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Note

Combinatorial proofs of generating function identities for F-partitions

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Abstract

In his memoir in 1984, George E. Andrews introduces many general classes of Frobenius partitions (simply F-partitions). Especially, he focuses his interest on two general classes of F-partitions: one is F-partitions that allow up to k repetitions of an integer in any row, and the other is F-partitions whose parts are taken from k copies of the nonnegative integers. The latter are called k colored F-partitions or F-partitions with k colors. Andrews derives the generating functions of the number of F-partitions with k repetitions and F-partitions with k colors of n and leaves their purely combinatorial proofs as open problems. The primary goal of this article is to provide combinatorial proofs in answer to Andrews' request.

1. Introduction

For a nonnegative integer n, a generalized Frobenius partition or simply an F-partition of n is a two-rowed array of nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r, \end{pmatrix},$$

where each row is of the same length, each is arranged in nonincreasing order and $n = r + \sum_{i=1}^{r}(a_i + b_i)$.

Frobenius studied F-partitions [4] in his work on group representation theory under the additional assumptions $a_1 > a_2 > \cdots > a_r \geqslant 0$ and $b_1 > b_2 > \cdots > b_r \geqslant 0$. By considering the Ferrers graph of an ordinary partition, we see that a_i form rows to

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the right of the diagonal and b_i form columns below the diagonal. For example, the F-partition for $7+7+5+4+2+2$ is

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix},$$

as is seen easily from the Ferrers graph in Fig. 1. Thus F-partitions are another representation of ordinary partitions.

In his memoir [2], Andrews introduces many general classes of F-partitions. Especially, he focuses his interest on two general classes of F-partitions: one is F-partitions that allow up to k repetitions of an integer in any row, and the other is F-partitions whose parts are taken from k copies of the nonnegative integers. The latter are called k colored F-partitions or F-partitions with k colors. Andrews [2] derives the generating functions of the number of F-partitions with k repetitions and F-partitions with k colors of n and leaves their purely combinatorial proofs as open problems. More precisely, he [2, p. 40] offers ten series of problems; in this paper we offer solutions to two series of problems, comprising a total of five separate problems. The primary goal of this article is to provide combinatorial proofs in answer to Andrews' request.

In Section 2, we interpret the enumerative proof of Jacobi's triple product of E. M. Wright in another way using the generating function for two-rowed arrays of nonnegative integers. In Section 3, we prove the generating function for F-partitions with k colors in a combinatorial way, which was independently and earlier established by Garvan [6], and in Section 4, we prove identities arising in the study on F-partitions with k colors of Andrews. We prove the generating function for F-partitions with k repetitions combinatorially in Section 5.

2. Wright's enumerative proof of Jacobi's triple product

The well-known Jacobi triple product identity is

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) \tag{2.1}$$

for $z \neq 0$, $|q|<1$. There are two proofs of Andrews [1,2]. A proof of Wright [8] is combinatorial, and involves a direct bijection of bipartite partitions. Proofs due to Cheema [3] and Sudler [7] are variations of Wright's proof. Here we describe Wright's proof, which we use in the following sections.

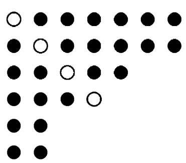


Fig. 1. Ferrers graph of $7+7+5+4+2+2$.

By substituting zq for z and q for q^2 in (2.1), and then dividing by $\prod_{n=1}^{\infty}(1-q^n)$ on both sides, we obtain

$$\frac{\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}}{\prod_{n=1}^{\infty}(1-q^n)} = \prod_{n=1}^{\infty} (1+zq^n)(1+z^{-1}q^{n-1}), \tag{2.2}$$

which we use below. Let $\mathscr{A}$ be the set of two-rowed arrays of nonnegative integers

$$(u;v) := \begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ v_1 & v_2 & \cdots & v_t, \end{pmatrix},$$

where $u_1 > u_2 > \cdots > u_s \geqslant 0$, $v_1 > v_2 > \cdots > v_t \geqslant 0$, and s and t can be distinct. We define the weight of $(u;v) \in \mathscr{A}$ by

$$|(u;v)| := s + \sum_{i=1}^{s} u_i + \sum_{i=1}^{t} v_i.$$

When $z = 1$, the right-hand side of (2.2) is the generating function for $(u;v) \in \mathscr{A}$,

$$\sum_{(u;v)\in\mathscr{A}} q^{|(u;v)|}.$$

Thus, we interpret Wright's 1-1 correspondence using $\mathscr{A}$. To do that, we first need to describe a graphical representation of a two-rowed array

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ v_1 & v_2 & \cdots & v_t \end{pmatrix}$$

that is analogous to the Ferrers graph of an ordinary partition. Put s circles on the diagonal, and then put u_j nodes in row j to the right of the diagonal and v_j nodes in column $(s-t+j)$ below the diagonal. The diagram of a two-rowed array coincides with the Ferrers graph when the lengths of the two rows are equal. Fig. 2 shows the diagram of

$$\begin{pmatrix} * & * & 6 & 5 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & 0 \end{pmatrix}.$$

We are ready to construct Wright's 1-1 correspondence. Let $r = s - t$. Suppose that $r \geqslant 0$. For the given diagram of $(u;v)$, we separate the diagram into two sets of nodes along the vertical line between column r and column $r+1$ to obtain the Ferrers graph of an ordinary partition and a right angled triangle of nodes consisting of r rows. Meanwhile, if $r<0$, then we separate the diagram along the horizontal line

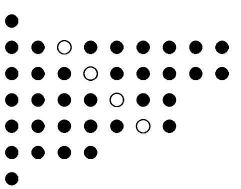


Fig. 2.

between row 0 and row 1. The triangle of nodes consists of $-(r+1)$ rows. Note that the number of the nodes in the triangle is $r(r+1)/2$ in both cases. The ordinary partitions and triangles contribute to $\prod_{n=1}^{\infty}(1-q^n)^{-1}$ and $\sum_{r=-\infty}^{\infty} z^r q^{r(r+1)/2}$, respectively, on the left side of (2.2). Clearly, the process can be carried out in reverse, and gives us a bijection to explain (2.2). For example, we consider the diagram of

$$\begin{pmatrix} 9 & 8 & 5 & 3 & 2 & 1 \\ * & * & 5 & 4 & 1 & 0 \end{pmatrix},$$

which splits into the partition $8+8+6+5+5+5+2+2$ and the triangle consisting of 2 rows in Fig. 3.

3. Generating function of F-partitions with k colors

We begin this section by defining F-partitions with k colors. We consider k copies of the nonnegative integers written j_i, where $j \geqslant 0$ and $1 \leqslant i \leqslant k$. We call j_i an integer j in color i, and we use the order

$$j_i < l_h \Leftrightarrow j < l, \quad \text{or} \quad j = l \text{ and } i < h.$$

For a nonnegative integer n, an F-partition of n with k colors is an F-partition of n into nonnegative integers with k colors. For example, when $k=4$,

$$\begin{pmatrix} 2_1 & 1_3 & 1_1 & 0_2 & 0_1 \\ 1_2 & 1_1 & 0_4 & 0_2 & 0_1 \end{pmatrix}$$

is an F-partition of 11 with 4 colors, since $5+(2+1+1+0+0)+(1+1+0+0+0)=11$.

Let λ be an F-partition with k colors of N

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

For convenience, we call a_j and b_j the parts of λ. We define two statistics. Let

$$A_i := A_i(\lambda) := \# \text{ of } a_j \text{ in color } i$$

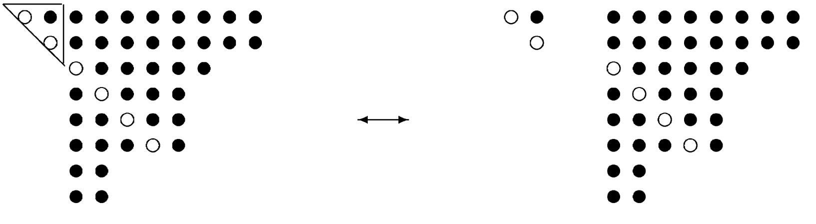


Fig. 3.

and

$$B_i := B_i(\lambda) := \#\text{ of } b_j \text{ in color } i$$

for $i = 1, 2, \ldots, k$.

Let $c\phi_k(n)$ be the number of F-partitions with k colors of n, and define the generating function $C\Phi_k(q)$ of $c\phi_k(n)$ by

$$C\Phi_k(q) = \sum_{n=0}^{\infty} c\phi_k(n)q^n \quad \text{for } |q| < 1.$$

In Section 5 in [2], $C\Phi_k(q)$ is given in the following theorem.

Theorem 3.1 (Theorem 5.2, Andrews [2]). *For* $|q| < 1$,

$$C\Phi_k(q) = \frac{\sum_{m_1, m_2, \ldots, m_{k-1} = -\infty}^{\infty} q^{Q(m_1, m_2, \ldots, m_{k-1})}}{\prod_{n=1}^{\infty} (1 - q^n)^k}, \tag{3.1}$$

where

$$Q(m_1, m_2, \ldots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \leqslant i < j \leqslant k-1} m_i m_j. \tag{3.2}$$

Proof. To show (3.1), we establish a bijection between F-partitions with k colors and $(k-1)$-tuples of integers and k ordinary partitions. Let

$$m_i := A_i(\lambda) - B_i(\lambda)$$

for $i = 1, 2, \ldots, k$.

We first consider the case when $k = 2$. The identity (3.1) gives us

$$C\Phi_2(q) = \frac{\sum_{m=-\infty}^{\infty} q^{m^2}}{\prod_{n=1}^{\infty} (1 - q^n)^2}. \tag{3.3}$$

Note that $A_1 + A_2 = B_1 + B_2$ and $m_1 = -m_2$. We split λ according to the colors of the parts: $\lambda = \mu + \nu$, where μ and ν are the two-rowed arrays consisting of parts of λ in colors 1 and 2, respectively, and each row of μ and λ is arranged in nonincreasing order. Say

$$\mu = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{A_1} \\ \beta_1 & \beta_2 & \cdots & \beta_{B_1} \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{A_2} \\ \delta_1 & \delta_2 & \cdots & \delta_{B_2} \end{pmatrix}.$$

We apply Wright's 1-1 correspondence described in Section 2 to μ and ν. We consider the diagrams of μ and ν. By symmetry it is sufficient to consider the case when $m_1 \geqslant 0$. Since $m_1 \geqslant 0$, the number of the parts of μ in the first row is greater than or equal to the number of the parts of μ in the second row. Thus μ has a triangle to the left of its diagram, and ν has a triangle at the top of its diagram. We obtain two ordinary partitions by separating the nodes and circles in the triangle from the diagrams of μ and ν. Since the triangle from μ has $m_1(m_1 + 1)/2$ of nodes and circles, and the triangle from ν has $m_1(m_1 - 1)/2$ nodes, we see that the sum of the weights of the two partitions newly obtained is $N - m_1^2$ as required. The process can be carried out in reverse, and gives us a bijection between the set of F-partitions with 2 colors and the set of triples of an integer and two ordinary partitions.

We consider the general case. Let λ be an F-partition of N with k colors, and split λ according to the colors of the parts: $\lambda = \lambda^1 + \lambda^2 + \cdots + \lambda^k$, where some λ^i may be empty. By applying Wright's process to each λ^i, we obtain k ordinary partitions. The number of the nodes and circles in the k triangles is

$$\sum_{i=1}^{k} \frac{m_i(m_i+1)}{2}.$$

Since $\sum_{i=1}^{k} m_i = 0$ by the definition of m_i, the number of the nodes and circles in the k triangles is $Q(m_1, m_2, \ldots, m_{k-1})$ as desired, which completes the proof. $\square$

Garvan in his Ph.D. thesis [6, pp. 55–68] established a combinatorial proof of Theorem 3.1.

4. F-partitions with k colors and q-series

In the sequel, we use the customary notation for q-series

$$(a)_0 := (a;q)_0 := 1, \quad (a)_n := (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{if } n \geqslant 1,$$

and

$$(a;q)_\infty := \lim_{n \to \infty} (a;q)_n, \quad |q| < 1.$$

Theorem 4.1 (Theorem 8.2, Andrews [2]). *For* $|q| < 1$,

$$C\Phi_k(q) = \frac{1}{(q)_\infty} \sum_{n_1,\ldots,n_{k-1} \geqslant 0} \; \sum_{m_1,\ldots,m_{k-1} \geqslant 0} \frac{q^{R(n_1,\ldots,n_{k-1},m_1,\ldots,m_{k-1})}}{(q)_{n_1} \cdots (q)_{n_{k-1}} (q)_{m_1} \cdots (q)_{m_{k-1}}},$$

where

$$R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1}) = \sum_{i=1}^{k-1} (n_i^2 + m_i^2) + \sum_{1 \leqslant i < j \leqslant k-1} (n_i n_j + m_i m_j) - \sum_{1 \leqslant i,j \leqslant k-1} n_i m_j. \quad (4.1)$$

Proof. Let λ be an F-partition with k colors of N

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

Let $n_i = A_i(\lambda)$ and $m_i = B_i(\lambda)$ for $1 \leqslant i \leqslant k$. Then $(n_1 + \cdots + n_k) - (m_1 + \cdots + m_k) = 0$. To obtain the generating function for F-partitions with k colors, we consider the generating function for two-rowed arrays made of parts of λ in color i, $i = 1, \ldots, k-1$,

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in_i} \\ b_{i1} & b_{i2} & \cdots & b_{im_i} \end{pmatrix}$$

by splitting λ according to colors, and then we combine the generating functions together. Note that the parts a_{ij} are distinct and so are the b_{ij}. Since the generating function for partitions into n distinct parts is $q^{n(n-1)/2}/(q)_n$, we see that the generating function for pairs of partitions consisting of a_{ij} and b_{ij} is

$$\frac{q^{n_i(n_i-1)/2+m_i(m_i-1)/2}}{(q)_{n_i}(q)_{m_i}}.$$

However, since we need to count the nodes of λ on the diagonal, the generating function for two-rowed arrays made of parts of λ in color i, $i = 1, \ldots, k-1$, is

$$\frac{q^{n_i(n_i+1)/2+m_i(m_i-1)/2}}{(q)_{n_i}(q)_{m_i}}.$$

Meanwhile, for color k, we obtain

$$\sum_{n_k \geqslant 0} \frac{q^{n_k(n_k+1)/2+m_k(m_k-1)/2}}{(q)_{n_k}(q)_{m_k}} = \frac{q^{(n_k-m_k)(n_k-m_k+1)/2}}{(q)_\infty},$$

by applying Wright's 1-1 correspondence for a fixed m_k. Thus,

$$\begin{aligned} C\Phi_k(q) &= \prod_{i=1}^{k} \sum_{\substack{n_i, m_i \geqslant 0 \\ (n_1+\cdots+n_k)-(m_1+\cdots+m_k)=0}} \frac{q^{n_i(n_i+1)/2+m_i(m_i-1)/2}}{(q)_{n_i}(q)_{m_i}} \\ &= \sum_{\substack{n_1,\ldots,n_k,m_1,\ldots,m_k \geqslant 0 \\ (n_1+\cdots+n_k)-(m_1+\cdots+m_k)=0}} \frac{q^{n_1(n_1+1)/2+\cdots+n_k(n_k+1)/2+m_1(m_1-1)/2+\cdots+m_k(m_k-1)/2}}{(q)_{n_1}\cdots(q)_{n_k}(q)_{m_1}\cdots(q)_{m_k}} \\ &= \frac{1}{(q)_\infty} \sum_{\substack{n_1,\ldots,n_k,m_1,\ldots,m_k \geqslant 0 \\ (n_1+\cdots+n_k)-(m_1+\cdots+m_k)=0}} \frac{q^{\sum_{i=1}^{k-1} (n_i(n_i+1)/2+m_i(m_i-1)/2)+(n_k-m_k)(n_k-m_k+1)/2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{m_1}\cdots(q)_{m_{k-1}}}. \end{aligned} \tag{4.2}$$

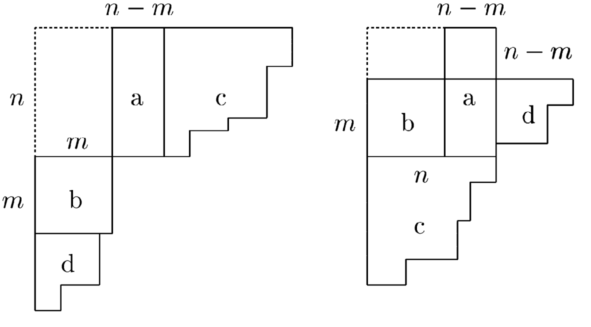
Since $(n_1 + \cdots + n_k) - (m_1 + \cdots + m_k) = 0$, by substituting $(n_1 + \cdots + n_{k-1}) - (m_1 \cdots + m_{k-1})$ for $n_k - m_k$, we see that the exponent of q in the numerator of each term on the right side of (4.2) is $R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1})$, which completes the proof. $\square$

In our proofs of Theorems 3.1 and 4.1 we have answered Andrews' request for combinatorial proofs. Andrews also asked for combinatorial proofs of the next two corollaries. Indeed, our proofs below are perhaps the combinatorial proofs which Andrews sought.

Corollary 4.2 (Corollary 8.2, Andrews [2]). *For* $|q|<1$,

$$\sum_{n,m\geqslant 0} \frac{q^{n^2+m^2-nm}}{(q)_n(q)_m} = \frac{\sum_{s=-\infty}^{\infty} q^{s^2}}{(q)_\infty}. \tag{4.3}$$

Proof. By symmetry, it suffices to consider the case when $n\geqslant m$. On the left-hand side of (4.3), $q^{n^2-nm}/(q)_n$ counts the partitions with part n as the largest part at least $(n-m)$ times, and $q^{m^2}/(q)_m$ counts the partitions with part m as the largest part at least m times. Thus the coefficient of q^N on the left-hand side of (4.3) is the number of partitions of $N+nm$ with the first Durfee square n^2 and the second Durfee square m^2. Meanwhile, the right-hand side of (4.3) is the generating function for pairs of partitions and squares. For a given partition of $N+nm$, we change the Ferrers graph by pushing area b up and switching areas c and d to obtain a partition of $N+nm-(n-m)^2$ and a square $(n-m)^2$. Since the process is reversible, we complete the proof. $\square$



Andrews noted that some nice combinatorial aspects of the previous corollary are presented in [5]. We consider the generalization of Corollary 4.2.

Corollary 4.3 (Corollary 8.1, Andrews [2]). *For* $|q|<1$,

$$\sum_{n_1,\dots,n_{k-1}\geqslant 0}\ \sum_{m_1,\dots,m_{k-1}\geqslant 0} \frac{q^{R(n_1,\dots,n_{k-1},m_1,\dots,m_{k-1})}}{(q)_{n_1}\cdots(q)_{n_{k-1}}(q)_{m_1}\cdots(q)_{m_{k-1}}} = \frac{\sum_{s_1,\dots,s_{k-1}=-\infty}^{\infty} q^{Q(s_1,\dots,s_{k-1})}}{(q)_\infty^{k-1}},$$

where $R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1})$ *and* $Q(s_1, \ldots, s_{k-1})$ *are defined in* (4.1) *and* (3.2).

Proof. We examine $R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1})$. By definition,

$$\begin{aligned} &R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1}) \\ &\quad = \sum_{i=1}^{k-1} (n_i^2 + m_i^2 - n_i m_i) + \sum_{1 \leqslant i < j \leqslant k-1} (n_i n_j + m_i m_j - n_i m_j - n_j m_i) \\ &\quad = \sum_{i=1}^{k-1} (n_i^2 + m_i^2 - m_i n_i) + \sum_{1 \leqslant i < j \leqslant k-1} (n_i - m_i)(n_j - m_j). \end{aligned}$$

By applying the argument we used in the proof of Corollary 4.2, we see that

$$\sum_{n_i, m_i \geqslant 0} \frac{q^{n_i^2 + m_i^2 - m_i n_i}}{(q)_{n_i} (q)_{m_i}} = \frac{\sum_{s_i = -\infty}^{\infty} q^{s_i^2}}{(q)_\infty},$$

where $s_i = n_i - m_i$. Thus,

$$\begin{aligned} &\sum_{n_1, \ldots, n_{k-1} \geqslant 0} \; \sum_{m_1, \ldots, m_{k-1} \geqslant 0} \frac{q^{R(n_1, \ldots, n_{k-1}, m_1, \ldots, m_{k-1})}}{(q)_{n_1} \cdots (q)_{n_{k-1}} (q)_{m_1} \cdots (q)_{n_{k-1}}} \\ &\quad = \sum_{n_1, \ldots, n_{k-1} \geqslant 0} \; \sum_{m_1, \ldots, m_{k-1} \geqslant 0} \frac{q^{\sum_{i=1}^{k-1} (n_i^2 + m_i^2 - n_i m_i) + \sum_{1 \leqslant i < j \leqslant k-1} (n_i - m_i)(n_j - m_j)}}{(q)_{n_1} \cdots (q)_{n_{k-1}} (q)_{m_1} \cdots (q)_{n_{k-1}}} \\ &\quad = \frac{\sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} q^{\sum_{i=1}^{k-1} s_i^2 + \sum_{1 \leqslant i < j \leqslant k-1} s_i s_j}}{(q)_\infty^{k-1}} \\ &\quad = \frac{\sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} q^{Q(s_1, \ldots, s_{k-1})}}{(q)_\infty^{k-1}}, \end{aligned}$$

which completes the proof. $\square$

In fact, Corollary 4.2 can be directly generalized for any k.

Corollary 4.4. *For* $|q| < 1$,

$$\sum_{n_1, \ldots, n_k \geqslant 0} \; \sum_{m_1, \ldots, m_k \geqslant 0} \frac{q^{\sum_{i=1}^{k} (n_i^2 + m_i^2 - n_i m_i)}}{(q)_{n_1} \cdots (q)_{n_k} (q)_{m_1} \cdots (q)_{m_k}} = \frac{\sum_{s_1, \ldots, s_k = -\infty}^{\infty} q^{s_1^2 + \cdots + s_k^2}}{(q)_\infty^k}.$$

5. F-partitions with k repetitions and q-series

In this section, we consider F-partitions that allow up to k repetitions of an integer in any row. For a nonnegative integer n, an F-partition with k repetitions is an F-partition of n into nonnegative integers with k repetitions in any row. For example,

$$\begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

is an F-partition of 12 with 3 repetitions since $5 + (3 + 1 + 1 + 0 + 0) + (1 + 1 + 0 + 0 + 0) = 12$. Let $\phi_k(n)$ be the number of F-partitions of n with k repetitions, and define the generating function $\Phi_k(q)$ of $\phi_k(n)$ by

$$\Phi_k(q) = \sum_{n=0}^{\infty} \phi_k(n) q^n \quad \text{for } |q| < 1. \tag{5.1}$$

In Section 8 in [2], Andrews gives a generating function $\Phi_k(q)$ for F-partitions with k repetitions in the following theorem.

Theorem 5.1 (Theorem 8.1, Andrews [2]). *For* $|q| < 1$,

$$\Phi_k(q) = \sum_{r,s,t \geqslant 0} \frac{(-1)^{r+s} q^{(k+1)r(r+1)/2 + (k+1)s(s-1)/2 + t}}{(q^{k+1}; q^{k+1})_r (q^{k+1}; q^{k+1})_s (q)_t (q)_{(r-s)(k+1)+t}}. \tag{5.2}$$

Proof. We first need to consider F-partitions whose parts are allowed to repeat infinitely many times, and then use the principle of inclusion and exclusion for F-partitions with at most k repetitions allowed. Let $\mathscr{RP}$ be the set of F-partitions with repetitions in parts, i.e., the set of two-rowed arrays of nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix},$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m \geqslant 0$. Since the generating function for ordinary partitions into nonnegative parts is

$$\sum_{m=0}^{\infty} \frac{1}{(q)_m},$$

we see that the generating function for F-partitions with repetitions is

$$\sum_{m=0}^{\infty} \frac{q^m}{(q)_m (q)_m},$$

where the numerator q^m comes from the m nodes on the diagonal in the Ferrers graph of $\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}$.

Let $\mathscr{RP}(1;0)$ be the set of F-partitions with at least one part a appearing more than k times in the top row. By breaking up the partition into the $k+1$ a's and the remaining parts, we can write the partition as

$$\underbrace{a + \cdots + a}_{k+1 \ \ times} \quad \begin{pmatrix} a_1 a_2 \cdots a_{m-k-1} \\ b_1 b_2 \cdots b_m \end{pmatrix},$$

where the a_i's do not necessarily coincide with the old a_i's. Recall that there are m nodes on the diagonal in the Ferrers graph of $\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}$. Then, we see that the generating function for $\mathcal{RP}(1;0)$ is

$$\frac{q^{(k+1)}}{(1-q^{k+1})} \sum_{t=0}^{\infty} \frac{q^t}{(q)_t (q)_{t+k+1}}.$$

On the other hand, let $\mathcal{RP}(0;1)$ be the set of F-partitions with at least one part b appearing more than k times in the bottom row. By breaking up the partition into the $k+1$ b's and the remaining parts, we can write the partition as

$$\underbrace{b + \cdots + b}_{k+1 \ \ times} \quad \begin{pmatrix} a_1 a_2 \cdots a_m \\ b_1 b_2 \cdots b_{m-k-1} \end{pmatrix},$$

where the b_i's do not necessarily coincide with the old b_i's. Then, we see that the generating function for $\mathcal{RP}(0;1)$ is

$$\frac{1}{(1-q^{k+1})} \sum_{t=0}^{\infty} \frac{q^t}{(q)_t (q)_{t+k+1}}.$$

In general, for nonnegative integers r and s, let $\mathcal{RP}(r;s)$ be the set of F-partitions with at least r and s distinct parts appearing more than k times in the top and bottom row, respectively. Since the generating function for partitions with exactly n_1 distinct parts, with each distinct part repeated n_2 times, is

$$\frac{q^{n_2 n_1 (n_1+1)/2}}{(q^{n_2};q^{n_2})_{n_1}},$$

we see that the sum of the generating functions for the F-partitions in $\mathcal{RP}(r;s)$ is

$$\frac{q^{(k+1)(r(r+1)/2+s(s-1)/2)}}{(q^{k+1};q^{k+1})_r (q^{k+1};q^{k+1})_s} \sum_{t=0}^{\infty} \frac{q^t}{(q)_t (q)_{(r-s)(k+1)+t}}.$$

Now, we apply the principle of inclusion and exclusion to obtain $\Phi_k(q)$. Thus,

$$\Phi_k(q) = \sum_{r,s \geqslant 0} \frac{(-1)^{r+s} q^{(k+1)(r(r+1)/2+s(s-1)/2)}}{(q^{k+1};q^{k+1})_r (q^{k+1};q^{k+1})_s} \sum_{t=0}^{\infty} \frac{q^t}{(q)_t (q)_{(r-s)(k+1)+t}},$$

which completes the proof. □

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