\rho_t\mathrm{d}x+(k_2+k_3)\int_{\mathbb{R}}\rho^{2p-1}\rho u_x\mathrm{d}x+k_2\int_{\mathbb{R}}\rho^{2p-1}\rho_x u\mathrm{d}x=0. \tag{2.1}$$

The estimates

$$\begin{aligned}\int_{\mathbb{R}}\rho^{2p-1}\rho_t\mathrm{d}x&=\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{2p}^{2p}=\|\rho(t)\|_{2p}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{2p},\\
\int_{\mathbb{R}}\rho^{2p-1}\rho u_x\mathrm{d}x&\le\|u_x(t)\|_\infty\|\rho(t)\|_{2p}^{2p},\\
\int_{\mathbb{R}}\rho^{2p-1}\rho_x u\mathrm{d}x&=\frac{1}{2p}\int_{\mathbb{R}}u(\rho^{2p})_x\mathrm{d}x=-\frac{1}{2p}\int_{\mathbb{R}}\rho^{2p}u_x\mathrm{d}x\\
&\le\frac{1}{2p}\|u_x(t)\|_\infty\|\rho(t)\|_{2p}^{2p}\end{aligned}$$

and Hölder's inequality in (2.1) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{2p}&\le2(|k_2|+|k_3|)\|u_x(t)\|_\infty\|\rho(t)\|_{2p}\\
&=2K\|u_x(t)\|_\infty\|\rho(t)\|_{2p}.\end{aligned}$$

For any $m\ge\frac{1}{2}$, the Sobolev embedding theorem $H^m\hookrightarrow L^\infty$ holds. Thus, $\|u_x(t)\|_\infty\le\|u(t)\|_{H^s}\le M$ holds for $s\ge2$. By Gronwall's inequality and Sobolev embedding theorem, one finds that

$$\|\rho(t)\|_{2p}\le\|\rho_0\|_{2p}\mathrm{e}^{2KMt}.$$

Taking the limit as $p\uparrow\infty$, we obtain

$$\|\rho(t)\|_\infty\le\|\rho_0\|_\infty\mathrm{e}^{2KMt}.$$

Multiplying the equation (1.3) by $u^{2p-1}$ with $p\in\mathbb{Z}^+$ and integrating the result in the $x$-variable, one has

$$\int_{\mathbb{R}}u^{2p-1}u_t\mathrm{d}x+\int_{\mathbb{R}}u^{2p-1}uu_x\mathrm{d}x+\int_{\mathbb{R}}u^{2p-1}\partial_xG*F(u,\rho)\mathrm{d}x=0. \tag{2.2}$$

The estimates
$$\int_{\mathbb{R}} u^{2p-1}u_t\mathrm{d}x = \frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2p}^{2p} = \|u(t)\|_{2p}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2p}$$
and
$$\left|\int_{\mathbb{R}} u^{2p-1}uu_x\mathrm{d}x\right| \le \|u_x(t)\|_\infty\|u(t)\|_{2p}^{2p},$$
and Hölder's inequality in (2.2) yield
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{2p} \le \|u_x(t)\|_\infty\|u(t)\|_{2p} + \|\partial_x G * F(u,\rho)(t)\|_{2p}.$$

By Gronwall's inequality and Sobolev embedding theorem, we obtain
$$\|u(t)\|_{2p} \le (\|u(0)\|_{2p} + \int_0^t \|\partial_x G * F(u,\rho)(\tau)\|_{2p}\mathrm{d}\tau)\mathrm{e}^{Mt}. \tag{2.3}$$
Because $f \in L^2 \cap L^\infty$ implies
$$\lim_{q\uparrow\infty}\|f\|_q = \|f\|_\infty,$$
taking the limits in (2.3) (note that $\partial_x G \in L^1, F(u,\rho) \in L^1 \cap L^\infty$), we obtain
$$\|u(t)\|_\infty \le (\|u(0)\|_\infty + \int_0^t \|\partial_x G * F(u,\rho)(\tau)\|_\infty\mathrm{d}\tau)\mathrm{e}^{Mt}.$$
Next, differentiating (1.3) in the $x$-variable yields
$$u_{xt} + uu_{xx} + u_x^2 + \partial_x^2 G * F(u,\rho) = 0. \tag{2.4}$$
Again, multiplying equation (2.4) by $u_x^{2p-1}$ ($p \in \mathbb{Z}^+$), integrating the result in the $x$-variable and using integration by parts, we obtain
$$\int_{\mathbb{R}} uu_{xx}u_x^{2p-1}\mathrm{d}x = \frac{1}{2p}\int_{\mathbb{R}} u(u_x^{2p})_x\mathrm{d}x = -\frac{1}{2p}\int_{\mathbb{R}} u_x^{2p+1}\mathrm{d}x,$$
and so,
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(t)\|_{2p} \le 2\|u_x(t)\|_\infty\|u_x(t)\|_{2p} + \|\partial_x^2 G * F(u,\rho)(t)\|_{2p}.$$
Therefore, as before,
$$\|u_x(t)\|_{2p} \le (\|u_x(0)\|_{2p} + \int_0^t \|\partial_x^2 G * F(u,\rho)(\tau)\|_{2p}\mathrm{d}\tau)\mathrm{e}^{2Mt}.$$
Because $\partial_x^2 G * f = G * f - f$ holds for any $f$, we can take the limit as $p \uparrow \infty$ to obtain
$$\|u_x(t)\|_\infty \le (\|u_x(0)\|_\infty + \int_0^t \|\partial_x^2 G * F(u,\rho)(\tau)\|_\infty\mathrm{d}\tau)\mathrm{e}^{2Mt}.$$
We shall repeat the above arguments using the weight
$$\varphi_N(x) = \begin{cases} 1, & x \le 0, \\ \mathrm{e}^{\theta x}, & x \in (0,N), \\ \mathrm{e}^{\theta N}, & x \ge N. \end{cases}$$

For all $N>0$, we have $0\le \varphi'_N(x)\le \varphi_N(x)$. Multiplying (1.3) and (2.4) by $\varphi_N(x)$, we can obtain

$$\partial_t(u\varphi_N)+(u\varphi_N)u_x+\varphi_N\partial_xG*F(u,\rho)=0, \tag{2.5}$$

$$\partial_t(u_x\varphi_N)+(u_x\varphi_N)u_x+uu_{xx}\varphi_N+\varphi_N\partial_x^2G*F(u,\rho)=0. \tag{2.6}$$

Again, multiplying (2.5) by $(u\varphi_N)^{2p-1}$ and (2.6) by $(u_x\varphi_N)^{2p-1}$, respectively, then integrating the result in $x$-variable, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}}u_{xx}u\varphi_N(u_x\varphi_N)^{2p-1}\mathrm{d}x\Big| &= \Big|\int_{\mathbb{R}}u(u_x\varphi_N)^{2p-1}[\partial_x(u_x\varphi_N)-u_x\varphi'_N]\mathrm{d}x\Big|\\ &= \Big|\frac{1}{2p}\int_{\mathbb{R}}u\partial_x[(u_x\varphi_N)^{2p}]\mathrm{d}x-\int_{\mathbb{R}}uu_x\varphi'_N(u_x\varphi_N)^{2p-1}\mathrm{d}x\Big|\\ &= \Big|\frac{1}{2p}\int_{\mathbb{R}}u_x(u_x\varphi_N)^{2p}\mathrm{d}x+\int_{\mathbb{R}}uu_x\varphi'_N(u_x\varphi_N)^{2p-1}\mathrm{d}x\Big|\\ &\le 2(\|u(t)\|_\infty+\|u_x(t)\|_\infty)\|u_x\varphi_N\|_{2p}^{2p},\end{aligned}$$

after integration by parts. Hence, as in weightless case, we obtain

$$\begin{aligned}\|u\varphi_N\|_\infty+\|u_x\varphi_N\|_\infty &\le \mathrm{e}^{2Mt}(\|u_0\varphi_N\|_\infty+\|u_x(0)\varphi_N\|_\infty)\\ &\quad+\mathrm{e}^{2Mt}\int_0^t(\|\varphi_N\partial_xG*F(u,\rho)\|_\infty+\|\varphi_N\partial_x^2G*F(u,\rho)\|_\infty)\mathrm{d}\tau.\end{aligned} \tag{2.7}$$

Multiplying the second equation in (1.1) by $\varphi_N$, we have

$$(\rho\varphi_N)_t+(k_2+k_3)u_x(\rho\varphi_N)+k_2\rho_xu\varphi_N=0.$$

Again, multiplying the above equation by $(\rho\varphi_N)^{2p-1}$, then integrating the result in $x$-variable and using integration by parts, we obtain

$$\int_{\mathbb{R}}(\rho\varphi_N)^{2p-1}(\rho\varphi_N)_t\mathrm{d}x=\|\rho\varphi_N\|_{2p}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\varphi_N\|_{2p},$$

$$\begin{aligned}\int_{\mathbb{R}}\rho_xu\varphi_N(\rho\varphi_N)^{2p-1}\mathrm{d}x &= \int_{\mathbb{R}}u(\rho\varphi_N)^{2p-1}[\partial_x(\rho\varphi_N)-\rho\varphi'_N]\mathrm{d}x\\ &= \frac{1}{2p}\int_{\mathbb{R}}u\partial_x(\rho\varphi_N)^{2p}\mathrm{d}x-\int_{\mathbb{R}}u\rho\varphi'_N(\rho\varphi_N)^{2p-1}\mathrm{d}x\\ &= -\frac{1}{2p}\int_{\mathbb{R}}u_x(\rho\varphi_N)^{2p}\mathrm{d}x-\int_{\mathbb{R}}u\rho\varphi'_N(\rho\varphi_N)^{2p-1}\mathrm{d}x.\end{aligned}$$

Therefore, by Hölder's inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\varphi_N\|_{2p} &\le 2(|k_2|+|k_3|)(\|u\|_\infty+\|u_x\|_\infty)\|\rho\varphi_N\|_{2p}\\ &= 2K(\|u\|_\infty+\|u_x\|_\infty)\|\rho\varphi_N\|_{2p}.\end{aligned}$$

So,

$$\|\rho\varphi_N\|_{2p}\le \mathrm{e}^{2KMt}\|\rho_0\varphi_N\|_{2p}.$$

Letting $p\uparrow\infty$, one finds that

$$\|\rho\varphi_N\|_\infty\le \mathrm{e}^{2KMt}\|\rho_0\varphi_N\|_\infty. \tag{2.8}$$

A calculation shows that there exists $c > 0$, depending on $\theta \in (0,1)$, such that, for any $N \in \mathbb{Z}^+$,

$$\varphi_N(x)\int_{\mathbb{R}} \mathrm{e}^{-|x-y|}\frac{1}{\varphi_N(y)}\mathrm{d}y \le c.$$

Thus, for $F(u,\rho)$, we obtain

$$\begin{aligned}|\varphi_N\partial_x G * F(u,\rho)| &= \left|\frac{1}{2}\varphi_N(x)\int_{\mathbb{R}}\mathrm{sgn}(x-y)\mathrm{e}^{-|x-y|}\left(\frac{3}{2}u^2-\frac{k_1}{2}\rho^2\right)\mathrm{d}y\right|\\ &\le \frac{1}{2}\varphi_N(x)\left|\int_{\mathbb{R}}\mathrm{e}^{-|x-y|}\frac{1}{\varphi_N(y)}\varphi_N(y)\left(\frac{3}{2}u^2-\frac{k_1}{2}\rho^2\right)\mathrm{d}y\right|\\ &\le c(\|\varphi_N u\|_\infty\|u\|_\infty+\|\rho\varphi_N\|_\infty\|\rho\|_\infty).\end{aligned}$$

Using $\partial_x^2 G * f = G * f - f$ for any $f$ again, one sees that

$$\begin{aligned}|\varphi_N\partial_x^2 G * F(u,\rho)| &= \left|\frac{1}{2}\varphi_N(x)\int_{\mathbb{R}}\mathrm{e}^{-|x-y|}F(u,\rho)\mathrm{d}y-\varphi_N(x)\left(\frac{3}{2}u^2-\frac{k_1}{2}\rho^2\right)(x)\right|\\ &\le \frac{1}{2}\varphi_N(x)\left|\int_{\mathbb{R}}\mathrm{e}^{-|x-y|}\frac{1}{\varphi_N(y)}\varphi_N(y)\left(\frac{3}{2}u^2-\frac{k_1}{2}\rho^2\right)\mathrm{d}y\right|\\ &\quad+\left|\varphi_N(x)\left(\frac{3}{2}u^2-\frac{k_1}{2}\rho^2\right)(x)\right|\\ &\le c(\|\varphi_N u\|_\infty\|u\|_\infty+\|\rho\varphi_N\|_\infty\|\rho\|_\infty).\end{aligned}$$

Thus, by inserting the above estimates into (2.7) and using (2.8), it follows that there exists a constant $C = C(M,T,K) > 0$, such that

$$\begin{aligned}&\|\varphi_N u\|_\infty+\|\varphi_N u_x\|_\infty+\|\varphi_N\rho\|_\infty\\ \le\ & C(\|\varphi_N u_0\|_\infty+\|\varphi_N u_x(0)\|_\infty+\|\varphi_N\rho_0\|_\infty)\\ &+C\int_0^t(\|u\|_\infty+\|u_x\|_\infty+\|\rho\|_\infty)(\|\varphi_N u\|_\infty+\|\varphi_N u_x\|_\infty+\|\varphi_N\rho\|_\infty)\mathrm{d}\tau.\end{aligned}$$

Hence, for any $N \in \mathbb{Z}^+$ and any $t \in [0,T]$, we have

$$\begin{aligned}&\|\varphi_N u\|_\infty+\|\varphi_N u_x\|_\infty+\|\varphi_N\rho\|_\infty\\ &\le C(\|\varphi_N u_0\|_\infty+\|\varphi_N u_x(0)\|_\infty+\|\varphi_N\rho_0\|_\infty)\\ &\le C(\|\mathrm{e}^{\theta x}u_0\|_\infty+\|\mathrm{e}^{\theta x}u_x(0)\|_\infty+\|\mathrm{e}^{\theta x}\rho_0\|_\infty).\end{aligned}$$

Finally, taking the limit as $N$ goes to infinity, we obtain

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\mathrm{e}^{\theta x}u\|_\infty+\|\mathrm{e}^{\theta x}u_x\|_\infty+\|\mathrm{e}^{\theta x}\rho\|_\infty)\\ &\le C(\|\mathrm{e}^{\theta x}u_0\|_\infty+\|\mathrm{e}^{\theta x}u_x(0)\|_\infty+\|\mathrm{e}^{\theta x}\rho_0\|_\infty).\end{aligned}$$

Therefore,

$$|u(t,x)|, |\partial_x u(t,x)|, |\rho(t,x)| \sim O(\mathrm{e}^{-\theta x}) \quad \text{as} \quad x \uparrow \infty,$$

uniformly in the time interval $[0,T]$.

## 3 Proofs of Theorems 1.3 and 1.4

In this section, we are dedicated to prove the unique continuation property and its optimality.

### 3.1 Proof of Theorem 1.3

**Proof** For the case $k_1 < 0$, integrating equation (1.3) over the time interval $[0, t_1]$, we have

$$u(x,t_1) - u(x,0) + \int_0^{t_1} uu_x(x,\tau)\mathrm{d}\tau = -\int_0^{t_1} \partial_x G * \left(\frac{3}{2}u^2 - \frac{k_1}{2}\rho^2\right)\mathrm{d}\tau.$$

By hypothesis of Theorem 1.3, we find that

$$u(x,t_1) - u(x,0) \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

From Theorem 1.2, it follows that

$$\int_0^{t_1} uu_x(x,\tau)\mathrm{d}x \sim O(\mathrm{e}^{-2\alpha x}) \quad \text{as} \quad x \uparrow \infty,$$

and so,

$$\int_0^{t_1} uu_x(x,\tau)\mathrm{d}x \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

For the right-hand side, we have

$$\int_0^{t_1} \partial_x G * \left(\frac{3}{2}u^2 - \frac{k_1}{2}\rho^2\right)\mathrm{d}\tau = \partial_x G * \int_0^{t_1} \left(\frac{3}{2}u^2 - \frac{k_1}{2}\rho^2\right)\mathrm{d}\tau = \partial_x G * p(x),$$

where, by Theorem 1.2 and $k_1 < 0$,

$$0 \le p(x) \sim O(\mathrm{e}^{-2\alpha x}), \quad \text{so that} \quad p(x) \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

Therefore,

$$\begin{aligned} \partial_x G * p(x) &= -\frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(x-y)\mathrm{e}^{-|x-y|}p(y)\mathrm{d}y \\ &= -\frac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{y}p(y)\mathrm{d}y + \frac{1}{2}\mathrm{e}^{x}\int_x^{\infty} \mathrm{e}^{-y}p(y)\mathrm{d}y, \end{aligned}$$

$$\mathrm{e}^{x}\int_x^{\infty} \mathrm{e}^{-y}p(y)\mathrm{d}y = o(1)\mathrm{e}^{x}\int_x^{\infty} \mathrm{e}^{-2y}\mathrm{d}y = o(1)\mathrm{e}^{-x} \sim o(\mathrm{e}^{-x}).$$

But, if $p \neq 0$, then there exists $c_0 > 0$ such that

$$\int_{-\infty}^{x} \mathrm{e}^{y}p(y)\mathrm{d}y \ge c_0 \quad \text{for} \quad x \gg 1.$$

Hence,

$$-\partial_x G * p(x) \ge \frac{c_0}{2}\mathrm{e}^{-x} \quad \text{for} \quad x \gg 1,$$

which, combined with the above estimates, yields a contradiction. Thus, $p(x) \equiv 0$ and consequently, $u \equiv 0, \rho \equiv 0$.

For the case $k_1 = 0$, by the same step as for the above case $k_1 < 0$, we can obtain $u \equiv 0$. Inserting $u \equiv 0$ into the second equation in (1.1), it follows that $\rho_t = 0$, and so $\rho(t,x) = \rho_0(x)$, which completes the proof of Theorem 1.3.

### 3.2 Proof of Theorem 1.4

**Proof** For any $t \in (0,T]$, integrating equation (1.3) over the time interval $[0,t]$, we obtain

$$u(x,t) + \frac{k_1}{2}\int_0^t \partial_x G * \rho^2 \mathrm{d}\tau = u(x,0) - \int_0^t uu_x(x,\tau)\mathrm{d}\tau + \frac{3}{2}\int_0^t \partial_x G * u^2 \mathrm{d}\tau.$$

By the hypothesis of Theorem 1.3, we have

$$u(x,0) \sim O(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

From Theorem 1.2, it follows that

$$\int_0^t uu_x(x,\tau)\mathrm{d}\tau \sim O(\mathrm{e}^{-2\alpha x}) \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty,$$

$$\frac{3}{2}\int_0^{t_1} \partial_x G * u^2(\tau,x)\mathrm{d}\tau = \partial_x G * q(x),$$

where

$$0 \le q(x) \sim O(\mathrm{e}^{-2\alpha x}) \quad \text{as} \quad x \uparrow \infty.$$

So,

$$\partial_x G * q(x) = -\frac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^x \mathrm{e}^y q(y)\mathrm{d}y + \frac{1}{2}\mathrm{e}^x\int_x^\infty \mathrm{e}^{-y}q(y)\mathrm{d}y.$$

As before,

$$\mathrm{e}^x\int_x^\infty \mathrm{e}^{-y}q(y)\mathrm{d}y \sim O(\mathrm{e}^{-2\alpha x}) \sim o(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

But

$$-\mathrm{e}^{-x}\int_{-\infty}^x \mathrm{e}^y q(y)\mathrm{d}y \le -c_0\mathrm{e}^{-x} \quad \text{for} \quad x \gg 1.$$

Therefore,

$$\frac{3}{2}\int_0^{t_1} \partial_x G * u^2(\tau,x)\mathrm{d}\tau \sim O(\mathrm{e}^{-x}).$$

Similarly,

$$\frac{k_1}{2}\int_0^t \partial_x G * \rho^2(\tau,x)\mathrm{d}\tau \sim O(\mathrm{e}^{-x}).$$

Therefore,

$$u(x,t) \sim O(\mathrm{e}^{-x}) \quad \text{as} \quad x \uparrow \infty.$$

By Theorem 1.2, we can obtain

$$\rho(x,t) \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty$$

if

$$\rho_0(x) \sim O(\mathrm{e}^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty.$$

Thus, we complete the proof of Theorem 1.4.

**Remark 3.1** In this article, we establish the unique continuation only for the case $k_1 \leq 0$. For the case $k_1 > 0$, the two terms in the last term of the left-hand side in (1.3) have different signs. Therefore, if they possess same decaying condition, the lower order infinitesimal terms $O(\mathrm{e}^{-x})$ will be canceled. Thus, the contradiction cannot be derived. So, the unique continuation property for the case $k_1 > 0$ may not hold. It will be an interesting issue with regard to the unique continuation property for the case $k_1 > 0$.

## References


[1] Bourgain J. On the compactness of the support of solutions of dispersive equations. Internat Math Res Notices, 1997, **9**: 437–447

[2] Carleman T. Sur les systèmes Lineaires aux dérivées partielles du premier ordre à deux variables. C R Acad Sci Paris, 1939, **97**: 471–474

[3] Carvajal X, Panthee M. Unique continuation property for a higher order nonlinear Schrödinger equation. J Math Anal Appl , 2005, **303**: 188–207

[4] Escauriaza L, Kenig C E, Ponce G, Vega L. On uniqueness properties of solutions of the k-generalized KdV equations. J Funct Anal, 2007, **244**: 504–535

[5] Escauriaza L, Kenig C E, Ponce G, Vega L. On unique continuation of solutions of Schrödinger equations. Comm Partial Differential Equations, 2006, **31**: 1811–1823

[6] Escher J, Lechtenfeld O, Yin Zhaoyang. Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation. Discrete Continuous Dynamical Systems, 2007, **17**: 493–513

[7] Escher J, Liu Yue, Yin Zhaoyang. Global weak solutions and blow-up structure for the Degasperis-Procesi equation. J Funct Anal, 2006, **241**: 457–485

[8] Guo Zhengguang, Zhou Yong. On solutions to a two-component generalized Camassa-Holm equation. Stud Appl Math, 2010, DOI: 10.1111/j.1467-9590.2009.00472.x

[9] Himonas A, Misiolek G, Ponce G, Zhou Yong. Persistence properties and unique continuation of solutions of the Camassa-Holm equations. Comm Maths Phys, 2007, **271**: 511–522

[10] Isakov V. Carleman type estimates in an anisotropic case and applications. J Differential Equations, 1993, **105**: 217–238

[11] Kato T. Quasi-linear equations of evolution, with applications to partial differential equations//Spectral Theory and Differential Equations. Lecture Notes in Math, Vol **448**. Berlin: Springer Verlag, 1975: 25–70

[12] Kenig C E, Ponce G, Vega L. On the support of solutions to the generalized KdV equation. Ann Inst H Poincaré Anal Non Linéaire , 2002, **19**: 191–208

[13] Kenig C E, Ponce G, Vega L. On unique continuation for nonlinear Schrödinger equation. Comm Pure Appl Math, 2003, **56**: 1247–1262

[14] Panthee M. A note on the unique continuation property for Zakharov-Kuznetsov equation. Nonlinear Analysis: Theory, Methods and Applications, 2004, **59**: 425-438

[15] Popowicz Z. A 2-component generalization of the Degasperis-Procesi equation. J Phys A: Math Gen, 2006, **39**: 13717–13726

[16] Saut J C, Scheurer B. Unique continuation for some evolution equations. J Differential Equations, 1987, **77**: 118–139

[17] Tataru D. Carleman type estimates and unique continuation for the Schrödinger equation. Differential Integral Equations, 1995, **8**: 901–905

[18] Zhang Bingyu. Unique continuation for the Kortemeg-de Vries equation. SIAM J Math Anal, 1992, **23**: 55–71

[19] Zhou Yong. Blow-up phenomenon for the integrable Degasperis-Procesi equation. Phys Lett A, 2004, **328**: 157–162