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Letter to the Editor  
 A note on compact normal operators

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**Abstract**

In this note, we present three simple necessary and sufficient conditions for a linear compact operator on a Hilbert space  $H$  to be normal.

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It is well known that linear compact operators on a Hilbert space form one of the most important class of bounded linear operators and have many more excellent properties, see [1]. Thus, it is very important to consider some conditions under which a linear compact operator is normal. Recently, in [3] Sadkane presented three simple necessary and sufficient conditions under which a  $n$ -by- $n$  complex matrix is normal. Since  $n$ -by- $n$  complex matrices can be considered as the class of linear compact operators on Hilbert space  $\mathbb{C}^n$ , it is natural to ask whether those results in [3] hold for all linear compact operators on a separable complex Hilbert space. In this note, we give an affirmative answer to this question, and our proofs are different from [3].

Let  $\mathbb{N}, \mathbb{C}$  be the sets of all positive integers and all complex numbers, respectively. And denote by  $\aleph_0$  the cardinal of  $\mathbb{N}$ . Let  $H$  be a separable complex Hilbert space and  $I$  be the identity operator on  $H$ . In the note, we always assume that  $A: H \rightarrow H$  is a linear compact operator on  $H$ , and let  $r(A), N(A)$  be the spectral radius and the null space of  $A$ , respectively. We denote by  $\lambda_1(A), \lambda_2(A), \lambda_3(A), \dots$  all the eigenvalues of  $A$ . We assume that the eigenvalues are ordered such that

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq |\lambda_3(A)| \geq \dots, \tag{1}$$

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where each eigenvalue is repeated as many times as the value of its multiplicity. Let  $\lambda_1(A^*A), \lambda_2(A^*A), \dots$  be the sequence of eigenvalues of  $A^*A$  such that

$$\lambda_1(A^*A) \geq \lambda_2(A^*A) \geq \lambda_3(A^*A) \geq \dots, \tag{2}$$

where each eigenvalue is repeated as many times as its multiplicity. By definition, for  $j = 1, 2, \dots$  the  $j$ th singular value of  $A$  is the number  $s_j(A) \stackrel{\text{def}}{=} (\lambda_j(A^*A))^{1/2}$ , see for example [2, p. 96]. Without loss of generality, we assume that there exists a sequence  $\{m_i\}_{i=1}^v \subseteq \mathbb{N}$  with  $m_l < m_k$ , for  $l < k$  such that  $\lambda_{m_l}(A) \neq \lambda_{m_k}(A)$ , for  $l \neq k$  and  $\lambda_j(A) = \lambda_{m_i}(A)$  for  $j \in \{m_{i-1} + 1, m_{i-1} + 2, \dots, m_i\}$ , where  $i = 1, 2, \dots, v$  and  $m_0 = 0$ . Clearly  $v$  may be a finite number or  $\aleph_0$ . With these symbols, we have

**Lemma 1.** *Suppose  $A$  be a compact operator on  $H$ . If  $\lambda_j(A) \neq 0$  and  $s_j(A) = |\lambda_j(A)|$ , for all  $j \in \mathbb{N}$ , then, for each  $s \in \{1, \dots, v\}$  with  $s < +\infty$ , there exists an orthonormal set  $\{x_1, x_2, \dots, x_{m_s}\}$  of  $H$  such that*

$$A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s$$

and  $\{\lambda_{m_1}(A), \dots, \lambda_{m_s}(A)\} \subseteq \mathbb{C} \setminus \sigma(A_s)$ , where  $A_s$  is a compact operator on  $\bigcap_{i=1}^s N(A - \lambda_{m_i}(A)I)^\perp$ .

**Proof.** First, we will prove that  $N(A - \lambda_{m_1}(A)I)$  is a reduced subspace for  $A$ . To do this, it is enough to show that  $N(A - \lambda_{m_1}(A)I)$  is invariant for  $A^*$ . Note that  $\lambda_1(A) = \dots = \lambda_{m_1}(A)$ . For  $j = 1$ , we have

$$|\lambda_1(A)| = s_1(A) = \max_{0 \neq x \in H} \frac{\|Ax\|}{\|x\|} = \|A\|.$$

Then for each  $x \in N(A - \lambda_{m_1}(A)I)$  with  $\|x\| = 1$ , we have

$$\|A\| = |\lambda_1(A)| = |\langle Ax, x \rangle| = |\langle x, A^*x \rangle| \leq \|A^*x\| \leq \|A^*\| = \|A\|.$$

So  $|\langle x, A^*x \rangle| = \|A^*x\|$ . By Schwarz inequality, there exists  $h \in \mathbb{C}$  such that  $A^*x = hx$ . Furthermore, we can get  $h = \overline{\lambda_{m_1}(A)}$ . Thus  $A^*x \in N(A - \lambda_{m_1}(A)I)$ . This shows that  $N(A - \lambda_{m_1}(A)I)$  is a reduced subspace for  $A$ . Therefore the restriction of  $A$  to  $N(A - \lambda_{m_1}(A)I)$  is  $\lambda_{m_1}(A)I_1$ , where  $I_1$  is the identity operator on  $N(A - \lambda_{m_1}(A)I)$ . Hence  $A = \lambda_{m_1}(A)P_1 \oplus A_1$ , and  $\lambda_{m_1}(A) \in \mathbb{C} \setminus \sigma(A_1)$ , where  $P_1$  is the orthogonal projection on  $N(A - \lambda_{m_1}(A)I)$ . Moreover, we assume that  $\{x_1, \dots, x_{m_1}\}$  is an orthonormal basis of  $N(A - \lambda_{m_1}(A)I)$ , then  $\lambda_{m_1}(A)P_1 = \sum_{j=1}^{m_1} \lambda_{m_1}(A)(x_j \otimes x_j)$ , that is,  $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$ . Repeat the process above, we can obtain  $A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s$ .  $\square$

The following is the main result of this paper.

**Theorem 2.** *Let  $A$  be a compact operator on  $H$ , then the following statements are equivalent:*

- (a)  $A$  is normal
- (b) For each  $j \in \mathbb{N}$ ,

$$s_j(A^{m+n}) = \sqrt{s_j(A^{2n})s_j(A^{2m})}, \quad \forall m, n = 0, 1, 2, \dots \tag{3}$$

- (c)  $s_j(A) = |\lambda_j(A)|$ , for  $j = 1, 2, \dots$ .
- (d) For each  $x \in H$ ,

$$\|A^{n+m}x\| \leq \sqrt{\|A^{2n}x\| \cdot \|A^{2m}x\|}, \quad \forall n, m = 0, 1, 2, \dots \tag{4}$$

**Proof.** If  $H$  is finite dimensional, then  $A$  can be viewed as a matrix, the proof can be found in [3]. Next, we only consider the case  $H$  is infinite dimensional.

(a)  $\Rightarrow$  (b): Suppose that  $A$  is normal, then  $A = \sum_{i=1}^v \lambda_{m_i}(A)P_i$ , where  $P_i$  is the orthogonal projection onto  $N(A - \lambda_{m_i}(A)I)$ . Hence  $A^*A = \sum_{i=1}^v |\lambda_{m_i}(A)|^2 P_i$  and  $(A^n)^*A^n = (A^*A)^n = \sum_{i=1}^v |\lambda_{m_i}(A)|^{2n} P_i$ . Therefore

$$\{s_{m_i}(A): i = 1, 2, \dots, v\} = \{|\lambda_{m_i}(A)|: i = 1, 2, \dots, v\}.$$

Clearly,

$$\begin{aligned} \dim(N((A^*A)^n - (s_{m_i}(A))^{2n})) &= \dim(N(A^*A - (s_{m_i}(A))^2)) \\ &= \sum_{k \in D} \dim(N(A - \lambda_k(A))), \end{aligned}$$

where  $D = \{k: |\lambda_k(A)| = s_{m_i}(A)\}$ . Since  $(|\lambda_j(A)|)_{j=1}^\infty$  and  $(s_j(A))_{j=1}^\infty$  are ordered as (1) and (2), respectively, we have

$$s_j(A^n) = (s_j(A))^n = |\lambda_j(A)|^n. \text{ Therefore}$$

$$s_j(A^{m+n}) = \sqrt{s_j(A^{2n})s_j(A^{2m})}, \quad \forall j \in \mathbb{N}, \quad \forall n, m = 0, 1, 2, \dots$$

(b)  $\Rightarrow$  (c): Under condition (3), we have

$$s_j(A) = (s_j(A^2))^{1/2} = (s_j(A^4))^{1/4} = \dots = (s_j(A^{2^k}))^{1/2^k} = \dots \quad \forall k \in \mathbb{N}.$$

If  $j = 1$ , then  $s_1(A^m) = \max_{0 \neq x} \|A^m x\|/\|x\| = \|A^m\|$ , hence

$$s_1(A) = \lim_{k \rightarrow \infty} (s_1(A^{2^k}))^{1/2^k} = \lim_{k \rightarrow \infty} \|A^{2^k}\|^{1/2^k} = \mathbf{r}(A) = |\lambda_1(A)|.$$

Hence  $s_1(A) = \dots = s_{m_1}(A) = |\lambda_1(A)| = \dots = |\lambda_{m_1}(A)|$ . From the proof of Lemma 1, we get  $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$  and  $s_{m_1+1}(A) = s_1(A_1)$ ,  $\lambda_{m_1+1}(A) = \lambda_1(A_1)$ . Hence,

$$s_1(A_1) = (s_1(A_1^2))^{1/2} = \dots = (s_1(A_1^{2^k}))^{1/2^k} = \dots \quad \forall k \in \mathbb{N}$$

and

$$s_1(A_1) = \lim_{k \rightarrow \infty} (s_1(A_1^{2^k}))^{1/2^k} = \lim_{k \rightarrow \infty} \|A_1^{2^k}\|^{1/2^k} = |\lambda_1(A_1)|.$$

So  $s_{m_1+1}(A) = \dots = s_{m_2}(A) = |\lambda_{m_1+1}(A)| = \dots = |\lambda_{m_2}(A)|$ . For each  $j > 1$ , by induction and simple discussion, we can obtain  $s_j(A) = |\lambda_j(A)|$ .

(c)  $\Rightarrow$  (a): By Lemma 1, for each  $s \in \{1, \dots, v\}$  with  $s < +\infty$ , we have

$$A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s.$$

Let  $B_s = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j)$ , then  $A = B_s \oplus A_s$ . If  $v$  is a finite number, then clearly  $A = B_v \oplus 0$  is normal. In the case  $v = \aleph_0$ , let  $(\alpha_{sk})_{k=1}^\infty$  be the sequence of distinct eigenvalues of  $A_s^* A_s$ , with  $\alpha_{sk} \geq \alpha_{si}$ ,  $k > i$  and  $P_{sk}$  be the orthogonal projection from  $\bigcap_{i=1}^s N(A - \lambda_{m_i}(A)I)^\perp$  onto  $N(A_s^* A_s - \alpha_{sk} I_s)$  and  $P_{sk} P_{st} = P_{st} P_{sk} = 0$  for  $k \neq t$ , where  $I_s$  is the identity operator on  $\bigcap_{i=1}^s N(A - \lambda_{m_i}(A)I)^\perp$ . Clearly  $(\alpha_{sk})_{k=1}^\infty$  is a subsequence of  $((s_j(A))^2)_{j=1}^\infty$ . Then  $A_s^* A_s = \sum_{k=1}^\infty \alpha_{sk} P_{sk}$ , since  $A_s^* A_s$  is compact and normal. For each  $x \in \bigcap_{i=1}^s N(A - \lambda_{m_i}(A)I)^\perp$ , we have

$$\|(A_s^* A_s)x\|^2 = \sum_{k=1}^\infty \alpha_{sk}^2 \|P_{sk}x\|^2 \leq \sup\{\alpha_{sk}^2 : k \geq 1\} \cdot \|x\|^2.$$

Hence  $\|(A_s^* A_s)\| = \sup_{\|x\|=1} \|(A_s^* A_s)x\| \leq \sup\{\alpha_{sk}^2 : k \geq 1\}$ . Since  $(s_j(A)) \rightarrow 0$  as  $j \rightarrow +\infty$ , we have  $\sup\{\alpha_{sk}^2 : k \geq 1\} \rightarrow 0$  as  $j \rightarrow \infty$ . So that  $\|A_s^* A_s\| \rightarrow 0$  as  $s \rightarrow +\infty$ . This implies  $\|A_s\| \rightarrow 0$ . Hence  $\|B_s \oplus 0 - A\| = \|0 \oplus A_s\| = \|A_s\| \rightarrow 0$ . Therefore  $A$  is normal, since each  $B_s \oplus 0$  is normal.

(a)  $\Rightarrow$  (d): If  $A$  is normal, then for each  $x \in H$ ,

$$\begin{aligned} \|A^{n+m}x\|^2 &= \langle A^{n+m}x, A^{n+m}x \rangle \\ &= \langle (A^*A)^n x, (A^*A)^m x \rangle \\ &\leq \|(A^*A)^n x\| \cdot \|(A^*A)^m x\| \\ &= \|A^{2n}x\| \cdot \|A^{2m}x\|, \end{aligned}$$

since  $\|(A^*A)^n x\|^2 = \langle (A^*A)^n x, (A^*A)^n x \rangle = \|A^{2n}x\|^2$ . Hence, (d) holds.

(d)  $\Rightarrow$  (c): condition (4) implies that

$$s_1(A^{n+m}) \equiv \max_{0 \neq x \in H} \frac{\|A^{n+m}x\|}{\|x\|} \leq \sqrt{s_1(A^{2n})s_1(A^{2m})}$$

and in particular, we have

$$|\lambda_1(A^n)| \leq s_1(A^n) \leq (s_1(A^{2n}))^{1/2} \leq (s_1(A^{4n}))^{1/4} \leq \dots \leq |\lambda_1(A^n)|.$$

The last inequality holds, since  $\lim_{m \rightarrow \infty} (s_1(A^{nm}))^{1/m} = |\lambda_1(A^n)|$ . In particular, for  $n=1$ , we obtain that  $s_1(A) = \dots = s_{m_1}(A) = |\lambda_1(A)| = \dots = |\lambda_{m_1}(A)|$ . From the proof of Lemma 1, we have  $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$ , and  $A_1$  is a compact operator on  $N(A - \lambda_{m_1}(A)I)^\perp$ . For each  $x \in N(A - \lambda_{m_1}(A)I)^\perp$ , we have  $A^{n+m}x = A_1^{n+m}x$ , hence

$$\|A_1^{n+m}x\| \leq \sqrt{\|A_1^{2n}x\| \|A_1^{2m}x\|}, \quad \forall n, m = 0, 1, 2, \dots$$

Similarly, we get  $|\lambda_1(A_1)| = s_1(A_1)$ . From the proof of Lemma 1 again, we obtain

$$s_{m_1+1}(A) = \dots = s_{m_2}(A) = s_1(A_1) = |\lambda_1(A_1)| = |\lambda_{m_1+1}(A)| = \dots = |\lambda_{m_2}(A)|.$$

For each  $j \in \mathbb{N}$ , by induction and simple discussion, we get  $s_j(A) = |\lambda_j(A)|$ . Hence (c) holds. This completes the proof.  $\square$

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### **References**

- [1] J.B. Conway, A Course in Functional Analysis, Springer, New York, 1985.
- [2] I. Gohberg, S. Goldberg, M.A. Kaashoek, Classes of Linear Operators, Vol. 1, Birkhäuser, Basel, Boston, Berlin, 1990.
- [3] M. Sdkane, A note on normal matrices, J. Comput. Appl. Math. 136 (2001) 185–187.