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SOME PROPERTIES OF COMMUTING AND ANTI-COMMUTING m-INVOLUTIONS*

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Abstract We define an m-involution to be a matrix $K \in \mathbb{C}^{n\times n}$ for which $K^m = I$. In this article, we investigate the class $S_m(A)$ of m-involutions that commute with a diagonalizable matrix $A \in \mathbb{C}^{n\times n}$. A number of basic properties of $S_m(A)$ and its related subclass $S_m(A, X)$ are given, where X is an eigenvector matrix of A. Among them, $S_m(A)$ is shown to have a torsion group structure under matrix multiplication if A has distinct eigenvalues and has non-denumerable cardinality otherwise. The constructive definition of $S_m(A, X)$ allows one to generate all m-involutions commuting with a matrix with distinct eigenvalues. Some related results are also given for the class $\tilde{S}_m(A)$ of m-involutions that anti-commute with a matrix $A \in \mathbb{C}^{n\times n}$.

Key words Centrosymmetric; skew-centrosymmetric; bisymmetric; involution; eigenvalues

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1 Introduction

Let J represent the exchange matrix of order n, defined by $J_{i,j} = \delta_{i,n-j+1}$ for $1 \le i, j \le n$, where $\delta_{i,j}$ is the Kronecker delta. If $A \in \mathbb{C}^{n\times n}$ commutes with J, then A is called centrosymmetric. Centrosymmetric matrices, which appear in numerous applications, include the class of symmetric Toeplitz matrices and the class of bisymmetric matrices. A number of articles (among them, [1, 2, 5, 6], and [9]) investigated the generalization, where J is replaced by an involutory matrix K.

More recently, W. F. Trench (in [7, 8]) investigated the set of complex matrices A that satisfy $AK = \zeta^j KA$, where ζ is an m-th root of unity, $0 \le j \le m-1$, and K's minimal polynomial is $x^m - 1$ for an integer $m > 1$. Trench referred to such K as m-involutions, but in this article, we use the term m-involution to refer to any matrix K for which $K^m = I$. We refer to the set of matrices whose minimal polynomial is $x^m - 1$ as the non-trivial m-involutions. This terminology is consistent with usage in the case $m = 2$, where the matrices $\pm I$ are regarded as trivial involutions.

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In this article, we define a K_m-symmetric matrix to be a complex matrix A that commutes with an m-involutory matrix K. In articles studying K_2-symmetric and K_m-symmetric matrices, the matrix K is usually considered fixed and the properties of the matrix A are studied. One difficulty in applying results of this nature is that, except in special cases, it is typically not easy to discern the K_m-symmetry of a matrix A by mere inspection. In this article, we take a somewhat different approach by fixing A and studying the class $S_m(A)$ of m-involutory matrices K that commute with it. To begin the investigation, we introduce a constructive subset of $S_m(A)$ that enables the easy generation of m-involutions that commute with A. Now, the task of identifying the K_m-symmetry of a matrix A can, to some extent, be reduced to a relatively straightforward computation. We also show that $S_m(A)$ has a torsion group structure under matrix multiplication if A has distinct eigenvalues and is infinite cardinality otherwise.

If $A \in \mathbb{C}^{n\times n}$ anti-commutes with the exchange matrix J, then A is called either skew-centrosymmetric or centroskew per reference [4]. As in the centrosymmetric case, the articles [5, 6, 9] investigated the case where J is replaced by a general involution, while [7] looked at a broader generalization (mentioned above) that includes the commuting and anti-commuting cases. This motivates a study of the anti-commuting case, and so we close this article by establishing some related results for the class $\tilde{S}_m(A)$ of m-involutory matrices K which anti-commute with A.

2 Preliminaries

In [5, 6, 9], several K_2-symmetric matrix analogs to well-known results for centrosymmetric matrices were established. For example, during the 1960s and 1970s, Collar [4], Andrew [2], and Cantoni and Butler [3] each noted (in slightly different contexts) that the eigenbasis $\{x_i\}_{i=1}^n$ of an $n \times n$ centrosymmetric matrix A is composed of $\lceil \frac{n}{2} \rceil$ symmetric eigenvectors (that is, $x_i = Jx_i$) and $\lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors (that is, $x_i = -Jx_i$). In [5], this observation was extended Tao and Yasuda to the situation where J is replaced by a non-trivial involutory matrix K and both A and K are real symmetric ([5, Lemma 3.11]). This was a direct generalization of the context considered in [3]. The article [9] extended this to the Hermitian case for A and K ([8, Propositions 3.5 and 4.1]). In [6], W. F. Trench strengthened this result ([6, Theorem 7]) by showing that K_2-symmetric matrices A have a basis consisting of K-symmetric (that is, $x_i = Kx_i$) and K-skew-symmetric eigenvectors (that is, $x_i = -Kx_i$) without the Hermitian condition assumed in [9]. Trench also established the converse, thereby generalizing a result of Andrew for centrosymmetric matrices ([2, Theorem 2]). We will refer to these two results of Trench collectively as the Eigenbasis Theorem for K_2-symmetric matrices.

More recently, in [7], Trench extended this result to the class of complex matrices A which commute with a non-trivial m-involutory matrix K. Let $\zeta = e^{2\pi \mathrm{i}/m}$. His result ([7, Theorem 13]) states that if $K \in \mathbb{C}^{n\times n}$ is non-trivial m-involutory, $A \in \mathbb{C}^{n\times n}$ is K_m-symmetric, and λ is an eigenvalue of A, then the λ-eigenspace of A has a basis in $Q_A = \bigcup_{j=0}^{m-1} \left\{x | Kx = \zeta^j x\right\}$. Conversely, he showed that if a matrix A has n linearly independent vectors in Q_A, then A is K_m-symmetric. Accordingly, we will collectively refer to these two more general statements as the Eigenbasis Theorem for K_m-symmetric matrices. Following Trench, we will refer to vectors satisfying $Kx = \zeta^r x$ as (K, r)-symmetric vectors.

In this article, we use the notation $S_m(A)$ to denote the set of complex m-involutory matrices K that commute with a complex matrix A. We use the notation $T_p(x)(A)$ to denote the set of complex m-involutory matrices with minimal polynomial $p(x)$ that commute with A. Note that

$$S_m(A) = \bigcup_{\substack{p(x)|x^m-1 \\ p(x)\in\mathbb{C}[x] \\ p(x)\text{monic}}} T_{p(x)}(A).$$

Let $\alpha, \mu \in \{0, 1, \cdots, k-1\}$ and let R and S be non-trivial m-involutions. In Trench's articles [7, 8], the class of matrices A satisfying the equation

$$RAS^{-\alpha} = \zeta^{\mu} A \tag{1}$$

are studied. When $\alpha = 1$, $\mu = 0$, and $R = S$, the matrices R and S belong to the set $T_{p(x)}(A)$ where $p(x) = x^m - 1$. Trench's focus is on the class of matrices A satisfying (1), and while he does provide some general results concerning non-trivial m-involutions, he does not attempt to investigate the collective properties of either the set $T_{p(x)}(A)$ or its superset $S_m(A)$.

3 Construction and Properties of $S_m(A, X)$

In what follows, we assume that A is diagonalizable. Let $A = X\Lambda X^{-1}$, where X is an eigenvector matrix of A and Λ is the diagonal eigenvalue matrix of A. For convenience, we will assume that all the columns of X are normalized and that there is a fixed ordering for the eigenvalues in Λ.

If we wish to construct a non-trivial subset of $S_m(A)$, consideration of simultaneous diagonalization and familiarity with the Eigenbasis Theorem for K_m-symmetric matrices leads naturally to the investigation of matrices of the form $K = XDX^{-1}$, where D is a diagonal matrix whose diagonal elements belong to the set of the m-th roots of unity.

Definition 3.1 For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n\times n}$, we define

$$S_m(A, X) = \left\{ XDX^{-1} | D \ \text{ diagonal with } \ D_{i,i} \in \left\{\zeta^j\right\}_{j=0}^{m-1} \right\},$$

where $\zeta = e^{2\pi \mathrm{i}/m}$.

The following theorem lists some of the more elementary properties concerning the set $S_m(A, X)$ and its relationship to $S_m(A)$.

Theorem 3.2 For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n\times n}$, we have

$$S_m(A, X) \subseteq S_m(A), \tag{2}$$

$$|S_m(A, X)| = m^n, \tag{3}$$

and

$$S_2(A) = \bigcup_X S_2(A, X) \tag{4}$$

where the union ranges over all possible eigenvector matrices X of A.

Proof (2) and (3) follow easily from the definition of $S_m(A, X)$.

To show (4), we first note that the trivial involutions $\pm I$ in $S_2(A)$ are obtained using $D=\pm I$ for any eigenvector matrix X of A. For any non-trivial involution K, the Eigenbasis Theorem for K_2-symmetric matrices states that there exists an eigenbasis for A consisting of K-symmetric and K-skew-symmetric eigenvectors. Let $X=\begin{pmatrix}x_1 \; x_2 \; \cdots \; x_n\end{pmatrix}$ have columns comprised of such a basis. If we choose $D=(d_{i,j})_{1\le i,j\le n}$ to be the diagonal matrix having $d_{j,j}=-1$ for each K-skew-symmetric eigenvector x_j and 1 for the remaining diagonal elements, then $K=XDX^{-1}$. That is, we recover K with this choice of D.

So, for any non-trivial involutory $K\in S_2(A)$, there exists an eigenvector matrix X of A and a choice of D such that $K=XDX^{-1}$. This shows that $S_2(A)\subseteq\bigcup\limits_X S_2(A,X)$, and (2) shows the reverse inclusion.

In the proof of the third assertion of 3.2, we showed how one could choose the elements of a diagonal matrix D to recover a matrix $K\in S_2(A)$, given an eigenvector matrix for A consisting of K-symmetric and K-skew-symmetric columns. We now state the analogous "recovery" theorem for a nontrivial m-involution $K\in S_m(A)$. The proof follows along the same lines as that for the $S_2(A)$ case so we omit it.

Theorem 3.3 Let $A\in\mathbb{C}^{n\times n}$ be K_m-symmetric, where K is a non-trivial m-involution. Suppose that $X=\begin{pmatrix}x_1 \; x_2 \; \cdots \; x_n\end{pmatrix}$ is an eigenvector matrix for A, where each column x_j is (K,r_j)-symmetric for some $0\le r_j\le m-1$. As A is K_m-symmetric, such an X is guaranteed to exist by the Eigenbasis Theorem for K_m-symmetric matrices. Then, there exists a diagonal matrix $D=(d_{i,j})_{1\le i,j\le n}$, such that $K=XDX^{-1}$ and whose diagonal elements $d(j,j)$ satisfy $d_{j,j}=\zeta^{n-r_j}$. As usual, $\zeta=e^{2\pi\mathrm{i}/m}$.

The following corollary is immediate from 3.3.

Corollary 3.4 Let $A\in\mathbb{C}^{n\times n}$ be K_m-symmetric, where K is a non-trivial m-involution. If X_a and X_b are eigenvector matrices of A, each of whose columns is (K,r_j)-symmetric for some $0\le r_j\le m-1$, then $K\in S_m(A,X_a)\cap S_m(A,X_b)$

Example If A is a centrosymmetric matrix and X is any eigenvector matrix of A comprised of symmetric and skew-symmetric vectors, then the exchange matrix J is guaranteed to be found in $S_2(A,X)$.

Equations (2) and (3) of 3.2 show that after one determines an eigenvector matrix X and its inverse X^{-1} for a diagonalizable matrix A, one can then identify m^n different K_m-symmetries of A by simply computing the matrix products XDX^{-1}, where one varies the diagonal elements of the matrix D over the different combinations of m-th roots of unity. This computation can be further made more efficient by making the following observations.

1. Let $K=XDX^{-1}$ and $\tilde{K}=X\tilde{D}X^{-1}$ be members of $S_m(A,X)$, where the diagonal matrices $D=\mathrm{diagonal}(d_{ii})$ and $\tilde{D}=\mathrm{diagonal}(\tilde{d}_{ii})$ differ only by a factor of ζ in their kth diagonal elements. If $\tilde{d}_{kk}=\zeta d_{kk}$, and we denote the (i,j)-th element of X^{-1} by y_{ij}, then element $\tilde{k}_{ij}$ of $\tilde{K}$ is related to element k_{ij} of K by

$$\tilde{k}_{ij}=k_{ij}+(\zeta-1)d_{kk}x_{ik}y_{kj}.$$

So, as one iterates through the products XDX^{-1} by varying D, if an update to D only involves a ζ-scaling of a single diagonal element, then deriving an updated $\tilde{K}$ from K requires just $3n^2$ multiplications and n^2+1 additions.

2. If $K \in S_m(A, X_j)$, then $\zeta^r K \in S_m(A, X_j)$ for $0 \le r \le m-1$. To avoid the unnecessary computation of these scalar multiples of K, we can simply fix a diagonal element of D (say d_{11}) at 1 and vary the other diagonal elements over the m-th roots of unity. To see this, let $K = XDX^{-1}$ be an element of $S_m(A, X_j)$ for some $D = \text{diagonal}\begin{pmatrix} \zeta^{r_1} & \zeta^{r_2} & \cdots & \zeta^{r_n} \end{pmatrix}$. Then,

$$\zeta^{-r_1} KX = \begin{pmatrix} x_1 & \zeta^{r_2 - r_1} x_2 & \cdots & \zeta^{r_n - r_1} x_n \end{pmatrix}.$$

This shows that there exists a matrix $\tilde{K} \in S_m(A, X_j)$, such that $K = \zeta^r \tilde{K}$, where $\tilde{K} = X\tilde{D}X^{-1}$ and $\tilde{D}$ has the value 1 for its $d_{1,1}$ element. Employing this observation reduces the number of matrix product computations needed to enumerate the elements of $S_m(A, X_j)$ by a factor of m.

After generating the elements of $S_m(A, X)$, one can then select distinguished elements K from the set and apply known theoretical results for K_m-symmetric matrices to a particular study involving A.

As $S_m(A, X)$ is finite, it is not surprising that one can make some fairly strong statements about its elements. The next theorem lists two such statements and shows that, when a matrix $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues, these statements apply to $S_m(A)$ itself.

Theorem 3.5 If $A \in \mathbb{C}^{n\times n}$ is diagonalizable with an eigenvector matrix X, then

1. If A has distinct eigenvalues (hence, is diagonalizable), then

$$S_m(A) = S_m(A, X), \tag{5}$$

2. The number of non-trivial m-involutory matrices in $S_m(A, X)$ is

$$\sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (m-j)^n. \tag{6}$$

Proof The proof of the first assertion (5) is elementary and has been omitted.

For the second assertion (6), we wish to compute the cardinality of $T_p(x)(A)$ when $p(x) = x^m - 1$ and $p(x)$ has distinct roots. To determine this, one simply needs to count the number of diagonal matrices D whose set of diagonal elements contain at least one of each of the m-th roots of unity. This can be performed by a standard inclusion-exclusion enumeration.

We now characterize the algebraic structure of $S_m(A, X)$, which, as we have shown in Theorem 3.5, equals $S_m(A)$ when A has distinct eigenvalues.

Theorem 3.6 Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable with an eigenvector matrix X. Then, the set $S_m(A, X)$ under matrix multiplication forms a torsion group $G_m(A, X)$ that is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}_m$.

Proof That $S_m(A, X)$ is an abelian group follows easily from its definition. The mapping

$$X \begin{pmatrix} \zeta^{j_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{j_n} \end{pmatrix} X^{-1} \mapsto (j_1) \oplus \cdots \oplus (j_n)$$

is clearly a group homomorphism from $G_m(A, X)$ to $\bigoplus_{i=1}^{n} \mathbb{Z}_m$ with a trivial kernel, thereby establishing the isomorphism.

Corollary 3.7 If A is centrosymmetric with distinct eigenvalues, then the involutory matrices that commute with it are centrosymmetric.

Proof This follows from equation (5) and Theorem 3.6 because the exchange matrix J is a member of the abelian group $G_2(A, X)$.

It is easy to produce examples that show that Corollary 3.7 fails if the restriction that A has distinct eigenvalues is violated. However, it is also not hard to show that Corollary 3.7 holds without the restriction to involutory matrices. We start by defining the simultaneously diagonalizable family $R(A, X)$.

Definition 3.8 For a fixed eigenvector matrix X of $A \in \mathbb{C}^{n\times n}$, let

$$R(A, X) = \left\{XDX^{-1} | D \in \mathbb{C}^{n\times n} \text{ diagonal}\right\}.$$

It is clear that $R(A, X)$ is a superset of $S_m(A, X)$ for every m, and that $R(A, X)$ is a commutative monoid under matrix multiplication.

Lemma 3.9 Suppose $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues. If $B \in \mathbb{C}^{n\times n}$ commutes with A, then $B \in R(A, X)$.

Proof As $\Lambda = X^{-1}AX$ has distinct entries and B commutes with A, it follows that $X^{-1}BX$ commutes with Λ. The only matrices which commute with diagonal matrices with distinct entries are diagonal matrices. Therefore $X^{-1}BX$ is diagonal, and so $B \in R(A, X)$.

Theorem 3.10 If A is centrosymmetric with distinct eigenvalues, then the matrices that commute with it are centrosymmetric.

Proof Suppose that matrices B_i and B_j commute with A. From Lemma 3.9, we verify that B_i commutes with B_j. Letting $B_i = J$ shows that B_j is centrosymmetric.

Corollary 3.11 If A is real, bisymmetric and has distinct eigenvalues, then the matrices that commute with it are bisymmetric.

Proof If A has real entries and $A = A^T$, then X can be taken to be a matrix with real entries and $X^{-1} = X^T$. If B commutes with A, then X^TBX must be a diagonal matrix by Lemma 3.9. Hence,

$$X^TB^TX = (X^TBX)^T = D^T = D = X^TBX.$$

Multiplying the left side by X and the right side by X^T gives $B^T = B$. This and Theorem 3.10 yield the result.

4 $S_m(A, X)$ and the Non-Distinct Eigenvalue Case

When a diagonalizable matrix A has distinct eigenvalues, equation (5) implies that Theorem 3.6 is a result about $S_m(A)$ itself. If, in contrast, A has at least one eigenvalue with multiplicity greater than one, several of the nice algebraic properties of Theorem 3.6 no longer hold. In particular, commutativity and closure do not hold in general for $S_m(A)$, and the cardinality of $S_m(A)$ will be infinite. The underlying reasons for the latter are that $\bigcup_X S_m(A, X) \subseteq S_2(A)$ from (2), that there are now infinitely many choices for the normalized columns of A's eigenvector matrix X, and that $S_m(A, X_i)$ and $S_m(A, X_j)$ can differ when X_i and X_j differ. We will be more precise about this in what follows.

First, however, we prove a statement about the commonality between the sets $S_m(A, X_i)$ and $S_m(A, X_j)$. As the identity matrix belongs to every set $S_m(A, X)$, they are clearly not disjoint. In fact, one can always find at least m elements in common between any pair of these sets, and when A does not have distinct eigenvalues a larger lower bound can be established. We begin by establishing a preliminary lemma.

Lemma 4.1 Let $X_i \in \mathbb{C}^{n\times n}$ and $X_j \in \mathbb{C}^{n\times n}$ be nonsingular matrices, where the first $0 \le r \le n$ columns of X_j are a linear combination of the first r columns of X_i arising from right multiplication by a nonsingular linear transformation which leaves the remaining columns of X_i fixed. Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix of the form

$$D = \begin{pmatrix} \alpha I_r & 0 \\ 0 & D_{22} \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and α is a fixed complex number. Then, $X_i D X_i^{-1} = X_j D X_j^{-1}$.

Proof By hypothesis, $X_j = X_i M$, where

$$M = \begin{pmatrix} M_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix}, \tag{7}$$

with $M_{11} \in \mathbb{C}^{r\times r}$ nonsingular and I_{n-r} is the $(n-r)\times(n-r)$ identity matrix.

Let

$$X_i = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \quad \text{and} \quad X_i^{-1} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \tag{8}$$

under the same partitioning as D and M. If we write D_{11} in place of αI_r to allow us to express the forthcoming equalities more generally, then we see that $K = X_j D X_j^{-1} = X_i M D M^{-1} X_i^{-1}$ is comprised of the blocks

$$K_{11} = X_{11} M_{11} D_{11} M_{11}^{-1} Y_{11} + X_{12} D_{22} Y_{21}, \tag{9}$$

$$K_{12} = X_{11} M_{11} D_{11} M_{11}^{-1} Y_{12} + X_{12} D_{22} Y_{22}, \tag{10}$$

$$K_{21} = X_{21} M_{11} D_{11} M_{11}^{-1} Y_{11} + X_{22} D_{22} Y_{21}, \tag{11}$$

$$K_{22} = X_{21} M_{11} D_{11} M_{11}^{-1} Y_{12} + X_{22} D_{22} Y_{22}. \tag{12}$$

This is easily shown to equal $X_i D X_i^{-1}$.

Theorem 4.2 Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable and let its eigenvalues of multiplicity greater than one comprise a set of values $V = \{v_k\}_{k=1}^{s}$ where $0 \le s < n$ (V may be empty). Let $\mu_k > 1$ be the multiplicity of A's eigenvalue with value v_k for $1 \le k \le$s. Then, if X_i and X_j are eigenvector matrices of A, we have $|S_m(A, X_i) \bigcap S_m(A, X_j)| \ge m^{n-r+s}$ where $r = \sum\limits_{i=1}^{s} \mu_i$.

Proof Recall that the eigenvector matrices X_i and X_j of A are assumed to have normalized columns. So, aside from scalar multiples of magnitude one, the columns of X_i and X_j can only differ in the columns corresponding to the eigenvalues of multiplicity greater than one.

For convenience, we will assume that the diagonal elements of the eigenvalue matrix $\Lambda = \operatorname{diag}(\lambda_k)_{1\le k\le n}$ are arranged so that the distinct eigenvalues λ_i are positioned in the last $n-r$

diagonal elements and the eigenvalues of multiplicity greater than one are set up consecutively in the first r diagonal elements as

$$\Lambda_{jj} = \begin{cases} v_1 & \text{for } \ 1 \le j \le \mu_1, \\ v_2 & \text{for } \ 1 + k_1 \le j \le \mu_1 + \mu_2, \\ \vdots & \\ v_s & \text{for } \ 1 + \sum\limits_{i=1}^{s-1} \mu_i \le j \le r. \end{cases}$$

This establishes an ordering for the columns of the matrices X_i and X_j.

Consider $X_i D X_i^{-1} \in S_m(A, X_i)$ and $X_j D X_j^{-1} \in S_m(A, X_j)$, where D has the form

$$D = \begin{pmatrix} \zeta^k I_{11} & 0 \\ 0 & D_{22} \end{pmatrix}, \tag{13}$$

$\zeta = e^{2\pi \mathrm{i}/m}$, and k is an integer in the range $0 \le k \le m-1$. Application of Lemma 4.1 shows that $X_i D X_i^{-1} = X_j D X_j^{-1}$. As there are m choices for $\zeta^k I_{11}$ and m^{n-r} choices for D_{22}, we have m^{n-r+1} matrices D of the form (13) for which $X_i D X_i^{-1} = X_j D X_j^{-1}$.

We now consider the 2×2 block repartitioning of D, where the new upper-left block is the upper-left $(r-\mu_s) \times (r-\mu_s)$ portion of the previous $r \times r$ upper-left block $\zeta^k I_{11}$. This new partitioning effectively means that the last μ_s elements of D_{11} in the previous partitioning have been transferred to the D_{22} block of the new partitioning. Another application of Lemma 4.1 shows that if D has the form (13) under the new partitioning, then here too $X_i D X_i^{-1} = X_j D X_j^{-1}$. The number of choices for D under this partitioning that are distinct from that found in the previous partitioning is $(m-1)m^{n-r+1}$. So, the number of choices for D under the two partitionings is $m^{n-r+1} + (m-1)m^{n-r+1} = m^{n-r+2}$.

We can now proceed inductively, reducing the upper-left block's size by μ_j at each stage, for $j = s-1$ through $j = 1$.

Corollary 4.3 Let X_i and X_j be any two eigenvector matrices of A.

1. In the case where all eigenvalues of A are the same (that is, multiplicity n),

$$\left| S_m(A, X_i) \bigcap S_m(A, X_j) \right| \ge m.$$

2. In the case where all eigenvalues of A are distinct,

$$S_m(A, X_i) = S_m(A, X_j).$$

Proof For the first assertion, apply Theorem 4.2 with $r = n$ and $s = 1$. The m elements found between any pair of $S_m(A, X_i)$ and $S_m(A, X_j)$ are precisely the matrices $\zeta^j I$ for $0 \le j \le m-1$.

For the second assertion, apply Theorem 4.2 with $r = 0$ and $s = 0$ together with equation (3).

A cursory study of the situation where A is a multiple of the identity matrix I_n for $n > 1$ is sufficient to generate examples where commutativity and closure fail for elements of $S_m(A)$ under matrix multiplication. It is also not hard to show that there are infinitely many m-involutions of every dimension greater than one, thereby showing that $|S_m(\alpha I_n)|$ is infinite for every $n > 1$. The latter statement, in fact, holds for $S_m(A)$, where A is any diagonalizable matrix in $\mathbb{C}^{n\times n}$ with an eigenvalue of multiplicity greater than one.

Theorem 4.4 Suppose that $A \in \mathbb{C}^{n\times n}$ is diagonalizable and has at least one eigenvalue of multiplicity greater than one. Then, the cardinality of the set $S_m(A)$ is non-denumerable.

Proof As in the proof of Theorem 4.2, we assume that the elements of the diagonal eigenvalue matrix $\Lambda = (\Lambda_{ij})_{1\le i,j\le n}$ are arranged so that any eigenvalues of multiplicity greater than one are placed consecutively in the upper-left. Let us reuse partitions (7) and (8), where we take the size of the upper-left block to be 2×2. We will also use the diagonal matrix partition

$$D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix},$$

where we take

$$D_{11} = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

with $\zeta = e^{2\pi \mathrm{i}/m}$ and consider D_{22} to be fixed.

Let

$$M_{11} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

and define $\tilde{M} = M_{11}D_{11}M_{11}^{-1}$.

Our expression for $K \in S_m(A)$ is comprised of the block equations (9), (10), (11), and (12). As the only degrees of freedom in these equations are the elements m_{11}, m_{12}, m_{21}, and m_{22}, we may confine our attention to the products

$$X_{11}\tilde{M}Y_{11}, \tag{14}$$

$$X_{11}\tilde{M}Y_{12}, \tag{15}$$

$$X_{21}\tilde{M}Y_{11}, \tag{16}$$

$$X_{21}\tilde{M}Y_{12}. \tag{17}$$

Computing $\tilde{M} = M_{11}D_{11}M_{11}^{-1}$, we find

$$\tilde{M} = \mu \begin{pmatrix} m_{11}m_{22} - \zeta m_{12}m_{21} & (\zeta - 1)m_{11}m_{12} \\ (1-\zeta)m_{21}m_{22} & -m_{12}m_{21} + \zeta m_{11}m_{22} \end{pmatrix}, \tag{18}$$

where $\mu = (m_{11}m_{22} - m_{12}m_{21})^{-1}$. For convenience, assume the partitioning

$$\tilde{M} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix}.$$

In (18), we note that the magnitude of the complex number $\tilde{m}_{12}$ ranges continuously over $[0, \infty)$ by (for example) setting $m_{22} = 0$, fixing m_{12} and m_{21} at nonzero values, and letting m_{11} vary. Similarly, we note that the magnitude of $\tilde{m}_{21}$ ranges continuously over $[0, \infty)$ by setting $m_{11} = 0$, fixing m_{12} and m_{21} at nonzero values, and letting m_{22} vary.

Let $z = m_{11}m_{22}$ and set $m_{12}m_{21} = 1$. Under this choice, $\tilde{m}_{11}$ becomes the complex function $\frac{z-\zeta}{z-1}$ and $\tilde{m}_{22}$ becomes the complex function $\frac{\zeta z-1}{z-1}$. From this, we see that here too, the magnitude of the elements $\tilde{m}_{11}$ and $\tilde{m}_{22}$ each range continuously over $[0,\infty)$.

From (8), we observe that the blocks X_{11} and X_{21} cannot be simultaneously zero. Similarly, Y_{11} and Y_{12} cannot be simultaneously zero. This leads us to consider four separate cases, namely,

1. $X_{11} \neq 0$ and $Y_{11} \neq 0$,
2. $X_{11} \neq 0$ and $Y_{12} \neq 0$,
3. $X_{21} \neq 0$ and $Y_{11} \neq 0$,
4. $X_{21} \neq 0$ and $Y_{12} \neq 0$.

At least one of these cases must be true.

For Case 1, we focus on equation (14). As $X_{11} \neq 0$, there exists at least one row of X_{11} to be nonzero. Pick one such row, and let it have values $x = \begin{pmatrix} \alpha & \beta \end{pmatrix}$. As $Y_{11} \neq 0$, there exists at least one column of Y_{11} that is nonzero. Pick one such column and let it have values $y^T = \begin{pmatrix} \gamma & \delta \end{pmatrix}^T$. Then,

$$x\tilde{M}y = \gamma\alpha\tilde{m}_{11} + \gamma\beta\tilde{m}_{21} + \delta\alpha\tilde{m}_{12} + \delta\beta\tilde{m}_{22} \tag{19}$$

As α and β are not simultaneously zero and γ and δ are not simultaneously zero, at least one of the four summands in (19) must not be identically nonzero. If only one summand is nonzero, as we know the magnitude of $\tilde{m}_{11}$, $\tilde{m}_{21}$, $\tilde{m}_{12}$, and $\tilde{m}_{22}$ each range continuously over $[0,\infty)$ for varying choices of m_{11}, m_{12}, m_{21}, and m_{22}, we verify that $x\tilde{M}y$ assumes a non-denumerable set of values. So, suppose at least two of the four summands in (19) are not identically zero. We wish to show that $x\tilde{M}y$ cannot be limited to a discrete set of values. Equation (18) shows that $\tilde{m}_{11}$, $\tilde{m}_{21}$, $\tilde{m}_{12}$, and $\tilde{m}_{22}$ are continuous complex valued functions, and only $\tilde{m}_{11}$ and $\tilde{m}_{22}$ can be linearly related (for example, when $\zeta = -1$). But if the coefficients of $\tilde{m}_{11}$ and $\tilde{m}_{22}$ are both nonzero in the linear combination (19), then the coefficients of $\tilde{m}_{21}$ and $\tilde{m}_{12}$ will also be nonzero. Therefore, if at least two of the four summands in (18) are nonzero, there will be a nonzero term involving either the product $\mu m_{11}m_{12}$ (that is, $\tilde{m}_{12}$) or the product $\mu m_{21}m_{22}$ (that is, $\tilde{m}_{21}$). As only one of them involves the product $\mu m_{11}m_{12}$ and only one of the four summands (19) involves the product $\mu m_{21}m_{22}$, the product $x\tilde{M}y$ cannot be limited to a discrete set of values. We conclude that the product in (14) assumes a non-denumerable number of values under the assumptions of Case 1.

The treatment of Cases 2, 3, and 4 may be treated in the same manner, with Case 2 dealing with equation (15), Case 3 with equation (16), and Case 4 with equation (17). As the form of D used in the demonstration occurs in every set $S_m(A, X)$ for which A has dimension two or more, and as at least one of the four cases must be true, this completes the proof.

5 The Anti-Commuting Case

In addition to studying K_2-symmetry, the articles [5, 6, 9] also investigated matrices A that satisfy the anti-commuting relationship $AK = -KA$, where K is an involution (the K_2-skew-centrosymmetric matrices). It would therefore be desirable to obtain results similar to those found for $S_m(A)$ that hold for the set of complex m-involutions that anti-commute with a fixed complex matrix $A \in \mathbb{C}^{n\times n}$.

Let us denote this set as $\tilde{S}_m(A)$ and suppose K is an m-involution. Some facts are immediately apparent. From a trivial determinant argument, we see that if n is odd and A is nonsingular, then $\tilde{S}_m(A)$ is empty. Also, as $A = K^m A = (-1)^m AK^m = (-1)^m A$, if m is odd, it then follows that $\tilde{S}_m(A)$ is empty except when A is the zero matrix, in which case $\tilde{S}_m(A)$ is non-denumerable for $n > 1$. So, we only need to focus on the case where m is even.

A deeper study of $\tilde{S}_m(A)$ requires a characterization of the structure of both A and K when $AK = -AK$. To accomplish this, we turn to the Jordan decomposition of A $\tilde{A} = S^{-1}AS$ of A, where

$$\tilde{A} = \mathrm{diag}(\mathcal{J}_1(\lambda_1), \cdots, \mathcal{J}_k(\lambda_k)),$$

the Jordan blocks $\mathcal{J}_i$ are $n_i \times n_i$, and $n = \sum n_i$. Let $\tilde{K}$ be any matrix of the same size as A, and block partition it so that its $\tilde{K}_{ij}$ block is $n_i \times n_j$. If let $K = S\tilde{K}S^{-1}$, then $AK = -KA$ if and only if $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$ if and only if $\mathcal{J}_i\tilde{K}_{ij} = -\tilde{K}_{ij}\mathcal{J}_i$ for all i, j. The following lemma helps describe the structure of the blocks $\mathcal{J}_i$ and $\tilde{K}_{ij}$ whenever $\tilde{A}\tilde{K} = -\tilde{K}\tilde{A}$.

Lemma 5.1 Suppose $B \in \mathbb{C}^{m\times n}$, I_m is the $m \times m$ identity matrix, and N_m is the $m \times m$ matrix with ones directly above the main diagonal and zeros elsewhere. Let $b_{i,j}$ denote the (i, j)-element of the matrix B. Then,

1. If $\lambda_i + \lambda_j \neq 0$, then

$$(\lambda_i I_m + N_m)B = -B(\lambda_j I_n + N_n) \tag{20}$$

if and only if $B = 0$.

2. If $\lambda_i + \lambda_j = 0$, then

$$(\lambda_i I_m + N_m)B = -B(\lambda_j I_n + N_n) \tag{21}$$

if and only if B is upper triangular, where the components of columns 1 through $n - m$ are zero if $n > m$, the components of rows $m + 1$ through n are zero if $n < m$, and each diagonal of the upper triangle has constant magnitude and alternates in sign along the diagonal (that is, $b_{i,j} = -b_{i+1,j+1}$).

Proof To prove the first assertion, first rewrite equation (20) as

$$((\lambda_i + \lambda_j)I_m + N_m)B + BN_n = 0. \tag{22}$$

Multiplying (22) the left-side by N_n^j for $1 \leq j \leq n-1$ gives $n - 1$ equalities of the form

$$((\lambda_i + \lambda_j)I_m + N_m)BN_n^j + BN_n^{j+1} = 0. \tag{23}$$

As $N_n^n = 0$ and $(\lambda_i + \lambda_j)I_m + N_m$ is nonsingular, if let $j = n-1$ in equation (23), we have $BN_n^{n-1} = 0$. Substituting this into equation (23) for $j = n-2$, we see that $BN_n^{n-2} = 0$. Continuing in this manner, we arrive at $BN_n = 0$ when $j = 1$ which, when substituting into (22), gives us $B = 0$.

To prove the second assertion, we first simplify equation (21) as

$$N_m B = -BN_n. \tag{24}$$

Comparison of the first column of each side of (24) shows that the matrix B is zero in the first column below the element $b_{1,1}$. Equating the j-th column of each side of (24) gives

$$(b_{2,j}, b_{3,j}, \cdots, b_{n,j}, 0)^T = -(b_{1,j-1}, b_{2,j-1}, \cdots, b_{n-1,j-1}, b_{n,j-1})^T. \tag{25}$$

This shows that the diagonals of B are of equal magnitude and alternate in sign, and are zero below the main diagonal. Finally, we note that the m-th row of (24) shows that $b_{m,1} = b_{m,2} = \cdots = b_{m,n-1} = 0$. Combining this with (25) shows that columns one through $n - m$ are zero when $n > m$.

The next lemma will be useful in the determination of the cardinality of $\tilde{S}_m(A)$.

Lemma 5.2 Suppose that $B \in \mathbb{C}^{n\times n}$ has the form described in Lemma 5.1 and $n > 1$. That is, B is upper triangular and each element satisfies $b_{i,j} = -b_{i+1,j+1}$. Then, there are uncountably many involutory B.

Proof Let $C = B^2$. Then, $c_{i,j} = \sum\limits_{k=i}^{j} b_{i,k}b_{k,j} = \sum\limits_{k=1}^{j-i+1} (-1)^{k+1} b_{1,k}b_{1,j-k-i+2}$. As $c_{i_1,j_1} = c_{i_2,j_2}$ whenever $j_2 - i_2 = j_1 - i_1$, we see that C is upper triangular Toeplitz and so we can restrict our attention to the values of the first row of C. For convenience, let us write c_j for $c_{1,j}$ and b_j for $b_{1,j}$. Then, $c_j = \sum\limits_{k=1}^{j} (-1)^{k+1} b_k b_{j-k+1}$. When j is even, $c_j = 0$. When $j = 2m+1$, we have

$$c_{2m+1} = (-1)^{m+1} b_{m+1}^2 + 2\sum_{k=1}^{m} (-1)^{k+1} b_k b_{2m-k+2}. \tag{26}$$

We want $C = I$. The constraint $c_1 = 1$ yields two solutions for b_1. If $n = 2$, then b_2 is unconstrained and we are done. Assume $n > 2$. As the solutions for b_1 can be substituted back into (26), the equation $c_3 = 0$ becomes a complex bivariate quadratic in b_2 and b_3, which has uncountably many solutions for b_2 and b_3. This process can be continued as m is incremented. Each time m is increased by one, the solutions of the previous iterations can be substituted into (26) and two new unknowns b_{2m} and b_{2m+1} appear, resulting in another bivariate quadratic for which there are uncountably many solutions.

Theorem 5.3 Given $A \in \mathbb{C}^{n\times n}$ and an m-involution $K \in \mathbb{C}^{n\times n}$:

1. Let $n = 1$. If A is the number zero, then $\tilde{S}_m(A)$ contains the m-th roots of unity. Otherwise, $\tilde{S}_m(A)$ is empty.

2. Suppose $n > 1$ and let $m > 1$ be an odd integer. If A is the zero matrix, then $\tilde{S}_m(A)$ is non-denumerable. Otherwise, $\tilde{S}_m(A)$ is empty.

3. Suppose $n > 1$ and let $m > 1$ be an even integer. If the non-zero eigenvalues of A come in pairs of opposite signs where the corresponding pairs have Jordan blocks of equal size, then the cardinality of the set $\tilde{S}_m(A)$ is non-denumerable. Otherwise, $\tilde{S}_m(A)$ is empty.

Proof The first assertion is trivial, while the second assertion's proof was given at the beginning of this section. We proceed with the proof of the third assertion.

From Lemma 5.1, we see that if A anti-commutes with K, then its nonzero eigenvalues must come in pairs of opposite signs where the corresponding pairs have Jordan blocks of equal size. So, to demonstrate the theorem's third assertion, we only need to show that $\tilde{S}_m(A)$ is non-denumerable when A satisfies this condition. For convenience, we will assume that the pairs of Jordan blocks with eigenvalues of opposite sign appear consecutively along the diagonal of $\tilde{A}$. The proof will be broken into several cases.

1. Case: $\tilde{A}$ is the zero matrix.

2. Case: $\tilde{A}$ has all eigenvalues zero and has at least Jordan block that is $m \times m$ where $m > 1$.

3. Case: $\tilde{A}$ has at least one pair of Jordan blocks for non-zero eigenvalues of opposite signs that are both $m \times m$ where $m > 1$.

4. Case: All Jordan blocks of $\tilde{A}$ corresponding to non-zero eigenvalues are 1×1.

Case 1 is trivial, so we move on to Case 2. For Case 2, we consider a particular class of block-diagonal matrices $\tilde{K}$ whose diagonal blocks are of the same size as those of $\tilde{A}$. Assume, without loss of generality, that an $m \times m$ Jordan block of $\tilde{A}$, where $m > 1$, occupies the $1, 1$ block position. Consider the set of block matrices whose $1, 1$ block is an upper triangular involution whose elements satisfy $k_{i,j} = -k_{i+1,j+1}$, and remaining diagonal blocks are diagonal matrices with alternating values of 1 and -1. The elements of this set are clearly involutions, they anti-commute with $\tilde{A}$ because their form satisfies the conditions of Lemma 5.1, and Lemma 5.2 shows that there are uncountably many choices for the $1, 1$ block. As m is even, this set of involutions belongs to $\tilde{S}_m(A)$ and so we are done with this case.

Case 3. Assume, without loss of generality, that the $1, 1$ and $2, 2$ Jordan block positions of $\tilde{A}$ are occupied by blocks corresponding to a positive-negative eigenvalue pair and whose block sizes are $m \times m$ where $m > 1$. We choose the $1, 2$ and $2, 1$ blocks of $\tilde{K}$, such that $\tilde{K}_{1,2} = \tilde{K}_{2,1}$ and $\tilde{K}_{1,2}$ and $\tilde{K}_{2,1}$ are upper triangular involutions whose elements satisfy $k_{i,j} = -k_{i+1,j+1}$. For the remaining $\tilde{A}$ Jordan block pairs in the i, i and $i+1, i+1$ positions, pick the $\tilde{K}$ blocks in the $i, i+1$ and $i+1, i$ positions to be identical diagonal matrices with alternating 1's and -1's on their diagonals. For the $\tilde{A}$ blocks in the i, i positions corresponding to zero eigenvalues, let the i, i block of $\tilde{K}$ also be diagonal with alternating 1's and -1's. Assume that the remaining blocks of $\tilde{K}$ are zero. Then, $\tilde{K}$ is an involution, Lemma 5.2 shows that there are uncountably many choices for $\tilde{K}$'s $1, 1$ and $2, 2$ blocks, and Lemma 5.1 shows that $\tilde{K}$ anti-commutes with $\tilde{A}$. As noted in Case 2, these involutions belong to $\tilde{S}_m(A)$.

Case 4. Assume, without loss of generality, that the $1, 1$ and $2, 2$ Jordan block positions of $\tilde{A}$ are occupied by 1×1 blocks corresponding to a positive-negative eigenvalue pair. Pick elements $k_{1,2}$ and $k_{2,1}$ of $\tilde{K}$, such that $k_{1,2}k_{2,1} = 1$. For the remaining 1×1 Jordan block pairs of $\tilde{A}$ in the i, i and $i + 1, i + 1$ positions, pick the elements $k_{i,i+1}$ and $k_{i+1,i}$ to be 1. For the $\tilde{A}$ blocks in the i, i positions corresponding to zero eigenvalues, let the i, i block of $\tilde{K}$ be diagonal with alternating 1's and -1's (as in Case 3) and assume the remaining blocks of $\tilde{K}$ are zero. Then, $\tilde{K}$ is an involution, there are uncountably many choices for $k_{1,2}$ and $k_{2,1}$, and Lemma 5.1 shows that $\tilde{K}$ anti-commutes with $\tilde{A}$. As noted in Case 2, these involutions belong to $\tilde{S}_m(A)$.

Because these results for $\tilde{A}$ and $\tilde{K}$ in Cases 1-4 translate directly into anti-commuting results for A and K, we are done.

Lemma 5.1, together with the m-involution constraint, characterizes the matrices $\tilde{K}$, which in turn characterizes the elements of $\tilde{S}_m(A)$. This characterization can then be used to develop an algorithm for generating subsets of $\tilde{S}_m(A)$. For example, after computing the Jordan decomposition $\tilde{A}$ of A, one could construct examples of $\tilde{K}$ as follows. For the $\tilde{A}$ Jordan block pairs corresponding to non-zero eigenvalues in the i, i and $i+1, i+1$ block positions, pick the $\tilde{K}$ blocks in the $i, i + 1$ and $i + 1, i$ positions to be identical diagonal matrices whose $1, 1$ diagonal element is an m-th root of unity and whose subsequent diagonal elements alternate in sign. For the $\tilde{A}$ blocks in the i, i positions corresponding to zero eigenvalues, let the i, i block of $\tilde{K}$ also be diagonal whose $1, 1$ diagonal element is an m-th root of unity and whose subsequent diagonal elements alternate in sign. Assume that the remaining blocks of $\tilde{K}$ are zero.

Of course, there are many other elements of $\tilde{S}_m(A)$ that this particular construction of $\tilde{K}$ does not consider. For example, other blocks of $\tilde{K}$ may be nonzero, in particular, if there are multiple Jordan blocks of $\tilde{A}$ corresponding to a particular eigenvalue value. Also, it is not necessary for the blocks of $\tilde{K}$ to be diagonal. The example of construction described in the previous paragraph could be modified to include such possibilities. The main drawback to this construction, however, is its reliance on the Jordan decomposition, which is known to be very sensitive numerically. Constructing an efficient and stable algorithm for generating elements of $\tilde{S}_m(A)$ that does not rely on first computing the Jordan decomposition of A is a possible topic for future investigation.

6 Concluding Remarks

This article has explored some of the fundamental properties of the class $S_m(A)$ of m-involutory matrices that commute with a given diagonalizable matrix A. The constructive nature of the definition of $S_m(A, X)$ allows one to easily generate numerous (in some cases all) m-involutions commuting with A. By providing a constructive means to generate elements of $S_m(A)$, it now becomes easier to identify the types K_m-symmetry satisfied by a matrix A and thereby use the body of results that accumulated in recent years regarding K_m-symmetry. Other results were given for the class $\tilde{S}_m(A)$ of m-involutory matrices that anti-commute with A. It is hoped that the results obtained in this article will lead to additional insights and research related to the class $S_m(A)$, $\tilde{S}_m(A)$, the study of K_m-symmetric matrices, and their applications.

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