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Partial sums of starlike harmonic multivalent functions

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Abstract In this paper, we study the ratio of starlike harmonic multivalent functions to its sequences of partial sums.

Keywords Harmonic · Multivalent · Starlike · Partial sums

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. The harmonic function $f = h + \overline{g}$ is sense preserving and locally one to one in D if $\left|h'(z)\right| > \left|g'(z)\right|$ in D. See Clunie and Sheil-Small [3].

For $m \geq 1$, denote by $SH(m)$ the class of functions $f = h + \overline{g}$ that are sense preserving, harmonic multivalent in the unit disk $U = \{z : |z| < 1\}$, where h and g defined by

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1 \tag{1}$$

which are analytic and multivalent functions in U.

Note that $SH(m)$ reduces to $S(m)$, the class of analytic multivalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero. The class $SH(1)$, the class of harmonic univalent functions, was introduced by Clunie and Sheil-Small [3].

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Let $SH^*(m,\alpha)$ denote the subclass of $SH(m)$ consisting of functions $f = h+\overline{g} \in SH(m)$ that satisfy the condition

$$Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}\right\} > m\alpha. \tag{2}$$
$$(0 \le \alpha < 1, \quad m \ge 1, \quad z \in U)$$

Denote by $TSH(m)$ the subclass of $SH(m)$, consist of harmonic functions $f = h + \overline{g}$ where h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|\, z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}|\, z^{n+m-1}. \tag{3}$$

Define $TSH^*(m,\alpha) := SH^*(m,\alpha) \cap TSH(m)$.

The class $SH^*(m,\alpha)$ was introduced and studied by Ahuja and Jahangiri ([1,2]). In particular, they stated the following sufficient coefficient condition:

Lemma 1.1 [1] *Let $f = h + \overline{g}$ be given by* (1). *If*

$$\sum_{n=1}^{\infty}\left[\frac{n-1+m(1-\alpha)}{m(1-\alpha)}\,|a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)}\,|b_{n+m-1}|\right] \le 2, \tag{4}$$

where $a_m = 1$, $m \ge 1$ and $0 \le \alpha < 1$ then the harmonic function f is sense-preserving, m-valent and $f \in SH^*(m,\alpha)$.

Ahuja and Jahangiri [1] also obtained the condition (4) is necessary for $f \in TSH^*(m,\alpha)$.

Porwal and Dixit [6] studied partial sums of the class $SH^*(1,\alpha)$ (the class of starlike harmonic univalent functions). In this paper, applying methods used by Silverman [7], Silvia [8], Porwal and Dixit [6] and Porwal [4,5] we investigate the ratio of a function of the form (1) satisfying the condition (4) to its sequence of partial sums

$$f_p(z) := f_{p+m-1}(z) = z^m + \sum_{n=2}^{p} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} \bar{z}^{n+m-1}$$
$$f_s(z) := f_{s+m-1}(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{s} \bar{b}_{n+m-1} \bar{z}^{n+m-1}$$
$$f_{p,s}(z) := f_{p+m-1,s+m-1}(z) = z^m + \sum_{n=2}^{p} a_{n+m-1} z^{n+m-1} + \sum_{n=1}^{s} \bar{b}_{n+m-1} \bar{z}^{n+m-1}.$$

Also, we determine the sharp lower bounds for $Re\left\{\frac{f(z)}{f_p(z)}\right\}$, $Re\left\{\frac{f_p(z)}{f(z)}\right\}$, $Re\left\{\frac{f_p'(z)}{f'(z)}\right\}$, $Re\left\{\frac{f(z)}{f_s(z)}\right\}$, $Re\left\{\frac{f_s(z)}{f(z)}\right\}$, $Re\left\{\frac{f_s'(z)}{f(z)}\right\}$, $Re\left\{\frac{f(z)}{f_{p,s}(z)}\right\}$, $Re\left\{\frac{f_{p,s}(z)}{f(z)}\right\}$, $Re\left\{\frac{f_{p,s}'(z)}{f'(z)}\right\}$, where $f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta})$.

2 Main results

Theorem 2.1 *If f of the form* (1) *satisfies condition* (4)*, then*

$$Re\left\{\frac{f(z)}{f_p(z)}\right\} \ge \frac{p}{p+m(1-\alpha)} (z \in U). \tag{5}$$

The result (5) *is sharp with the function*

$$f(z) = z^m + \frac{m(1-\alpha)}{p+m(1-\alpha)} z^{m+p}. \tag{6}$$

Proof We may write

$$\begin{aligned}
&\frac{1+w(z)}{1-w(z)} \\
&= \frac{p+m(1-\alpha)}{m(1-\alpha)} \left[\frac{f(re^{i\theta})}{f_p(re^{i\theta})} - \frac{p}{p+m(1-\alpha)} \right] \\
&= \frac{1+\sum_{n=2}^{p} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{p} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}} \\
&\quad + \frac{\frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta}}{1+\sum_{n=2}^{p} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}}.
\end{aligned}$$

So that

$$w(z) = \frac{\frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta}}{2+2\left(\sum_{n=2}^{p} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}\right) + \frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta}}.$$

Then

$$|w(z)| \leq \frac{\frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} |a_{n+m-1}|}{2-2\left(\sum_{n=2}^{p} |a_{n+m-1}| + \sum_{n=1}^{\infty} |b_{n+m-1}|\right) - \frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} |a_{n+m-1}|}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{p} |a_{n+m-1}| + \sum_{n=1}^{\infty} |b_{n+m-1}| + \frac{p+m(1-\alpha)}{m(1-\alpha)} \sum_{n=p+1}^{\infty} |a_{n+m-1}| \leq 1. \tag{7}$$

It suffices to show that the left hand side of (7) is bounded above by $\sum\limits_{n=2}^{\infty} \frac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1}| + \sum\limits_{n=1}^{\infty} \frac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1}|$, which is equivalent to

$$\sum_{n=2}^{p} \frac{n-1}{m(1-\alpha)} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{n-1+2m\alpha}{m(1-\alpha)} |b_{n+m-1}| + \sum_{n=p+1}^{\infty} \frac{n-p-1}{m(1-\alpha)} |a_{n+m-1}| \geq 0.$$

To see that $f(z) = z^m + \frac{m(1-\alpha)}{p+m(1-\alpha)} z^{m+p}$ gives the sharp result, we observe that for $z = re^{i\pi/p}$ that

$$\frac{f(z)}{f_p(z)} = 1 + \frac{m(1-\alpha)}{p+m(1-\alpha)} z^p \to 1 - \frac{m(1-\alpha)}{p+m(1-\alpha)} = \frac{p}{p+m(1-\alpha)},$$

when $r \to 1^-$. This completes the proof of Theorem 2.1

Theorem 2.2 *If f of the form* (1) *satisfies condition* (4), *then*

$$Re\left\{\frac{f_p(z)}{f(z)}\right\} \geq \frac{p+m(1-\alpha)}{p+2m(1-\alpha)} (z \in U). \tag{8}$$

The result (8) *is sharp with the function given by* (6).

Proof We may write

$$\frac{1+w(z)}{1-w(z)}$$
$$=\frac{p+2m(1-\alpha)}{m(1-\alpha)}\left[\frac{f_p(re^{i\theta})}{f(re^{i\theta})}-\frac{p+m(1-\alpha)}{p+2m(1-\alpha)}\right]$$
$$=\frac{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}$$
$$-\frac{\frac{p+m(1-\alpha)}{m(1-\alpha)}\sum_{n=p+1}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},$$

where

$$|w(z)|\le\frac{\frac{p+2m(1-\alpha)}{m(1-\alpha)}\sum_{n=p+1}^{\infty}|a_{n+m-1}|}{2-2\left(\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{\infty}|b_{n+m-1}|\right)-\frac{p}{m(1-\alpha)}\sum_{n=p+1}^{\infty}|a_{n+m-1}|}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{\infty}|b_{n+m-1}|+\frac{p+m(1-\alpha)}{m(1-\alpha)}\sum_{n=p+1}^{\infty}|a_{n+m-1}|\le 1. \tag{9}$$

It suffices to show that the left hand side of (9) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$\sum_{n=2}^{p}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|+\sum_{n=p+1}^{\infty}\frac{n-p-1}{m(1-\alpha)}|a_{n+m-1}|\ge 0.$$

This completes the proof of Theorem 2.2

Theorem 2.3 *If f of the form (1) satisfies condition (4), then*

$$Re\left\{\frac{f_p'(z)}{f'(z)}\right\}\ge\frac{p+m(1-\alpha)}{p+(p+m+1)m(1-\alpha)}\ (z\in U). \tag{10}$$

The result (8) is sharp with the function given by (6).

Proof We may write

$$\frac{1+w(z)}{1-w(z)}$$
$$=\frac{p+(p+m+1)m(1-\alpha)}{(p+m)m(1-\alpha)}\left[\frac{f_p'(re^{i\theta})}{f'(re^{i\theta})}-\frac{p+m(1-\alpha)}{p+(p+m+1)m(1-\alpha)}\right]$$
$$=\frac{1+\sum_{n=2}^{p}\frac{n+m-1}{m}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}-\sum_{n=1}^{\infty}\frac{n+m-1}{m}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty}\frac{n+m-1}{m}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}-\sum_{n=1}^{\infty}\frac{n+m-1}{m}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}$$
$$-\frac{\frac{p+m(1-\alpha)}{(p+m)m(1-\alpha)}\sum_{n=p+1}^{\infty}\frac{n+m-1}{m}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}}{1+\sum_{n=2}^{\infty}\frac{n+m-1}{m}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}-\sum_{n=1}^{\infty}\frac{n+m-1}{m}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},$$

where

$$|w(z)| \leq \frac{\frac{p+m(1-\alpha)}{(p+m)m(1-\alpha)} \sum_{n=p+1}^{\infty} \frac{n+m-1}{m} |a_{n+m-1}|}{2-2\left(\sum_{n=2}^{p} \frac{n+m-1}{m} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{n+m-1}{m} |b_{n+m-1}|\right) - \frac{p+m(1-\alpha)}{(p+m)m(1-\alpha)} \sum_{n=p+1}^{\infty} \frac{n+m-1}{m} |a_{n+m-1}|}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{p} \frac{n+m-1}{m} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{n+m-1}{m} |b_{n+m-1}| + \frac{p+m(1-\alpha)}{(p+m)m(1-\alpha)} \sum_{n=p+1}^{\infty} \frac{n+m-1}{m} |a_{n+m-1}| \leq 1. \tag{11}$$

It suffices to show that the left hand side of (11) is bounded above by $\sum_{n=2}^{\infty} \frac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1}|$, which is equivalent to

$$\sum_{n=2}^{p} \frac{(n-1)\alpha}{m(1-\alpha)} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{(n+2m-1)\alpha}{m(1-\alpha)} |b_{n+m-1}| + \sum_{n=p+1}^{\infty} \left[\frac{n+m-1}{m(1-\alpha)} \left(\frac{m(p+m-1+\alpha)-p}{m(p+m)}\right) - \frac{\alpha}{1-\alpha}\right] |a_{n+m-1}| \geq 0.$$

This completes the proof of Theorem 2.3

Theorem 2.4 *If f of the form* (1) *satisfies condition* (4), *then*

$$Re\left\{\frac{f(z)}{f_s(z)}\right\} \geq \frac{s+2\alpha m}{s+m(1+\alpha)} (z \in U). \tag{12}$$

The result (12) *is sharp with the function*

$$f(z) = z^m + \frac{m(1-\alpha)}{s+m(1+\alpha)} \bar{z}^{m+s}. \tag{13}$$

Proof We may write

$$\begin{aligned}
\frac{1+w(z)}{1-w(z)} &= \frac{s+m(1+\alpha)}{m(1-\alpha)} \left[\frac{f(re^{i\theta})}{f_s(re^{i\theta})} - \frac{s+2\alpha m}{s+m(1+\alpha)}\right] \\
&= \frac{1+\sum_{n=2}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{s} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}} \\
&\quad + \frac{\frac{s+m(1+\alpha)}{m(1-\alpha)} \sum_{n=s+1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty} a_{n+m-1} r^{n-1} e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1} r^{n-1} e^{-i(n+2m-1)\theta}},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{s+m(1+\alpha)}{m(1-\alpha)} \sum_{n=s+1}^{\infty} |b_{n+m-1}|}{2-2\left(\sum_{n=2}^{\infty} |a_{n+m-1}| + \sum_{n=1}^{s} |b_{n+m-1}|\right) - \frac{s+m(1+\alpha)}{m(1-\alpha)} \sum_{n=s+1}^{\infty} |b_{n+m-1}|}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{\infty}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|+\frac{s+m(1+\alpha)}{m(1-\alpha)}\sum_{n=s+1}^{\infty}|b_{n+m-1}|\leq 1. \tag{14}$$

It suffices to show that the left hand side of (14) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$\sum_{n=2}^{\infty}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{s}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|+\sum_{n=s+1}^{\infty}\frac{n-s-1}{m(1-\alpha)}|b_{n+m-1}|\geq 0.$$

To see that $f(z)=z^m+\frac{m(1-\alpha)}{s+m(1+\alpha)}\bar{z}^{m+s}$ gives the sharp result, we observe that for $z=re^{i\pi/2m+s}$ that

$$\frac{f(z)}{f_s(z)}=1+\frac{m(1-\alpha)}{s+m(1+\alpha)}r^s e^{-i(\frac{2m+s}{2m+s})\pi}\to 1-\frac{m(1-\alpha)}{s+m(1+\alpha)}=\frac{s+2m\alpha}{s+m(1+\alpha)},$$

when $r\to 1^-$. This completes the proof of Theorem 2.4

Theorem 2.5 *If f of the form (1) satisfies condition (4), then*

$$Re\left\{\frac{f_s(z)}{f(z)}\right\}\geq\frac{s+m(1+\alpha)}{s+2m}\ (z\in U). \tag{15}$$

The result (12) is sharp with the function $f(z)$ given by (13).

Proof We may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\frac{f_s(re^{i\theta})}{f(re^{i\theta})}-\frac{s+m(1+\alpha)}{s+2m}\right]\\ &= \frac{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}\\ &\quad -\frac{\frac{s+m(1+\alpha)}{m(1-\alpha)}\sum_{n=s+1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}.\end{aligned}$$

We omit the details of the proof, because it runs parallel to that from Theorem 2.4.

Theorem 2.6 *If f of the form (1) satisfies condition (4), then*

(i) *if $p\leq s+2m\alpha$ or $b_{n+m-1}=0$, $\forall n\geq 2$,*

$$Re\left\{\frac{f(z)}{f_{p,s}(z)}\right\}\geq\frac{p}{p+m(1-\alpha)}(z\in U), \tag{16}$$

(ii) *if $p\geq s+2m\alpha$ or $a_{n+m-1}=0$, $\forall n\geq 2$,*

$$Re\left\{\frac{f(z)}{f_{p,s}(z)}\right\}\geq\frac{s+2m\alpha}{s+m(1+\alpha)}\ (z\in U). \tag{17}$$

The results (16) and (17) are sharp with the functions given by (6) and (13), respectively.

Proof To prove (i) part, we may write

$$
\begin{aligned}
&\frac{1+w(z)}{1-w(z)}\\
&\quad=\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\frac{f(re^{i\theta})}{f_{p,s}(re^{i\theta})}-\frac{p}{p+m(1-\alpha)}\right]\\
&\quad=\frac{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}\\
&\qquad+\frac{\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=s+1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}\right]}{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},
\end{aligned}
$$

where

$$
|w(z)|\le\frac{\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}{2-2\left(\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|\right)-\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}.
$$

This last expression is bounded above by 1, if and only if

$$
\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|+\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]\le 1. \tag{18}
$$

It suffices to show that the left hand side of (18) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$
\begin{aligned}
&\sum_{n=2}^{p}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|\\
&\quad+\sum_{n=p+1}^{\infty}\frac{n-p-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}\frac{n-p-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|\ge 0.
\end{aligned}
$$

To see that $f(z)=z^m+\frac{m(1-\alpha)}{p+m(1-\alpha)}z^{m+p}$ gives the sharp result, we observe that for $z=re^{i\pi/p}$ that

$$
\frac{f(z)}{f_{p,s}(z)}=1+\frac{m(1-\alpha)}{p+m(1-\alpha)}z^p\to 1-\frac{m(1-\alpha)}{p+m(1-\alpha)},
$$

when $r\to 1^-$.

To prove second part, we write

$$
\begin{aligned}
&\frac{1+w(z)}{1-w(z)}\\
&\quad=\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\frac{f(re^{i\theta})}{f_{p,s}(re^{i\theta})}-\frac{s+2m\alpha}{s+m(1+\alpha)}\right]\\
&\quad=\frac{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}\\
&\qquad+\frac{\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=s+1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}\right]}{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},
\end{aligned}
$$

where

$$
|w(z)|\le\frac{\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}{2-2\left(\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|\right)-\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}.
$$

This last expression is bounded above by 1, if and only if

$$
\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|+\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]\le 1. \tag{19}
$$

It suffices to show that the left hand side of (19) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$
\begin{aligned}
&\sum_{n=2}^{p}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|\\
&\quad+\sum_{n=p+1}^{\infty}\frac{n-s-1-2m\alpha}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}\frac{n-s-1}{m(1-\alpha)}|b_{n+m-1}|\ge 0.
\end{aligned}
$$

To see that $f(z)=z^m+\frac{m(1-\alpha)}{s+m(1+\alpha)}\bar{z}^{m+s}$ gives the sharp result, we observe that for $z=re^{i\pi/2m+s}$ that

$$
\frac{f(z)}{f_{p,s}(z)}=1+\frac{m(1-\alpha)}{s+m(1+\alpha)}r^{s}e^{-i(\frac{2m+s}{2m+s})\pi}\to 1-\frac{m(1-\alpha)}{s+m(1+\alpha)},
$$

when $r\to 1^{-}$.

Theorem 2.7 *If f of the form (1) satisfies condition (4), then*

(i) *if $p\le s+2\alpha m$ or $b_{n+m-1}=0$, $\forall n\ge 2$,*

$$
Re\left\{\frac{f_{p,s}(z)}{f(z)}\right\}\ge\frac{p+m(1-\alpha)}{p+2m(1-\alpha)}\ (z\in U), \tag{20}
$$

(ii) *if* $p \geq s + 2\alpha m$ *or* $a_{n+m-1} = 0$, $\forall n \geq 2$,

$$Re\left\{\frac{f_{p,s}(z)}{f(z)}\right\} \geq \frac{s+m(1+\alpha)}{s+2m} \ (z \in U). \tag{21}$$

The results (20) *and* (21) *are sharp with the functions given by* (6) *and* (13)*, respectively.*

Proof To prove (i) part, we may write

$$\begin{aligned}
&\frac{1+w(z)}{1-w(z)}\\
&\quad= \frac{p+2m(1-\alpha)}{m(1-\alpha)}\left[\frac{f_{p,s}(re^{i\theta})}{f(re^{i\theta})} - \frac{p+m(1-\alpha)}{p+2m(1-\alpha)}\right]\\
&\quad= \frac{1+\sum_{n=2}^{p} a_{n+m-1}r^{n-1}e^{i(n-1)\theta} + \sum_{n=1}^{s} \bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty} a_{n+m-1}r^{n-1}e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}\\
&\qquad- \frac{\frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty} a_{n+m-1}r^{n-1}e^{i(n-1)\theta} + \sum_{n=s+1}^{\infty} \bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}\right]}{1+\sum_{n=2}^{\infty} a_{n+m-1}r^{n-1}e^{i(n-1)\theta} + \sum_{n=1}^{\infty} \bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},
\end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{p+2m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}| + \sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}{2-2\left(\sum_{n=2}^{p}|a_{n+m-1}| + \sum_{n=1}^{s}|b_{n+m-1}|\right) - \frac{p}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}| + \sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{p}|a_{n+m-1}| + \sum_{n=1}^{s}|b_{n+m-1}| + \frac{p+m(1-\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}| + \sum_{n=s+1}^{\infty}|b_{n+m-1}|\right] \leq 1. \tag{22}$$

It suffices to show that the left hand side of (22) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}| + \sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$\begin{aligned}
&\sum_{n=2}^{p}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}| + \sum_{n=1}^{\infty}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|\\
&\quad+ \sum_{n=p+1}^{\infty}\frac{n-p-1}{m(1-\alpha)}|a_{n+m-1}| + \sum_{n=s+1}^{\infty}\frac{n-p-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}| \geq 0.
\end{aligned}$$

To prove second part, we write

$$\begin{aligned}&\frac{1+w(z)}{1-w(z)}\\&=\frac{s+2m}{m(1-\alpha)}\left[\frac{f_{p,s}(re^{i\theta})}{f(re^{i\theta})}-\frac{s+m(1+\alpha)}{s+2m}\right]\\&=\frac{1+\sum_{n=2}^{p}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{s}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}}\\&\quad-\frac{\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=s+1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}\right]}{1+\sum_{n=2}^{\infty}a_{n+m-1}r^{n-1}e^{i(n-1)\theta}+\sum_{n=1}^{\infty}\bar{b}_{n+m-1}r^{n-1}e^{-i(n+2m-1)\theta}},\end{aligned}$$

where

$$|w(z)|\le\frac{\frac{s+2m}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}{2-2\left(\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|\right)-\frac{s+2m\alpha}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=2}^{p}|a_{n+m-1}|+\sum_{n=1}^{s}|b_{n+m-1}|+\frac{s+m(1+\alpha)}{m(1-\alpha)}\left[\sum_{n=p+1}^{\infty}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}|b_{n+m-1}|\right]\le 1. \tag{23}$$

It suffices to show that the left hand side of (23) is bounded above by $\sum_{n=2}^{\infty}\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+m(1+\alpha)}{m(1-\alpha)}|b_{n+m-1}|$, which is equivalent to

$$\begin{aligned}&\sum_{n=2}^{p}\frac{n-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{n-1+2m\alpha}{m(1-\alpha)}|b_{n+m-1}|\\&\quad+\sum_{n=p+1}^{\infty}\frac{n-s-1-2m\alpha}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=s+1}^{\infty}\frac{n-s-1}{m(1-\alpha)}|b_{n+m-1}|\ge 0.\end{aligned}$$

Theorem 2.8 *If f of the form* (1) *satisfies condition* (4)*, then*

$$Re\left\{\frac{f'_{p,s}(z)}{f'(z)}\right\}\ge\frac{p+m(1-\alpha)}{p+(p+m+1)m(1-\alpha)}(z\in U). \tag{24}$$

The result (24) *is sharp with the function given by* (6).

Proof The proof of the above theorem is similar to that of Theorem 2.3 so we omit the details involved.

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