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Coefficient estimates for a certain subclass of analytic and bi-univalent functions

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ABSTRACT

In this paper, we introduce and investigate an interesting subclass $\mathcal{H}_{\Sigma}^{h,p}$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{H}_{\Sigma}^{h,p}$, we obtain estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent work of Srivastava et al. [H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192].

1. Introduction and definitions

Let $\mathbb{R} = (-\infty, \infty)$ be the set of *real* numbers, $\mathbb{C}$ be the set of *complex* numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of *positive* integers. We let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of the *normalized* analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also *univalent* in $\mathbb{U}$ (see, for details, [1,2]; see also some of the recent investigations [3–8], dealing with various interesting subclasses of the analytic function class $\mathcal{A}$ and the univalent function class $\mathcal{A}$).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U})$$

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and

$$f^{-1}\big(f(w)\big) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [9] first investigated the bi-univalent function class Σ and showed that

$$|a_2| < 1.51.$$

Subsequently, Brannan and Clunie [10] conjectured that

$$|a_2| \leqq \sqrt{2}.$$

Netanyahu [11], on the other hand, showed that

$$\max_{f\in\Sigma} |a_2| = \frac{4}{3}.$$

The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\})$$

is presumably still an open problem. In [12] (see also [13] and [14]), certain subclasses of the bi-univalent function class Σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

Recently, Srivastava et al. [15] introduced the following subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$.

Definition 1 (*See [15]*)**.** A function $f(z)$ given by the Taylor–Maclaurin series expansion (1) is said to be in the class $\mathcal{H}_\Sigma^\alpha$ $(0 < \alpha \leqq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\big(f'(z)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{2}$$

and

$$\left|\arg\big(g'(w)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ 0 < \alpha \leqq 1), \tag{3}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

Theorem 1 (*See [15]*)**.** *Let the function $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{H}_\Sigma^\alpha$ $(0 < \alpha \leqq 1)$. Then*

$$|a_2| \leqq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \textit{and} \quad |a_3| \leqq \frac{\alpha(3\alpha+2)}{3}. \tag{5}$$

Definition 2 (*See [15]*)**.** A function $f(z)$ given by the Taylor–Maclaurin series expansion (1) is said to be in the class $\mathcal{H}_\Sigma^\beta$ $(0 \leqq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\big(f'(z)\big) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1) \tag{6}$$

and

$$\Re\big(g'(w)\big) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1), \tag{7}$$

where the function g is defined by (4).

Theorem 2 (*See [15]*). *Let the function $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_\Sigma^\beta$ $(0 \leqq \beta < 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leqq \frac{(1-\beta)(5-3\beta)}{3}. \tag{8}$$

Here, in our present sequel to some of the aforecited works (especially [15]), we introduce the following subclass of the analytic function class $\mathcal{A}$.

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\left\{\Re\big(h(z)\big), \Re\big(p(z)\big)\right\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$

Also let the function $f(z)$, defined by (1), be in the analytic function class $\mathcal{A}$. We say that $f \in \mathcal{H}_\Sigma^{h,p}$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad f'(z) \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{9}$$

and

$$g'(w) \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{10}$$

where the function g is given by (4).

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{11}$$

or

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1), \tag{12}$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1. If $f \in \mathcal{H}_\Sigma^{h,p}$, then

$$f \in \Sigma \quad \text{and} \quad \left|\arg\big(f'(z)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{13}$$

and

$$\left|\arg\big(g'(w)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{14}$$

or

$$f \in \Sigma \quad \text{and} \quad \Re\left(f'(z)\right) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1) \tag{15}$$

and

$$\Re\big(g'(w)\big) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1), \tag{16}$$

where the function g is given by (4). This means that

$$f \in \mathcal{H}_\Sigma^\alpha \quad (0 < \alpha \leqq 1) \quad \text{or} \quad f \in \mathcal{H}_\Sigma^\beta \quad (0 \leqq \beta < 1). \tag{17}$$

Motivated and stimulated especially by the work of Srivastava et al. [15], we propose to investigate the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$ introduced in Definition 3 and derive coefficient estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for a function $f \in \mathcal{H}_\Sigma^{h,p}$ given by (1). Our results for the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$ would generalize and improve the related work of Srivastava et al. [15].

2. A set of general coefficient estimates

In this section, we state and prove our general results involving the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$ given by Definition 3.

Theorem 3. *Let the function $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$. Then*

$$|a_2| \leqq \sqrt{\frac{|h''(0)| + |p''(0)|}{12}} \quad \text{and} \quad |a_3| \leqq \frac{|h''(0)|}{6}. \tag{18}$$

Proof. First of all, we write the argument inequalities in (9) and (10) in their equivalent forms as follows:

$$f'(z) = h(z) \quad (z \in \mathbb{U}) \quad \text{and} \quad g'(w) = p(w) \quad (w \in \mathbb{U}),$$

respectively, where h and p satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor–Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots,$$

respectively. Now, upon equating the coefficients of $f'(z)$ with those of $h(z)$ and the coefficients of $g'(w)$ with those of $p(w)$, we get

$$2a_2 = h_1, \tag{19}$$

$$3a_3 = h_2, \tag{20}$$

$$-2a_2 = p_1 \tag{21}$$

and

$$3(2a_2^2 - a_3) = p_2. \tag{22}$$

From (19) and (21), we find that

$$h_1 = -p_1 \quad \text{and} \quad 8a_2^2 = h_1^2 + p_1^2. \tag{23}$$

Also, from (20) and (22), we obtain

$$6a_2^2 = h_2 + p_2, \tag{24}$$

which gives us the desired estimate on the coefficient $|a_2|$ as asserted in (18).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (22) from (20). We thus get

$$6a_3 - 6a_2^2 = h_2 - p_2. \tag{25}$$

Upon substituting the value of a_2^2 from (24) into (25), it follows that

$$a_3 = \frac{h_2}{3}, \tag{26}$$

as claimed. This evidently completes the proof of Theorem 3. □

3. Corollaries and consequences

In view of Remark 1, if we set

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{27}$$

and

$$h(z) = p(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1) \tag{28}$$

in Theorem 1, we can readily deduce Corollaries 1 and 2, respectively, which we merely state here *without* proof.

Corollary 1. *Let the function $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leqq 1)$. Then*

$$|a_2| \leqq \frac{\sqrt{6}}{3}\alpha \quad \text{and} \quad |a_3| \leqq \frac{2}{3}\alpha^2. \tag{29}$$

Remark 2. It is easy to prove that

$$\frac{\sqrt{6}}{3}\alpha \leqq \alpha\sqrt{\frac{2}{\alpha+2}} \qquad \text{and} \qquad \frac{2}{3}\alpha^2 \leqq \frac{\alpha(3\alpha+2)}{3} \quad (0 < \alpha \leqq 1),$$

which, in conjunction with Corollary 1, would obviously yield an improvement of Theorem 1.

Corollary 2. *Let the function $f(z)$ given by the Taylor–Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\beta}$ $(0 \leqq \beta < 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leqq \frac{2(1-\beta)}{3}. \tag{30}$$

Remark 3. It is fairly straightforward to verify that

$$\frac{2(1-\beta)}{3} \leqq \frac{(1-\beta)(5-3\beta)}{3} \quad (0 \leqq \beta < 1),$$

which, in conjunction with Corollary 2, would lead us to an improvement of Theorem 2.

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