Applied Mathematics and Computation 218 (2012) 11461–11465

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems

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ARTICLE INFO

Keywords:
Normalized analytic functions in the open unit disk
Univalent and bi-univalent functions
Taylor–Maclaurin series expansion
Problems involving coefficient bounds and coefficient estimates
Taylor–Maclaurin coefficients of analytic function classes

ABSTRACT

Motivated essentially by the recent works of Srivastava et al. [10], Frasin and Aouf [6], Xu et al. [15], and other authors, the authors of the present sequel investigate the coefficient estimate problems associated with an interesting general subclass $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$, which is introduced here. In particular, for functions belonging to this general class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$, the problems involving the estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ are investigated. The results presented in this paper generalize and improve the aforecited recent works of Frasin and Aouf [6] and Xu et al. [15] (see also a closely-related earlier investigation by Srivastava et al. [10]).

1. Introduction, definitions and preliminaries

In the usual notation, let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the subclass of the *normalized* analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also *univalent* in $\mathbb{U}$ (see, for details, [5,11]; see also some of the recent investigations [1,9,12,14] dealing with various interesting subclasses of the analytic function class $\mathcal{A}$ and the univalent function class $\mathcal{A}$).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\Big(f(z)\Big) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}\Big(f(w)\Big) = w \quad \left(|w| < r_0(f); r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2012.05.034

By definition, a function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1). Many interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties and characteristics associated with the bi-univalent function class Σ (including also several open problems and conjectures involving estimates on the Taylor–Maclaurin coefficients of functions in Σ), can be found in the earlier works by Lewin [7], Brannan and Clunie [2], Netanyahu [8], and other authors (see, for example, [3]).

Various subclasses of the bi-univalent function class Σ were introduced and *non-sharp* estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion (1) were found in several recent investigations (see, for example, [4,13]; see also [10,15]). More recently, Frasin and Aouf [6] introduced the following subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$.

Definition 1 (*See [6]*). A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| \leqq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1;\ \lambda \geqq 1) \tag{2}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| \leqq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ 0 < \alpha \leqq 1;\ \lambda \geqq 1), \tag{3}$$

where the function $g(w)$ is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

Theorem 1 (*See [6]*). *Let $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the function class $\mathcal{B}_\Sigma(\alpha, \lambda)$. Then*

$$|a_2| \leqq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}} \tag{5}$$

and

$$|a_3| \leqq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{6}$$

Definition 2 (*See [6]*). A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1;\ \lambda \geqq 1) \tag{7}$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{z} + \lambda g'(w)\right) > \beta \quad (w \in \mathbb{U};\ 0 \leqq \beta < 1;\ \lambda \geqq 1), \tag{8}$$

where the function $g(w)$ is defined by (4).

Theorem 2 (*See [6]*). *Let $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the function class $\mathcal{B}_\Sigma(\beta, \lambda)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{2\lambda+1}} \tag{9}$$

and

$$|a_3| \leqq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1}. \tag{10}$$

Here, in our present sequel to some of the aforecited works (see, especially, [6,15]), we introduce an interesting general subclass $\mathcal{B}_\Sigma^{h,p}(\lambda)$ $(\lambda \geqq 1)$ of the analytic function class $\mathcal{A}$.

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\left\{\Re\Big(h(z)\Big), \Re\Big(p(z)\Big)\right\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1, \tag{11}$$

Also let the function $f(z)$, defined by (1), be in the analytic function class $\mathcal{A}$. We say that $f \in \mathcal{B}_\Sigma^{h,p}(\lambda)$ $(\lambda \geqq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \in h(\mathbb{U}) \quad (z \in \mathbb{U};\ \lambda \geqq 1) \tag{12}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \in p(\mathbb{U}) \quad (w \in \mathbb{U};\ \lambda \geqq 1), \tag{13}$$

where the function $g(w)$ is given by (4).

We note that, in the special case when $\lambda = 1$, the class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ reduces to the class $\mathcal{H}_{\Sigma}^{h,p}$ which was introduced and studied by Xu et al. [15].

Remark 1. From among the many choices of the functions h and p which would provide interesting subclasses of analytic functions, we set

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{14}$$

or

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1). \tag{15}$$

In each of the examples (14) and (15), it is easily verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. Clearly, therefore, if $f \in \mathcal{B}_{\Sigma}^{h,p}(\lambda)$, then we have

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| \leqq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1;\ \lambda \geqq 1) \tag{16}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| \leqq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ \ 0 < \alpha \leqq 1;\ \lambda \geqq 1), \tag{17}$$

or

$$f \in \Sigma \quad \text{and} \quad \Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1) \tag{18}$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right) > \beta \quad (w \in \mathbb{U};\ 0 \leqq \beta < 1;\ 0 \leqq \beta < 1), \tag{19}$$

where the function $g(w)$ is given by (4). This means that

$$f \in \mathcal{B}_{\Sigma}(\alpha, \lambda) \qquad \text{or} \qquad f \in \mathcal{B}_{\Sigma}(\beta, \lambda). \tag{20}$$

Motivated essentially by the recent works [6,15] (see also [10]), for the general subclass $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ $(\lambda \geqq 1)$ of the bi-univalent function class Σ, which is introduced here by Definition 3, we solve the interesting problem involving estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion given by (1). The results presented in this paper generalize and improve the related works of Frasin et al. [6] and Xu et al. [15] (see also the closely-related earlier work by Srivastava et al. [10]).

Throughout our present investigation, we let $\mathbb{R} = (-\infty, \infty)$ be the set of *real* numbers, $\mathbb{C}$ be the set of *complex* numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of *positive* integers.

2. Main results and their demonstration

Our main results involving the general bi-univalent function class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ $(\lambda \geqq 1)$, given by Definition 3, are contained in Theorem 3 below.

Theorem 3. *Suppose that $f(z)$ given by its Taylor–Maclaurin series expansion (1) is in the function class $f \in \mathcal{B}_{\Sigma}^{h,p}(\lambda)$ $(\lambda \geqq 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)}} \qquad \text{and} \qquad |a_3| \leqq \frac{|h''(0)|}{2(1+2\lambda)}. \tag{21}$$

Proof. It follows from the conditions (12) and (13) that

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)=h(z) \quad (z\in\mathbb{U}) \tag{22}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)=p(w) \quad (w\in\mathbb{U}), \tag{23}$$

where h and p satisfy the hypotheses of Definition 3, Furthermore, the functions $h(z)$ and $p(w)$ have the following series expansions:

$$h(z)=1+h_1z+h_2z^2+\cdots \tag{24}$$

and

$$p(w)=1+p_1w+p_2w^2+\cdots, \tag{25}$$

respectively. Now, in view of the series expansions (24) and (25), by equating the coefficients in (22) and (23), we get

$$(1+\lambda)a_2=h_1, \tag{26}$$

$$(1+2\lambda)a_3=h_2, \tag{27}$$

$$-(1+\lambda)a_2=p_1 \tag{28}$$

and

$$(1+2\lambda)(2a_2^2-a_3)=p_2. \tag{29}$$

We find from (26) and (28) that

$$h_1=-p_1 \quad \text{and} \quad 2(1+\lambda)^2a_2^2=h_1^2+p_1^2. \tag{30}$$

Also, from (27) and (29), we obtain

$$2(1+2\lambda)a_2^2=h_2+p_2, \tag{31}$$

which gives us the desired estimate on $|a_2|$ as asserted in (21).

Next, in order to find the bound on $|a_3|$, by subtracting (29) from (27), we get

$$2(1+2\lambda)a_3-2(1+2\lambda)a_2^2=h_2-p_2. \tag{32}$$

Thus, upon substituting the value of a_2^2 from (31) into (32), it follows that

$$a_3=\frac{h_2}{1+2\lambda}, \tag{33}$$

as claimed. This completes the proof of Theorem 3. □

3. Corollaries and consequences

Just as we observed in Remark 1 above, if we specialize the function $h(z)$ by means of (14) and (15), then Theorem 3 is easily seen to yield Corollaries 1 and 2, respectively. These two consequences of Theorem 3 are being stated here *without* proof.

Corollary 1. *If $f(z)$ given by its Taylor–Maclaurin series expansion (1) is in the bi-univalent function class $\mathcal{B}_\Sigma(\alpha,\lambda)$ $(\lambda\geqq 1)$, then*

$$|a_2|\leqq\left(\sqrt{\frac{2}{2\lambda+1}}\right)\alpha \qquad \textit{and} \qquad |a_3|\leqq\frac{2\alpha^2}{2\lambda+1}. \tag{34}$$

Remark 2. Since

$$\left(\sqrt{\frac{2}{1+2\lambda}}\right)\alpha\leqq\frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}} \quad (0<\alpha\leqq 1;\ \lambda\geqq 1) \tag{35}$$

and

$$\frac{2\alpha^2}{1+2\lambda} \leqq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1} \quad (0 < \alpha \leqq 1;\ \lambda \geqq 1), \tag{36}$$

an improvement of Theorem 1 can be accomplished by applying the inequalities (35) and (36) in Corollary 1.

Corollary 2. *If $f(z)$ given by the Taylor–Maclaurin series expansion* (1) *is in the bi-univalent function class $\mathcal{B}_\Sigma(\beta,\lambda)$ $(\lambda \geqq 1)$, then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{2\lambda+1}} \quad \text{and} \quad |a_3| \leqq \frac{2(1-\beta)}{2\lambda+1}. \tag{37}$$

Remark 3. It is obvious that

$$\frac{2(1-\beta)}{2\lambda+1} \leqq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1} \quad (0 \leqq \beta < 1;\ \lambda \geqq 1), \tag{38}$$

which, in conjunction with Corollary 2, would lead us to an improvement of Theorem 2.

By setting $\lambda = 1$ in Theorem 3, we get the following estimates, which were obtained by Xu et al. [15].

Corollary 3 (*See [15]*). *Let $f(z)$ given by the Taylor–Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$ $(\lambda \geqq 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{|h''(0)| + |p''(0)|}{12}} \quad \text{and} \quad |a_3| \leqq \frac{|h''(0)|}{6}. \tag{39}$$

Acknowledgements

This work was supported by the National Natural Science Foundation of the People's Republic of China (Grant No. 11061015), the Jiangxi Provincial Natural Science Foundation of the People's Republic of China (Grant No. 2010GZS0096), and the Natural Science Foundation of the Department of Education of Jiangxi Province of the People's Republic of China (Grant No. GJJ 09149).

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