The Ordered Weighted Geometric Averaging Operators

Z. S. Xu,* Q. L. Da
College of Economics and Management, Southeast University, Nanjing, Jiangsu, 210096, People's Republic of China

The ordered weighted averaging (OWA) operator was introduced by Yager.[1] The fundamental aspect of the OWA operator is a reordering step in which the input arguments are rearranged in descending order. In this article, we propose two new classes of aggregation operators called ordered weighted geometric averaging (OWGA) operators and study some desired properties of these operators. Some methods for obtaining the associated weighting parameters are discussed, and the relationship between the OWA and DOWGA operators is also investigated.

1. INTRODUCTION

The ordered weighted averaging (OWA) operator was introduced by Yager[1] to provide for aggregation lying between the max and min operators, and has received more and more attention[1−13] since its appearance. The OWA operator has been used in a wide range of applications such as neural networks,[14−15] database systems,[16,17] fuzzy logic controllers,[18,19] decision-making,[1,7,20] expert systems,[21] market research,[22] linguistic quantified propositions,[23,24] mathematical programming,[25] and lossless image compression.[26] In this article, we propose two new classes of aggregation operators called ordered weighted geometric averaging (OWGA) operators (i.e., DOWGA and AOWGA operators) and study their desired properties, including monotonicity, commutativity, idempotency, and that they are bounded by the max and min operators. The problem of obtaining the associated weighting parameters is discussed, and the relationship between the OWA and DOWGA operators is also studied.

*To whom correspondence should be addressed; e-mail: xu_zeshui@263.net.
The work was supported by the National Natural Science Foundation of China (NSFC) under Project 79970093, and the Ph.D. Dissertation Foundation of Southeast University-NARI-Relays Electric Co. Ltd.

INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 17, 709–716 (2002)
 Published online in Wiley InterScience (www.interscience.wiley.com). • DOI: 10.002/int.10045

2. THE CONCEPTS OF ORDERED WEIGHTED GEOMETRIC AVERAGING OPERATORS

An OWA operator[1] of dimension n is a mapping, $f : R^n \to R$, that has an associated n vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Furthermore, $f(a_1, \ldots, a_n) = \sum_{i=1}^n w_j b_j$, where b_j is the jth largest of the a_i. In the following we introduce two new classes of operators for aggregations.

DEFINITION 1. *A DOWGA operator of dimension n is a mapping* $g : R^{+^n} \to R^+$ *that has associated with it a weighting vector* $w = (w_1, w_2, \ldots, w_n)^T$, *with* $w_i \in [0, 1]$ *and* $\sum_{i=1}^n w_i = 1$, *such that*:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = \prod_{j=1}^n b_j^{w_j}$$

where b_j *is the* j*th largest of the* α_i $(i = 1, 2, \ldots, n)$.

Clearly, the elements b_j $(j = 1, 2, \ldots, n)$ in Definition 1 are arranged in descending order:

$$b_1 \geq b_2 \geq \cdots \geq b_n$$

and the *DOWGA* operator takes the form of an ordered weighted geometric averaging function.

Similarly, we have the following definition:

DEFINITION 2. *An AOWGA operator of dimension n is a mapping* $h : R^{+^n} \to R^+$ *that has associated with it a weighting vector* $w = (w_1, w_2, \ldots, w_n)^T$, *with* $w_i \in [0, 1]$ *and* $\sum_{i=1}^n w_i = 1$, *such that*:

$$h(\alpha_1, \alpha_2, \ldots, \alpha_n) = \prod_{j=1}^n d_j^{w_j}$$

where d_j *is the* j*th smallest of the* α_i $(i = 1, 2, \ldots, n)$. *Here, the elements* d_j $(j = 1, 2, \ldots, n)$ *are arranged in ascending order* $b_1 \leq b_2 \leq \cdots \leq b_n$.

LEMMA 1.[27] *Let* $x_j > 0, \lambda_j > 0,\ j = 1, 2, \ldots, n$, *and* $\sum_{j=1}^n \lambda_j = 1$; *then*:

$$\prod_{j=1}^n x_j^{\lambda_j} \leq \sum_{j=1}^n \lambda_j x_j$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

By Lemma 1, we get the relationship between the *OWA* and *DOWGA* operators:

THEOREM 1. *Assume g and h are the OWA and DOWGA operators, respectively. Let* $(\alpha_1, \ldots, \alpha_n)$ *be an argument vector*; *then* $f(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq g(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

The following example illustrates the use of these operators.

Example. Assume $w = (0.4, 0.1, 0.2, 0.3)^T$; then the OWA, DOWGA, and AOWGA operators are respectively:

$$f(7, 18, 6, 2) = 0.4 \times 18 + 0.1 \times 7 + 0.2 \times 6 + 0.3 \times 2 = 9.70$$
$$g(7, 18, 6, 2) = 18^{0.4} \times 7^{0.1} \times 6^{0.2} \times 2^{0.3} = 6.80$$
$$h(7, 18, 6, 2) = 2^{0.4} \times 6^{0.1} \times 7^{0.2} \times 18^{0.3} = 5.54$$

All the OWA, DOWGA, and AOWGA operators have a fundamental aspect; that is, the re-ordering step. In particular, an argument α_i is not associated with a particular weight w_i, but rather a weight w_i is associated with a particular ordered position i of the arguments.

We can easily prove that the DOWGA and AOWGA operators satisfy the following properties:

THEOREM 2. *Assume f, g, and h are the OWA, DOWGA, and AOWGA operators, respectively. Let $(\alpha_1, \ldots, \alpha_n)$ be an argument vector. Let $(\beta_1, \ldots, \beta_n)$ be the vector consisting of the arguments of $(\alpha_1, \ldots, \alpha_n)$ in descending order, and $(\gamma_1, \ldots, \gamma_n)$ be the vector consisting of the arguments of $(\alpha_1, \ldots, \alpha_n)$ in ascending order; then:*

$$f(\beta_1, \ldots, \beta_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g(\beta_1, \ldots, \beta_n),$$
$$h(\alpha_1, \ldots, \alpha_n) = h(\gamma_1, \ldots, \gamma_n)$$

THEOREM 3. *Assume f, g, and h are the OWA, DOWGA, and AOWGA operators, respectively. Let $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha'_1, \ldots, \alpha'_n)$ be two descending ordered argument vectors, where $\alpha_j \geq \alpha'_j$, for each j. Let $(\beta_1, \ldots, \beta_n)$ and $(\beta'_1, \ldots, \beta'_n)$ be two ascending ordered argument vectors, where $\beta_j \geq \beta'_j$, for each j; then:*

$$f(\alpha'_1, \ldots, \alpha'_n) \leq f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) \geq g(\alpha'_1, \ldots, \alpha'_n),$$
$$h(\beta_1, \ldots, \beta_n) \geq h(\beta'_1, \ldots, \beta'_n)$$

COROLLARY 1 (*Monotonicity*). *Assume f, g, and h are the OWA, DOWGA, and AOWGA operators, respectively. Let $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ be two arbitrary argument vectors; then:*

$$f(\beta_1, \ldots, \beta_n) \geq f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) \leq g(\beta_1, \ldots, \beta_n),$$
$$h(\alpha_1, \ldots, \alpha_n) \leq h(\beta_1, \ldots, \beta_n)$$

when $\alpha_j \leq \beta_j$ for all $j = 1, 2, \ldots, n$.

COROLLARY 2 (*Commutativity*). *Assume f, g, and h are the OWA, DOWGA, and AOWGA operators, respectively; then:*

$$f(\beta_1, \ldots, \beta_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g(\beta_1, \ldots, \beta_n),$$
$$h(\alpha_1, \ldots, \alpha_n) = h(\beta_1, \ldots, \beta_n)$$

where $(\beta_1, \ldots, \beta_n)$ is any permutation of the elements in $(\alpha_1, \ldots, \alpha_n)$.

THEOREM 4 (*Idempotency*). *Assume f, g, and h are the OWA, DOWGA, and AOWGA operators, respectively. If $\alpha_j = \alpha$, for all $j = 1, 2, \ldots, n$, then*:

$$f(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_n) = h(\alpha_1, \ldots, \alpha_n) = \alpha$$

The DOWGA and AOWGA operators, similar to the OWA operators, also include the max, min, and arithmetic mean operators for the appropriate selection of the weighting vector w:

(1) For $w = w^* = (1, 0, \ldots, 0)^T$, $f^*(\alpha_1, \ldots, \alpha_n) = g^*(\alpha_1, \ldots, \alpha_n) = \max_i(\alpha_i)$, and $h_*(\alpha_1, \ldots, \alpha_n) = \min_i(\alpha_i)$.
(2) For $w = w_* = (0, 0, \ldots, 1)^T$, $f_*(\alpha_1, \ldots, \alpha_n) = g_*(\alpha_1, \ldots, \alpha_n) = \min_i(\alpha_i)$, and $h^*(\alpha_1, \ldots, \alpha_n) = \max_i(\alpha_i)$.
(3) For $w = w_{Ave} = (1/n, 1/n, \ldots, 1/n)^T$, $\frac{1}{n}\sum_{j=1}^{n} a_j = f_{Ave}(\alpha_1, \ldots, \alpha_n) \geq g_{Ave}(\alpha_1, \ldots, \alpha_n) = h_{Ave}(\alpha_1, \ldots, \alpha_n) = \prod_{j=1}^{n} a_j^{1/n}$.

For any DOWGA operator g and AOWGA operator h, we have:

$$\begin{aligned} g^*(\alpha_1, \ldots, \alpha_n) = f^*(\alpha_1, \ldots, \alpha_n) &\geq f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) \\ &\geq g_*(\alpha_1, \ldots, \alpha_n) = f_*(\alpha_1, \ldots, \alpha_n) \\ h^*(\alpha_1, \ldots, \alpha_n) &\geq h(\alpha_1, \ldots, \alpha_n) \geq h_*(\alpha_1, \ldots, \alpha_n) \end{aligned}$$

Based on the results of Yager[3] and Theorem 1, we can draw the following important conclusions:

THEOREM 5. *If $w_j = 1, w_i = 0$, and $i \neq j$, then $f(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_n) = b_j$, and $h(\alpha_1, \ldots, \alpha_n) = d_j$, where b_j and d_j are the jth-largest and smallest of the α_i $(i = 1, 2, \ldots, n)$ respectively. Especially*,

$$\begin{gathered} f(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_n) = g^*(\alpha_1, \ldots, \alpha_n) = f^*(\alpha_1, \ldots, \alpha_n), \\ h(\alpha_1, \ldots, \alpha_n) = h_*(\alpha_1, \ldots, \alpha_n), \quad when \;\; j = 1 \\ f(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_n) = g_*(\alpha_1, \ldots, \alpha_n) = f_*(\alpha_1, \ldots, \alpha_n), \\ h(\alpha_1, \ldots, \alpha_n) = h^*(\alpha_1, \ldots, \alpha_n), \quad when \;\; j = n \end{gathered}$$

The operator f in Theorem 5 is called a Step-OWA operator. Similarly, we call g and h in Theorem 5 Step-DOWGA and Step-AOWGA operators, respectively.

THEOREM 6. *If $w_1 = \lambda, w_i = 0, i = 2, \ldots, n-1, w_n = 1 - \lambda$, and $\lambda \in [0, 1]$, then*:

$$\begin{gathered} \lambda f^*(\alpha_1, \ldots, \alpha_n) + (1-\lambda) f_*(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) \\ = g^{*\lambda}(\alpha_1, \ldots, \alpha_n) g_*^{1-\lambda}(\alpha_1, \ldots, \alpha_n) \\ h(\alpha_1, \ldots, \alpha_n) = h_*^{\lambda}(\alpha_1, \ldots, \alpha_n) h^{*1-\lambda}(\alpha_1, \ldots, \alpha_n) \end{gathered}$$

Obviously, the choice of OWA operator f in Theorem 6 leads to the so-called Hurwicz decision technique, while the choice of g and h in Theorem 6 are also a form of Hurwicz method.

THEOREM 7.

(1) *If* $w_1 = \frac{1}{n}(1-\lambda)+\lambda$, $w_i = \frac{1}{n}(1-\lambda)$, $i \neq 1$, *and* $\lambda \in [0, 1]$, *then*:

$$\lambda f^*(\alpha_1, \ldots, \alpha_n) + (1-\lambda) f_{Ave}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g^{*\lambda}(\alpha_1, \ldots, \alpha_n) g_{Ave}^{1-\lambda}(\alpha_1, \ldots, \alpha_n)$$

$$h(\alpha_1, \ldots, \alpha_n) = h_*^{\lambda}(\alpha_1, \ldots, \alpha_n) h_{Ave}^{1-\lambda}(\alpha_1, \ldots, \alpha_n)$$

Especially,

$$f_{Ave}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g_{Ave}(\alpha_1, \ldots, \alpha_n),$$
$$h(\alpha_1, \ldots, \alpha_n) = h_{Ave}(\alpha_1, \ldots, \alpha_n), \quad \textit{when } \lambda = 0$$
$$f^*(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g^*(\alpha_1, \ldots, \alpha_n),$$
$$h(\alpha_1, \ldots, \alpha_n) = h_*(\alpha_1, \ldots, \alpha_n), \quad \textit{when } \lambda = 1$$

(2) *If* $w_i = \frac{1}{n}(1-\mu)$, $i \neq n$, $w_n = \frac{1}{n}(1-\mu)+\mu$, *and* $\mu \in [0, 1]$, *then*:

$$\mu f_*(\alpha_1, \ldots, \alpha_n) + (1-\mu) f_{Ave}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g_*^{\mu}(\alpha_1, \ldots, \alpha_n) g_{Ave}^{1-\mu}(\alpha_1, \ldots, \alpha_n)$$

$$h(\alpha_1, \ldots, \alpha_n) = h^{*\mu}(\alpha_1, \ldots, \alpha_n) h_{Ave}^{1-\mu}(\alpha_1, \ldots, \alpha_n)$$

Especially,

$$f_{Ave}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g_{Ave}(\alpha_1, \ldots, \alpha_n)$$
$$= h_{Ave}(\alpha_1, \ldots, \alpha_n) = h(\alpha_1, \ldots, \alpha_n), \quad \textit{when } \mu = 0$$
$$f_*(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g_*(\alpha_1, \ldots, \alpha_n),$$
$$h(\alpha_1, \ldots, \alpha_n) = h^*(\alpha_1, \ldots, \alpha_n), \; \textit{when } \mu = 1$$

(3) *If* $w_1 = \frac{1}{n}[1-(\lambda+\mu)]+\lambda$, $w_i = \frac{1}{n}[1-(\lambda+\mu)]$, $i = 2, \ldots, n-1$, $w_n = \frac{1}{n}[1-(\lambda+\mu)] + \mu$, $\lambda, \mu \in [0, 1]$, *and* $\lambda + \mu \leq 1$, *then*:

$$\lambda f^*(\alpha_1, \ldots, \alpha_n) + \mu f_*(\alpha_1, \ldots, \alpha_n) + [1-(\lambda+\mu)] f_{Ave}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n) \geq g(\alpha_1, \ldots, \alpha_n) = g^{*\lambda}(\alpha_1, \ldots, \alpha_n) g_*^{\mu}(\alpha_1, \ldots, \alpha_n) g_{Ave}^{[1-(\lambda+\mu)]}(\alpha_1, \ldots, \alpha_n)$$

$$h(\alpha_1, \ldots, \alpha_n) = h_*^{\lambda}(\alpha_1, \ldots, \alpha_n) h^{*\mu}(\alpha_1, \ldots, \alpha_n) h_{Ave}^{[1-(\lambda+\mu)]}(\alpha_1, \ldots, \alpha_n)$$

Especially, we get case (1) *when* $\mu = 0$, *and case* (2) *when* $\lambda = 0$.

The operator f in Theorem 7 is called a S-OWA operator. Similarly, we call g and h in Theorem 7 S-DOWGA and S-AOWGA operators, respectively.

THEOREM 8.

(1) *If*

$$w_i = \begin{cases} 0, & i < k \\ \dfrac{1}{m}, & k \leq i < k+m \\ 0, & i \geq k+m \end{cases}$$

where k and m are positive integers such that $k+m \le n+1$, *then*:

$$\frac{1}{m}\sum_{j=k}^{k+m-1} b_j = f(\alpha_1,\ldots,\alpha_n) \ge g(\alpha_1,\ldots,\alpha_n) = \prod_{j=k}^{k+m-1} b_j^{\frac{1}{m}},$$

$$h(\alpha_1,\ldots,\alpha_n) = \prod_{j=k}^{k+m-1} d_j^{\frac{1}{m}}$$

where b_j *and* d_j *are the* j*th-largest and smallest of the* α_i $(i = 1, 2, \ldots, n)$, *respectively.*

(2) *If*

$$w_i = \begin{cases} 0, & i < k-m \\ \dfrac{1}{2m+1}, & k-m \le i < k+m \\ 0, & i \ge k+m \end{cases}$$

where k and m are positive integers such that $k+m \le n+1$ *and* $k \ge m+1$, *then*:

$$\frac{1}{2m+1}\sum_{j=k-m}^{k+m-1} b_j = f(\alpha_1,\ldots,\alpha_n) \ge g(\alpha_1,\ldots,\alpha_n) = \prod_{j=k-m}^{k+m-1} b_j^{\frac{1}{2m+1}},$$

$$h(\alpha_1,\ldots,\alpha_n) = \prod_{j=k-m}^{k+m-1} d_j^{\frac{1}{2m+1}}$$

where b_j *and* d_j *are the* j*th-largest and smallest of the* α_i $(i = 1, 2, \ldots, n)$, *respectively.*

(3) *If*

$$w_i = \begin{cases} \dfrac{1}{k}, & i \le k \\ 0, & i > k \end{cases}$$

then:

$$\frac{1}{k}\sum_{j=1}^{k} b_j = f(\alpha_1,\ldots,\alpha_n) \ge g(\alpha_1,\ldots,\alpha_n) = \prod_{j=1}^{k} b_j^{\frac{1}{k}},$$

$$h(\alpha_1,\ldots,\alpha_n) = \prod_{j=1}^{k} d_j^{\frac{1}{k}}$$

where b_j *and* d_j *are the* j*th-largest and smallest of the* α_i $(i = 1, 2, \ldots, n)$, *respectively.*

(4) *If*

$$w_i = \begin{cases} 0, & i < k \\ \dfrac{1}{(n+1)-k}, & i \ge k \end{cases}$$

then:

$$\frac{1}{(n+1)-k}\sum_{j=k}^{n} b_j = f(\alpha_1,\ldots,\alpha_n) \geq g(\alpha_1,\ldots,\alpha_n) = \prod_{j=k}^{n} b_j^{\frac{1}{(n+1)-k}},$$

$$h(\alpha_1,\ldots,\alpha_n) = \prod_{j=k}^{n} d_j^{\frac{1}{(n+1)-k}}$$

where b_j *and* d_j *are the* j*th-largest and smallest of the* α_i $(i = 1, 2, \ldots, n)$, *respectively.*

The operator f in Theorem 8 is called a Window-OWA operator. Similarly, we call g and h in Theorem 8 Window-DOWGA and Window-AOWGA operators, respectively.

References

1. Yager RR. On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans Syst Man Cybernet 1988;18:183–190.
2. Yager RR. Applications and extensions of OWA aggregation. Int J Man-Machine Studies 1992;37:103–132.
3. Yager RR. Families of OWA operators. Fuzzy Sets Syst 1993;59:125–148.
4. Filev D, Yager RR. Analytic properties of maximum entropy OWA operators. Info Sci 1995;85:11–27.
5. Yager RR, Filev DP. Parametrized "andlike" and "orlike" OWA operators. Int J General Syst 1994;22:297–316.
6. Yager RR, Filev DP. Generalizing the modelling of fuzzy logic controllers by parameterized aggregation operators. Fuzzy Sets Syst 1995;70:303–313.
7. Yager RR. A general approach to criteria aggregation using fuzzy measures. Int J Man-Machine Studies 1993;38:187–213.
8. Yager RR. Measures of entropy and fuzziness related to aggregation operators. Info Sci 1995;82:147–166.
9. Fodor J, Marichal JL, Roubens M. Characterization of the ordered weighted averaging operators. IEEE Trans Fuzzy Syst 1995;3:236–240.
10. Yager RR. On the inclusion of importances in multi-criteria decision-making in the fuzzy set framework. Int J Expert Syst 1992;5:211–228.
11. Torra V. The weighted OWA operator. Int J Intell Syst 1997;12:153–166.
12. Filev D, Yager RR. On the issue of obtaining OWA operator weights. Fuzzy Sets Syst 1998;94:157–169.
13. Mitchell HB, Estrakh DD. An OWA operator with fuzzy ranks. Int J Intell Syst 1998; 13:69–81.
14. Yager RR. Fuzzy aggregation of modular neural networks with ordered weighted averaging operators. Int J Approx Reas 1995;13:359–375.
15. Yager RR. On a semantics for neural networks based on fuzzy quantifiers. Int J Intell Syst 1992;7:765–786.
16. Bordogna G, Pasi G. Linguistic aggregation operators of selection criteria in fuzzy information retrieval. Int J Intell Syst 1995;10:233–248.
17. Yager RR. A note on weighted queries in information retrieval systems. J Am Soc Info Sci 1987;28:23–24.
18. Yager RR, Filev DP. Generalizing the modelling of fuzzy logic controllers by parameterized aggregation operators. Fuzzy Sets Syst 1995;70:303–313.
19. Yager RR, Filev DP, Sadeghi T. Analysis of flexible structured fuzzy logic controllers. IEEE Trans Syst Man Cybernet 1994;24:1035–1043.

20. Cutello V, Montero J. Hierarchies of aggregation operators. Int J Intell Syst 1994;9:1025–1045.
21. O'Hagan M. Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic. In: Proc 22nd Annual IEEE Asilomar Conf on Signals, Systems and Computers, Pacific Grove, CA; 1988. pp 681–689.
22. Yager RR, Goldstein LS, Mendels E. FUZMAR: An approach to aggregating market research data based on fuzzy reasoning. Fuzzy Sets Syst 1994;68:1–11.
23. Yager RR. Quantifier guided aggregation using OWA operators. Int J Intell Syst 1996; 11:49–73.
24. Yager RR. Interpreting linguistically quantified propositions. Int J Intell Syst 1994; 9:541–569.
25. Yager RR. Solving mathematical programming problems with OWA operators as objective functions. In: Proc 1995 IEEE Int Conf on Fuzzy Systems. pp 1441–1446.
26. Mitchell HB, Estrakh DD. A modified OWA operator and its use in lossless DPCM image compression. Int J Uncertainty Fuzziness Knowledge-Based Syst 1997;5:429–436.
27. Xu ZS. On consistency of the weighted geometric mean complex judgment matrix in AHP. Euro J Operational Research 2000;126:683–687.