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Approximations for Constant e and Their Applications

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Two equalities for the approximation of constant e and their applications are considered. © 2001 Academic Press

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1. INTRODUCTION

It is well known that the constant e plays an important role in many areas of mathematics. It is involved in many inequalities, identities, series expansions, and some special functions. The well known Hardy's inequality and Carleman's inequality are good examples of applications of approximation of e.

Recently, there have been many results in generalizing the above mentioned two inequalities by using better approximations of e. In [1], Yang and Debnath obtained the inequalities

$$e\left[1-\frac{1}{2(n+\frac{5}{6})}\right] < \left(1+\frac{1}{n}\right)^n < e\left[1-\frac{1}{2(n+1)}\right]. \tag{1.1}$$

As an application of (1.1), they proved the following strengthened Carleman's inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}\left[1-\frac{1}{2(n+1)}\right]a_n, \tag{1.2}$$

where $a_n \geq 0$, $n = 1, 2, \ldots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

In [2], Yang showed that for $x > 0$,

$$e\left[1-\frac{1}{2x+1}\right] < \left(1+\frac{1}{x}\right)^x < e\left[1-\frac{1}{2(x+1)}\right]. \tag{1.3}$$

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As an application of (1.3), he obtained the following strengthened Hardy's inequality

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\sigma_n} < e \sum_{n=1}^{\infty} \left[1 - \frac{\lambda_n}{2(\sigma_n + \lambda_n)}\right] \lambda_n a_n, \tag{1.4}$$

where $0 < \lambda_{n+1} \leq \lambda_n$, $\sigma_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$.

Sandor and Debnath [3] showed that

$$e\sqrt{\frac{x}{1+x}} < \left(1 + \frac{1}{x}\right)^x < e\left(\frac{2x+1}{2(x+1)}\right) \tag{1.5}$$

holds for all $x > 0$.

In [4], Xie and Zhong improved (1.3) and obtained the inequalities

$$e\left(1 - \frac{7}{14x + 12}\right) < \left(1 + \frac{1}{x}\right)^x < e\left(1 - \frac{6}{12x + 11}\right), \quad \text{for } x > 0. \tag{1.6}$$

As an application of (1.6), they replaced the right side of (1.4) by a sharper one: $e \sum_{n=1}^{\infty} [1 - 6\lambda_n/(12\sigma_n + 11\lambda_n)] \lambda_n a_n$.

The author of this paper [5] obtained a better approximation of e than (1.3),

$$\left(1 + \frac{1}{x}\right)^x < e\left[1 - \sum_{k=1}^{6} \frac{b_k}{(1+x)^k}\right], \tag{1.7}$$

where $x > 0$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}$, $b_4 = \frac{73}{5760}$, $b_5 = \frac{11}{1280}$, $b_6 = \frac{1945}{580608}$.

It is conjectured in [5] that if the following equality holds

$$\left(1 + \frac{1}{x}\right)^x = e\left[1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+x)^k}\right), \qquad x > 0. \tag{1.8}$$

then $b_k > 0$, $k = 1, 2, \ldots$.

As an application of (1.7), Yang [5] obtained the following strengthened Carleman's inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)} - \frac{1}{24(n+1)^2} - \frac{1}{48(n+1)^3}\right) a_n, \tag{1.9}$$

where $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n < \infty$.

In this paper, two equalities are given, which improve the inequalities (1.3) and (1.7); the conjecture in [5] is also proved. As applications, Hardy's inequality and Carleman's inequality are strengthened.

2. MAIN RESULTS

THEOREM 1. *For $x > 0$, let*

$$\left(\frac{1+x}{x}\right)^x = e\left[1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+x)^k}\right]. \tag{2.1}$$

Then $b_k > 0$, $k = 1, 2, \ldots$, and $\{b_k\}_{k=1}^n$ satisfy the following recursion formula:

$$\begin{aligned} b_1 &= \frac{1}{2}, \\ b_{n+1} &= \frac{1}{n+1}\left[\frac{1}{n+2} - \sum_{j=1}^{n} \frac{b_j}{n+2-j}\right], \qquad n = 1, 2, \ldots. \end{aligned} \tag{2.2}$$

THEOREM 2. *For $x > 0$, let*

$$\left(\frac{1+x}{x}\right)^x = e\left[1 - \sum_{k=1}^{\infty} \frac{a_k}{(2x+1)^k}\right]. \tag{2.3}$$

Then the $\{a_k\}_{k=1}^n$ satisfy the following recursion formula

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= -c_{n+1} + \sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) a_j c_{n+1-j}, \qquad n = 1, 2, \ldots, \end{aligned} \tag{2.4}$$

where

$$c_m = \begin{cases} -\frac{1}{m}, & \text{if } m \text{ is odd,} \\ \frac{1}{m+1}, & \text{if } m \text{ is even.} \end{cases}$$

Moreover, for $n = 1, 2, \ldots$,

$$a_{2n} = -a_{2n+1} < 0.$$

Proof of Theorem 1. Let $y = \frac{1}{1+x}$. Then $0 < y < 1$ and (2.1) is written as

$$\sum_{k=1}^{\infty} b_k y^k = 1 - \left(\frac{1}{1-y}\right)^{\frac{1-y}{y}} \Big/ e. \tag{2.5}$$

Denote $h(y) = (\frac{1}{1-y})^{(1-y)/y}/e$. Then $\ln h(y) = -1 + (\frac{1-y}{y}) \ln(1-y) = -1 + (1-y)\sum_{n=1}^{\infty}(y^{n-1}/n) = -\sum_{n=1}^{\infty}(y^n/n(n+1))$, hence

$$h(y) = e^{-\sum_{n=1}^{\infty} \frac{y^n}{n(n+1)}}.$$

If we write $g(y) = -\sum_{n=1}^{\infty}(y^n/n(n+1))$, then $h(y) = e^{g(y)}$ and $g^{(n)}(0) = -(n-1)!/(n+1)$, $n = 1, 2, \ldots$; Moreover,

$$h'(y) = h(y)g'(y) \tag{2.6}$$

$$h^{(n+1)}(y) = \sum_{j=0}^{n} C_n^j h^{(j)}(y) g^{(n+1-j)}(y). \tag{2.7}$$

By (2.5), (2.6), and (2.7), we obtain

$$\begin{aligned} b_1 &= -h'(0) = -h(0)g'(0) = \frac{1}{2} \\ b_{n+1} &= -h^{(n+1)}(0)/(n+1)! \\ &= -\sum_{j=0}^{n} C_n^j h^{(j)}(0) g^{(n+1-j)}(0)/(n+1)! \\ &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(-h^{(j)}(0))\left(\frac{-(n-j)!}{(n+2-j)}\right)\Big/(n+1)! \\ &= \frac{1}{n+1}\sum_{j=0}^{n}\left(\frac{-h^{(j)}(0)}{j!}\right)\cdot\frac{-1}{(n+2-j)} \\ &= \frac{1}{n+1}\left[\frac{1}{n+2} - \sum_{j=1}^{n}\frac{b_j}{n+2-j}\right]. \end{aligned}$$

Therefore (2.2) is proved. Next we show $0 < b_n < \frac{1}{n(n+1)}$, $n = 2, 3, \ldots$. From (2.2), it is easy to obtain the first 17 values of b_n's:

$$b_1 = \frac{1}{2},\ b_2 = \frac{1}{24},\ b_3 = \frac{1}{48},\ b_4 = \frac{73}{5760},\ b_5 = \frac{11}{1280},$$

$$b_6 = \frac{1945}{580608},\ b_7 = 0.00496,\ b_8 = 0.00386,\ b_9 = 0.00311,$$

$$b_{10} = 0.00257,\ b_{11} = 0.00216,\ b_{12} = 0.00191,\ b_{13} = 0.00159,$$

$$b_{14} = 0.00139,\ b_{15} = 0.00126,\ b_{16} = 0.00088,\ b_{17} = 0.00063.$$

Therefore $0 < b_n < \frac{1}{n(n+1)}$ holds for $n = 2, 3, \ldots, 16$. For $n \geq 17$, we have by (2.2), $b_{n+1} < \frac{1}{(n+1)(n+2)}$, if $b_1, \ldots, b_n > 0$.

Suppose, by induction, that $0 < b_m < \frac{1}{m(m+1)}$, $m = 2, 3, \ldots, n$. Then by (2.2), $b_{n+1} < \frac{1}{(n+1)(n+2)}$, and from (2.2), we need only to show

$$\sum_{j=1}^{n}\frac{b_j}{n+2-j} < \frac{1}{n+2}. \tag{2.8}$$

Since $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}$, (2.8) is equivalent to

$$\sum_{j=4}^{n} \frac{b_j}{n+2-j} < \frac{1}{n+2} - \frac{1}{2(n+1)} - \frac{1}{24n} - \frac{1}{48(n-1)}$$

$$= \frac{21n^3 - 31n^2 + 4}{48(n-1)n(n+1)(n+2)}. \tag{2.9}$$

Next we show the following stronger inequalities (since by assumption $0 < b_j < \frac{1}{j(j+1)}$, $j = 2, 3, \ldots, n$):

$$\sum_{j=4}^{n} \frac{b_j}{n+2-j} < \sum_{j=4}^{n} \frac{1}{j(j+1)(n+2-j)}$$

$$< \frac{21n^3 - 31n^2 + 4}{48(n-1)n(n+1)(n+2)}. \tag{2.10}$$

Since

$$\sum_{j=4}^{n} \frac{1}{j(j+1)(n+2-j)} = \sum_{j=4}^{n} \frac{j+1-j}{j(j+1)(n+2-j)}$$

$$= \sum_{j=4}^{n} \left[\frac{1}{j(n+2-j)} - \frac{1}{(j+1)(n+2-j)} \right]$$

$$= \sum_{j=4}^{n} \left[\frac{1}{(n+2)} \cdot \frac{n+2-j+j}{j(n+2-j)} - \frac{1}{(n+3)} \cdot \frac{n+2-j+j+1}{(j+1)n+2-j} \right]$$

$$= \frac{1}{n+2} \left(\sum_{j=4}^{n} \frac{1}{j} + \sum_{j=4}^{n} \frac{1}{n+2-j} \right) - \frac{1}{n+3} \left(\sum_{j=4}^{n} \frac{1}{j+1} + \sum_{j=4}^{n} \frac{1}{n+2-j} \right)$$

$$= \left(\frac{1}{n+2} \sum_{j=4}^{n} \frac{1}{j} - \frac{1}{n+3} \sum_{j=5}^{n+1} \frac{1}{j} \right) + \left(\frac{1}{n+2} \sum_{j=2}^{n-2} \frac{1}{j} - \frac{1}{n+3} \sum_{j=2}^{n-2} \frac{1}{j} \right)$$

$$= \frac{1}{(n+2)(n+3)} \left(\sum_{j=5}^{n} \frac{1}{j} + \sum_{j=2}^{n-2} \frac{1}{j} \right) + \frac{n^2 - 5}{4(n+1)(n+2)(n+3)}, \tag{2.11}$$

from (2.10) and (2.11), we need only to show

$$\frac{1}{(n+2)(n+3)} \left(\sum_{j=5}^{n} \frac{1}{j} + \sum_{j=2}^{n-2} \frac{1}{j} \right) < \frac{21n^3 - 31n^2 + 4}{48(n-1)n(n+1)(n+2)}$$

$$- \frac{n^2 - 5}{4(n+1)(n+2)(n+3)}$$

$$= \frac{9n^3 + 35n^2 - 68n + 12}{48(n-1)n(n+2)(n+3)}. \tag{2.12}$$

That is, we need only to show

$$\sum_{j=5}^{n}\frac{1}{j}+\sum_{j=2}^{n-2}\frac{1}{j}<\frac{9n^3+35n^2-68n+12}{48(n-1)n}. \tag{2.13}$$

Since $\sum_{j=n_1}^{n_2}\frac{1}{j}<\int_{n_1-1}^{n_2-1}\frac{1}{x}\,dx=\ln((n_2-1)/(n_1-1))$, we have

$$\sum_{j=5}^{n}\frac{1}{j}+\sum_{j=2}^{n-2}\frac{1}{j}<\ln\frac{n-1}{4}+\ln\frac{n-3}{1}=\ln\frac{(n-1)(n-3)}{4}. \tag{2.14}$$

We show next, for $n\geq 17$,

$$\ln\frac{(n-1)(n-3)}{4}<\frac{9n^3+35n^2-68n+12}{48(n-1)n}. \tag{2.15}$$

In fact, for $x\geq 4$, let

$$p(x)=\ln\frac{(x-1)(x-3)}{4},$$
$$q(x)=\frac{9x^3+35x^2-68x+12}{48(x-1)x},$$
$$r(x)=q(x)-p(x).$$

Then

$$r'(x)=\frac{3x^5-32x^4+98x^3-105x^2+28x-12}{16(x-1)^2x^2(x-3)}.$$

Let $s(x)=3x^5-32x^4+98x^3-105x^2+28x-12$. Then $s'(x)=15x^4-128x^3+294x^2-210x+28=(x-2)(15x^3-98x^2+98x-14)=(x-2)(x^2(15x-98)+(98x-14))>0$ for $x>7$. Since $s(7)=2242>0$, we have $s(x)>0$, if $x\geq 7$. This implies that $r'(x)>0$, if $x\geq 7$.

Since $r(17)=0.04848>0$, we obtain

$$r(x)>0 \qquad \text{for } x\geq 17.$$

This implies that (2.15) holds for $n\geq 17$. Combining (2.12), (2.13), (2.14), and (2.15), we have for $n\geq 17$, $0<b_{n+1}<\frac{1}{(n+1)(n+2)}$, by induction. Theorem 1 is thus proved.

Proof of Theorem 2. Similar to the proof of Theorem 1, let $z=\frac{1}{2x+1}$. Then (2.3) can be written as

$$\left(\frac{1+z}{1-z}\right)^{\frac{1-z}{2z}}=e\left[1-\sum_{k=1}^{\infty}a_kz^k\right]. \tag{2.16}$$

Let $F(z) = (\frac{1+z}{1-z})^{(1-z)/2z}/e$. Then

$$\begin{aligned}\ln F(z) &= \left(\frac{1-z}{2z}\right)[\ln(1+z) - \ln(1-z)] - 1 \\ &= (1-z)\sum_{k=1}^{\infty}\frac{z^{2k-2}}{2k-1} - 1 \\ &= \sum_{k=1}^{\infty} c_k z^k,\end{aligned}$$

where

$$c_k = \begin{cases} -\frac{1}{k}, & \text{if } k \text{ is odd,} \\ \frac{1}{k+1}, & \text{if } k \text{ is even.}\end{cases}$$

Hence $F(z) = e^{\sum_{k=1}^{\infty} c_k z^k}$ and

$$\sum_{k=1}^{\infty} a_k z^k = 1 - F(z). \tag{2.17}$$

Let $G(z) = \sum_{k=1}^{\infty} c_k z^k = \ln F(z)$. Then $F'(z) = F(z) \cdot G'(z)$ and

$$F^{(n+1)}(0) = \sum_{j=0}^{n} C_n^j F^{(j)}(0) G^{(n+1-j)}(0), \qquad n = 1, 2, \ldots. \tag{2.18}$$

From (2.17), (2.18) we obtain

$$\begin{aligned}a_1 &= -c_1 = 1, \\ a_{n+1} &= -F^{(n+1)}(0)/(n+1)! \\ &= \sum_{j=0}^{n} C_n^j (-F^{(j)}(0)) G^{(n+1-j)}(0)/(n+1)! \\ &= \sum_{j=0}^{n} \frac{n!}{(n+1)!}\left(\frac{-F^{(j)}(0)}{j!}\right) \cdot \frac{1}{(n-j)!} \cdot (n+1-j)! c_{n+1-j} \\ &= -c_{n+1} + \sum_{j=1}^{n} \frac{(n+1-j)}{n+1} a_j c_{n+1-j}.\end{aligned}$$

Therefore (2.4) is proved. From (2.4), let $n = 1, 2, \ldots, 6$. We obtain

$$a_1 = 1,\ a_2 = -\frac{5}{6},\ a_3 = \frac{5}{6},\ a_4 = \frac{-287}{360},$$
$$a_5 = \frac{287}{360},\ a_6 = -0.78097,\ a_7 = 0.78097.$$

Next we show by induction that

$$a_{2k-1} > 0, \qquad k = 1, 2, \ldots, \tag{2.19}$$

and

$$a_{2k} = -a_{2k+1}, \qquad k = 1, 2, \ldots. \tag{2.20}$$

Suppose for $k = 1, 2, \ldots, n$, (2.19) holds. Then for $n + 1 = 2m$ we have

$$a_{2m} = -c_{2m} + \sum_{j=1}^{2m-1} \frac{(2m-j)}{2m} a_j c_{2m-j}. \tag{2.21}$$

Since $c_{2k} > 0$, $a_{2k} < 0$, $c_{2k-1} < 0$, $a_{2k-1} > 0$, we see $a_{2m} < 0$. Similarly we can prove that $a_{2m+1} > 0$, $m = 1, 2, \ldots$.

Suppose for $k = 1, 2, \ldots, n$, (2.20) holds. Then for $n + 1 = 2m$, we have by (2.21),

$$\begin{aligned}
a_{2m} &= -\frac{1}{2m+1} + \frac{1}{2m}\Bigg[(2m-1)\frac{-1}{2m-1} + (2m-2)a_2\frac{1}{2m-1} \\
&\qquad - a_3 + \cdots + 2a_{2m-2}\frac{1}{3} + a_{2m-1}(-1)\Bigg] \\
&= -\frac{1}{2m+1} + \frac{1}{2m}\left[-1 + \frac{2m-2}{2m-1}a_2 - a_3 + \cdots - a_{2m-3} + \frac{2}{3}a_{2m-2} - a_{2m-1}\right] \\
&= -\frac{1}{2m+1} + \frac{1}{2m}\left[-1 + \left(\frac{2m-2}{2m-1}a_2 + a_2\right) + \cdots + \left(\frac{2}{3}a_{2m-2} + a_{2m-2}\right)\right] \\
&= -\frac{1}{2m+1} + \frac{1}{2m}\left[-1 + \frac{4m-3}{2m-1}a_2 + \cdots + \frac{5}{3}a_{2m-2}\right].
\end{aligned} \tag{2.22}$$

Similarly, by (2.4), we have

$$\begin{aligned}
a_{2m+1} &= \frac{1}{2m+1} + \frac{1}{2m+1}\Bigg[\frac{2m}{2m+1} - a_2 + \frac{2m-2}{2m-1} \\
&\qquad \times a_3 - \cdots - a_{2m-2} + \frac{2}{3}a_{2m-1} - a_{2m}\Bigg] \\
&= \frac{1}{2m+1} + \frac{1}{2m+1}\Bigg[\frac{2m}{2m+1} - \frac{4m-3}{2m-1} \\
&\qquad \times a_2 - \cdots - \frac{5}{3}a_{2m-2} - a_{2m}\Bigg].
\end{aligned} \tag{2.23}$$

Substituting (2.22) into (2.23), we obtain

$$a_{2m} + a_{2m+1} = 0.$$

By induction, (2.19), (2.20) hold for all $k \in N$. Theorem 2 is proved.

COROLLARY 1. *For $x > 0$, we have for $m, n \in N$*

$$e\left[1 - \sum_{k=1}^{2m-1} \frac{a_k}{(2x+1)^k}\right] < \left(\frac{1+x}{x}\right)^x < e\left[1 - \sum_{k=1}^{n} \frac{b_k}{(1+x)^k}\right],$$

where $a_1 = 1$, $a_2 = -a_3 = -\frac{5}{6}$, $a_4 = -a_5 = \frac{-287}{360}$, $a_6 = -a_7 = -0.78097$, a_k *satisfies* (2.4), (2.19), *and* (2.20), $k \geq 2$. $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}$, $b_4 = \frac{73}{5760}$, $b_5 = \frac{11}{1280}$, $b_6 = \frac{1945}{580608}, \ldots$, b_k *satisfies* (2.2), *and* $b_k > 0$, $k \geq 2$.

As applications, we obtain the following strengthened Hardy inequality and Carleman inequality:

COROLLARY 2. *Suppose* $a_n \geq 0$, $0 \leq \lambda_{n+1} \leq \lambda_n$, $\sigma_n = \sum_{m=1}^{n} \lambda_m$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. *Then for any $m \in N$, we have*

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\sigma_n} \leq \sum_{n=1}^{\infty} \left(1 + \frac{1}{\sigma_n/\lambda_n}\right)^{\sigma_n/\lambda_n} \lambda_n a_n$$
$$< e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+\sigma_n/\lambda_n)^k}\right) \lambda_n a_n,$$

where $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}, \ldots$, b_k *satisfies* (2.2), *and* $b_k > 0$, $k \geq 2$,

COROLLARY 3. *Suppose* $a_n \geq 0$, $n = 1, 2, \ldots$, *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then for any $m \in N$,*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k}\right) a_n,$$

where $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{24}$, $b_3 = \frac{1}{48}, \ldots$, b_k *satisfies* (2.2), *and* $b_k > 0$, $k \geq 2$.

Remark. By using a similar method as used in the proof of Theorem 2, the right side of (2.3) can be replaced by $e[1 - \sum_{k=1}^{\infty}(a_k/(x+\varepsilon)^k)]$, where $\varepsilon \in (0, 1]$, $a_k = a_k(\varepsilon)$, $k = 1, 2, \ldots$. In this case, the left side of (1.1), (1.5), and (1.6) can be replaced by equalities.

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