SOME DISTORTION THEOREMS FOR STARLIKE HARMONIC FUNCTIONS

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Dedicated to Professor Ya cedilla – s ar Polato ğ lu
the occasion of his 6zero – th birthday

Abstract

In this paper , we consider harmonic univalent mappings of the form
$f = h + \bar{g}$ defined on the unit disk $\mathbb{D}$ which are starlike . Distortion and growth theorems are obtained .

MSC 201 0 : 30 C 55 , 30 C 45

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growth theorem , distortion theorem

1 . Introduction

A continuous complex - valued function $f = u + iv$ defined in a simply connected domain $\mathfrak{D}$ is said t o be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$, that is , u, v satisfy , resp ectively the Laplace equations

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0.$$

There i s a well - known relation b etween analytic functions and harmonic functions . For example , for real harmonic functions u and v which are defined on a simply connected domain $\mathfrak{D}$ there exist analytic functions U
and V so that

$$u = \mathfrak{Re}(U) \text{and} v = \mathfrak{Im}(V).$$

 Therefore , it has a canonical decomposition

$$f = h + g \tag{1}$$

where h and g are , resp ectively , the analytic functions

$$h = U +_2 V \text{and} g = U - 2V.$$

We call h the analytic part and g the co - analytic part of f. It is fact that if $f = u + iv$ has continuous partial derivatives , then f i s analytic if and only if the Cauchy - Riemann equations are satisfied . It follows that every analytic function i s a complex - valued harmonic function . However , not every complex - valued harmonic function i s analytic .

The Jacobian J_f of a function $f = u + iv$ has a very important place in the theory of harmonic mappings , defined by

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

Or , in t erms of f_z and $f_{\bar{z}}$, we have

$$J_f = \mid f_z \mid^2 - \mid f_{\bar{z}} \mid^2 = \mid h'(z) \mid^2 - \mid g'(z) \mid^2,$$

where $f = h + g$ i s the harmonic function in $\mathfrak{D}$.

If $f = h + g$ is a harmonic function on $\mathfrak{D}$ with $J_f > 0$, then we say that f is a sense - preserving (or orientation preserving) harmonic function on $\mathfrak{D}$. In this case we have

$$\mid g'(z) \mid < \mid h'(z) \mid$$

for all $z \in \mathfrak{D}$. If f has $J_f < 0$, then f is sense preserving . For convenience , we will only examine sense preserving harmonic functions .

The mapping $z \to f(z)$ i s sense preserving and locally univalent in $\mathfrak{D}$ if and only if $J_f > 0$ in $\mathfrak{D}$. The function $f = h + \bar{g}$ i s said t o be harmonic univalent in $\mathfrak{D}$ if the mapping $z \to f(z)$ is sense preserving harmonic and univalent in $\mathfrak{D}$.

The second complex dilatation of a harmonic function $f = h + g$ i s the

$$\text{quantity} \quad \omega(z) = f_z^{f_{\bar{z}}} = g' h^{(z)}_{\prime(z)} \quad (z \in \mathfrak{D}). \quad (2)$$

Let $\mathcal{S}_{\mathcal{H}}$ denote the family of functions $f = h + g$ that are harmonic , sense preserving , and univalent in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : \mid z \mid < 1\}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{3}$$

It follows from the sense - preserving property if $f \in \mathcal{S}_{\mathcal{H}}$, then we have $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Thus , it is easy t o see that $|b_1| < 1$. Since the second complex dilatation ω of a sense preserving harmonic mapping f is always an analytic function of modulus less than one , then this function ω will be called the analytic dilatation of f. Also $f \in \mathcal{S}_{\mathcal{H}}$ reduces t o the class of normalized analytic univalent functions if the co - analytic part of it s members is zero . In 1984 Clunie and Sheil - Small [1] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as it s geometric subclasses and obtained some coefficient bounds . Many studies have been done on this class and it s subclasses , and continued taking place .

A sense - preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ i s in the class $\mathcal{S}^*_{\mathcal{H}}$ if the range $f(\mathbb{D})$ i s starlike with resp ect t o the origin . A function $f \in \mathcal{S}^*_{\mathcal{H}}$ is a called harmonic starlike mapping in $\mathbb{D}$. A function $f = h + g$ with such a property must satisfy the condition

$$\mathfrak{Re}\left(\frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right) > 0 \quad \text{for all} z \in \mathbb{D}.$$

In our proofs we use the following lemma :

Lemma 1 . 1 [2] . *If* $f = h + g \in \mathcal{S}^*_{\mathcal{H}}$, *then there exist angles* α *and* β *such that*

$$\mathfrak{Re}\{\left(\frac{e^{i\alpha}h(z)}{z} + \frac{e^{-i\alpha g(z)}}{z} \right) (e^{i\beta} - e^{-i\beta}z^2)\} > 0 \quad (4)$$

$$for all z \in \mathbb{D}.$$

Let $\mathcal{A}$ denote the class of all functions s_1 analytic in the open unit disk $\mathbb{D}$ with the usual normalization $s_1(0) = s_1'(0) - 1 = 0$. If s_1 and s_2 are analytic in $\mathbb{D}$, we say that s_1 is subordinate to s_2, written $s_1 \prec s_2$ or $s_1(z) \prec s_2(z)$, if s_2 is univalent , then we have $s_1(0) = s_2(0)$ and $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} pn^{z^n}$$

which are analytic in the open unit disk $\mathbb{D}$. If p in $\mathcal{P}$ satisfies $\mathfrak{Re}p(z) > 0$ for $z \in \mathbb{D}$, then we say that p i s the Carath é odory function . It has been shown that for a function $p(z) \in \mathcal{P}$, the following inequalities are satisfied ([3]) :

$$1^1 - +r^r \leq | p(z) | \leq \quad 1^1 +_- r^r, \tag{5}$$

and

$$|z^{p'(z)}_{p(z)}| \leq \quad 12^r_- r^2 \quad (6)$$

forall $| z |= r < 1$.

2 . Results

LEMMA 2 . 1 . *Let* $f = h + g$ *be an element of* $\mathcal{S}^*_{\mathcal{H}'}$ *then we have*

$$(A + (^{|B|)r}_{1+} r -)((^A_1 + - r^{|B|)r^2}_2) \quad \leq \leq |^{h(z\ )}_{(} A+ | B |)r + (A -^{(1+r)(1+r^2}_{-e^{-2i\alpha}g(z)|})^{|B|)r^2}, \tag{7}$$

for $| z | = r < 1$ *where* $A = \cos (\beta + \alpha) - a \cos (\beta - \alpha) + b \sin (\beta - \alpha) > 0$, $B = \sin (\beta + \alpha) - b \cos (\beta - \alpha) - a \sin (\beta - \alpha), g'(0) = b_1 = a + ib$ *for some choice of angles* α *and* β.

P r o o f . Since $f = h + g$ i s element of $\mathcal{S}^*_{\mathcal{H}'}$ then we have

$$h(_z z)|_{z=0} = 1, \quad g(_z z)|_{z=0} = b_1 = a + ib,$$

and if we consider (4) as a function with positive real part

$$p(z) = \begin{pmatrix} e^{i\alpha}h(z) & +e^{-i\alpha}g(z) \\ z & z \end{pmatrix} \quad (e^{i\beta} - e^{-i\beta}z^2) \tag{8}$$

has the properties $\mathfrak{Re}p(z) > 0$ and $p(0) = [\cos (\beta + \alpha) - a \cos (\beta - \alpha) + b \sin (\beta - \alpha)]i[\sin (\beta + \alpha) - b \cos (\beta - \alpha) - a \sin (\beta - \alpha)]$ where $b_1 = a + ib$ and α, β are angles .

On the other hand , the assumption $p(0) = 1$ i s not restriction for the Carath é dory class . Indeed , let $p(z)$ b e element of the Carath é dory class with $p(0) = A + iB, A > 0$, then the function

$$p1(z) = 1A(p(z) - iB)$$

satisfies the condition $p1(0) = 1$ and $\mathfrak{Re}p1(z) > 0$. This shows that $p1(z)$ is the element of the Carath é dory class . Therefore , the function

 $p1(z) = 1A(p(z) - iB) =$
$\cos\ (\beta+\alpha) - a\cos\ (1\beta-\alpha) + b\sin\ (\beta-\alpha)$

$$\times\ [\begin{pmatrix} e^{i\alpha}h(z) & -e^{-i\alpha}g(z) \\ z & z \end{pmatrix} \tag{9}$$

$-i(\sin\ (\beta+\alpha) - b\cos\ (\beta-\alpha) - a\sin\ (\beta-\alpha))]$ i s the Carath é dory function under the condition $A = \cos\ (\beta+\alpha) - a\cos\ (\beta-\alpha) + b\sin\ (\beta-\alpha) > 0.$ Then we have

$$1^1 - +r^r \leq |\, p1(z)\, | \leq\ 1^1 +_- r^r \tag{10}$$

for $p1(z) \in \mathcal{P}$ and $|\, z\, | = r < 1$. If we substitute (9) into (1 0) and after simple calculations we get

$$(A+ |\, B\, |)_1 - +(_r A- |\, B\, |)r \leq |\, p(z)\, | \leq\ (A+ |\, B\, |)_1 +_- (_r A- |\, B\, |)r. \tag{11}$$

Using (8) and (1 1) we obtain

$$(A + (_{1+} |\, B^{|)r-}_{r}\ _{)|e^i\beta}(^A_- e^{-|^B_{\beta_i}|)r^2}_{z^2|}\ \leq ||\, B\, |)r + (A- |\, B\, |^{r)|e^{i\beta}-e^{-i\beta}z^2}_{\leq\ (A+^{(1}_{e^{i\alpha}h(z)}-^{-}e^{-i\alpha}g(z)|}\ )_| r^2. \tag{12}$$

On the other hand , we have

$$11{+}r^2 \ \leq\ |\, e^{i\beta} - 1_{e-i\beta_z 2_|}\ \leq\ 11_- r^2. \tag{13}$$

Therefore , if we use (1 3) in (1 2) we obtain the desired result .

THEOREM 2 . 2 . *Let* $f = h + g$ *be element of* $\mathcal{S}^*_{\mathcal{H}}$, *then we have*

$$|\, h'(z) - e^{-2i\alpha}g'(z)\, | \leq\ (A+ |\, B\, |\, (^)_1 +_- (^{A-}_{r})^3 \,|\, B\, |)r$$

for $|\, z\, | \ = r <\ 1$ *where* $A = \cos\ (\beta+\alpha) - a\cos\ (\beta-\alpha) + b\sin\ (\beta-\alpha)\ > 0,$ $B = \sin\ (\beta+\alpha) - b\cos\ (\beta-\alpha) - a\sin\ (\beta-\alpha), g'(0) = b_1 = a + ib$ *for some choice of angles* α *and* β.

 P r o o f . Using Lemma 2 . 1 , we obtain that

$$z_{p(z)}^{p'(z)} = z_{Ap \quad 1(z)+iB}^{Ap_1'(z)}$$

for all z in the open unit disc . Also , we know that the following inequality satisfies for functions which in the Carath é dory class :

$$|z_{p(z)}^{p'(z)}| = |p' z_{p1\binom{1(z)}{(z)} + i_A^B}| \leq \quad 12_-^r r^2. \tag{14}$$

On the other hand , from the equation (8) we have

$$zh'^{(z)}_{h(z)} -^- e^{e^{-2i\alpha}_{-2i\alpha}} {}^{zg'(z)}_{g(z)} = 1^1 +_- e^{e^{-}\,2i\beta_z 2}_{-\;2i\beta_z 2} + z^{p'(z)}_{p(z)}. \tag{15}$$

Considering (14) and (1 5) t ogether , we obtain

$$|zh'^{(z)}_{h(z)} -_- e^{e^{-2i\alpha}_{-2i\alpha}} {}^{zg'(z)}_{g(z)}| \leq^= ||1^{1^1_1} +^{+-}_- e^{e^{e^{-\,-2i\beta}_{2i\beta}}_{e\,-2i\beta}{}_{-2\,i\beta}} {}^{z^2_{z^2}}_{z^2_{z^2}} |^{+z}_{+} |p'^{p^{(z)}_{(z)}}_{z^{p'(z)}_{p(z)}}||. \tag{16}$$

Also we know that

$$1^1 - +r^{r^2_2} \leq |1_1 +_- e^{-2i\beta}_{e^{-i2\beta}} z^2| \leq \quad 1^1 +_- r^{r^2_2}. \tag{17}$$

Using (1 7) and (14) in (1 6) , we get

$$|zh'^{(z)}_{h(z)} -^- e^{e^{-}\,2i\alpha_{zg'(z)}}_{-\;2i\alpha_{g}(z)}| \leq |1^1 +_- e^{e^{-}\,2i\beta_z 2}_{-\;2i\beta_z 2}| + |z^{p'(z)}_{p(z)}| = 1^1 +_- r^r. \tag{18}$$

Using Lemma 2 . 1 in (1 8) , we obtain that

$$| h'(z) - e^{-2i\alpha} g'(z) | \leq \quad (A + | B | \binom{)}{1} +_- \binom{A-}{r}{}^{)3} | B |)r.$$

Lemma 2 . 3 . *Let $\omega(z)$ be the analytic dilatation of $f = h + g \in \mathcal{S}_{\mathcal{H}}$ defined by $\omega(z) = g'(z)/h'(z)$ for all $z \in \mathbb{D}$, then we have*

$$(1 - r1^{)(1}_{+} {}_{|b_1} - |_r | b_1 |) \quad \leq | 1 - e^{-2i\alpha}\omega(z) | \leq \quad (1 +_1 r)(+^1_{|b_1} + |_r | b_1 |) \tag{19}$$

$$(| z |= r < 1) where g'(0) = b_1 \neq 0 and | b_1 |< 1.$$

P r o o f . Let we define the function

$$\phi(z) = \omega 1^{(z)}_{-} {}^{-\; b_1}_{b_1\omega(z)}$$

 where $g'(0) = b_1$ for all $z \in \mathbb{D}$. Since $\phi(z)$ is a transformation which maps $\mathbb{D}$ onto itself we have $| \phi(z) |< 1$ and $\phi(0) = 1$. Thus we can write

$$\omega(z) \prec \quad b_{1+}^{1+} z b_1 z.$$

On the other hand , the function $\omega(z) = \quad b_{1+b_1 z}^{1+z}$ maps $| z |= r$ into the circle centered at

$$C(r) = \{\mathfrak{Re}_1^{b_1(1}{}_{-|b_1} - |r^2 2_{r2}^{)}, \mathfrak{Im} 1_{-}^{b_1(1}{}_{|b_1} - |r^2 2_{r2}^{)}\},$$

having the radius

$$\rho(r) = (_1 1_- -_{|b \ \ 1|}^{|b_1 \ |^2)r} {}_{2_r 2}.$$

So we have

$$|\omega(z) - b_{1-}^{1(1} \mid -b_1 \mid^{r_2} {}_{r}^{2)} {}_2| \leq \quad (_1 1_- -_{|b \ \ 1|^2}^{|b_1 \ |^2)r} {}_{r^2}.$$

Therefore , we get the result after some simple calculations .

THEOREM 2 . 4 . *Let* $f = h + g$ *be an element of* $\mathcal{S}_{\mathcal{H}}^{*}$*, then we have*

$$| h'(z) |\leq \quad (1+ \mid b_{1(\ 1}^{|r)(A} {}_{-|b^1|r}^{+ \ |B|})(1 +_- (_r^A)_4 - \mid B \mid r)), \tag{20}$$

$$| g'(z) |\leq \quad (A+ \mid B \mid (_1^) +_- (_{r\ )^3}^{A-} \mid B \mid)r(1 + (1 -^1 +)_r(|_1^b \ _-^1 |^r \mid b_1 \mid r)) \tag{21}$$

for $| z | = r < 1$ *where* $A = \cos (\beta + \alpha) - a \cos (\beta - \alpha) + b \sin (\beta - \alpha) > 0$, $B = \sin (\beta + \alpha) - b \cos (\beta - \alpha) - a \sin (\beta - \alpha), g'(0) = b_1 = a + ib$ *for some choice of angles* α *and* β*.*

P r o o f . Let consider the analytic dilatation function $\omega = g'/h'$ of $f = h + g$. Then , we have

$$| h'(z) - e^{-2i\alpha} g'(z) |=_= |^{|} {}_{h'(z)}^{h'(z)} |^- | 1 - e_{-}{}_{e^{-2i\alpha}(z)|.}^{2i\alpha_\omega(z)h'(z)} | \tag{22}$$

Considering (19) and Theorem 2 . 2 in (22) we obtain ,

$$| h'(z) |\leq \quad (1+ \mid b_{1(\ 1}^{|r)(A} {}_{-|b^1|r}^{+ \ |B|})(1 +_- (_r^A)_4 - \mid B \mid r)),$$

and

$$| g'(z) |\leq \quad (A+ \mid B \mid (_1^) +_- (_{r\ )^3}^{A-} \mid B \mid)r(1 + (1 -^1 +)_r(|_1^b \ _-^1 |^r \mid b_1 \mid r))$$

$$\text{forall} \mid z \mid= r < 1.$$

COROLLARY 2 . 5 . *Let* $f = h + g \in \mathcal{S}^*_{\mathcal{H}},$ *then we have*

$$| f(z) | \le \int_0^r (1+ | b_1 | \rho (^{)((A+}_{1\;-} {}^{|B|)}_{|b_1|\rho})(1 +_- (^{A-}_{\rho\;)^4} | B |)\rho) d\rho$$

$$+ \int_0^r (A+ | B | (^{)}_{1} +_- (^{A-}_{\rho\;)^3} | B |)\rho(1 + (1 -^1 \rho +)(|^{b\;\;1|\rho}_{1\;-} | b_1 | \rho)) d\rho$$

for $| z | = r < 1$ *where* $A = \cos (\beta+\alpha) - a \cos (\beta-\alpha) + b \sin (\beta-\alpha) > 0,$ $B = \sin (\beta+\alpha) - b \cos (\beta-\alpha) - a \sin (\beta-\alpha), g'(0) = b_1 = a + ib$ *for some choice of angles* α *and* β.

P r o o f . For $f = h + g$, we have the following inequalities

$$f = h + g = \int_0^r h'(\rho e^{i\theta}) e^{i\theta} d\rho + \int_0^r g'(\rho e^{i\theta}) e^{i\theta} d\rho$$

$$= \int_0^r h'(\rho e^{i\theta}) e^{i\theta} d\rho + \int_0^r g'(\rho e^{i\theta}) e^{-i\theta} d\rho = \int_0^r f_z(\rho e^{i\theta}) e^{i\theta} d\rho + \int_0^r f_z(\rho e^{i\theta}) e^{-i\theta} d\rho.$$

Hence

$$| f | = | h + g | \le | h | + | g | \le \int_0^r | f_z(\rho e^{i\theta}) | d\rho + \int_0^r | f_z(\rho e^{i\theta}) | d\rho \Rightarrow$$

$$| f | \le \int_0^r | h'(\rho e^{i\theta}) | d\rho + \int_0^r | g'(\rho e^{i\theta}) | d\rho \Rightarrow .$$

Applying inequalities (20) and (2 1) t o the above , we obtain the result .

References

[1] J . Clunie and T . Sheil - Small , Harmonic univalent functions . *Ann . Acad . Aci . Fenn . Ser . A . I . Math .* **9** (1 984) , 3 - 25 .

[2] P . Duren , *Harmonic Mappings in the Plane* . Cambridge University Press , New York , 2004 . [3] A . W . Goodman , *Univalent Functions* , Volume 1 . Mariner Publishing Company , Inc . , Florida , 1983 .

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