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# Sharp coefficient estimates for certain subclasses of starlike functions of complex order

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ABSTRACT

In this paper, the authors derive several sharp coefficient estimates for the subclasses

$$\mathcal{S}(\lambda,\gamma,A,B) \quad \text{and} \quad \mathcal{K}(\lambda,\gamma,A,B,m;\mu)$$

of the class of normalized starlike functions of complex order $\gamma \in \mathbb{C}\setminus\{0\}$. The results presented here would generalize as well as improve the recent work by Srivastava et al. (2011) [20]. Some closely-related open problems are also posed.



## 1. Introduction and definitions

Throughout the present investigation, we denote by $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. Also let

$$\mathbb{N} := \{1,2,3,\ldots\} =: \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{S}$ is said to be starlike in $\mathbb{U}$ if the image $f(\mathbb{U})$ is starlike with respect to 0. It is well known that a function $f \in \mathcal{S}$ is starlike in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{2}$$

We denote by $\mathcal{S}^*$ the class of all functions in $\mathcal{S}$ which are starlike in $\mathbb{U}$.

A function $f \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if the image $f(\mathbb{U})$ is convex. In fact, a function $f \in \mathcal{S}$ is convex in $\mathbb{U}$ if and only if

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$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in\mathbb{U}). \tag{3}$$

We denote by $\mathcal{K}$ the class of all functions in $\mathcal{S}$ which are also convex in $\mathbb{U}$.

It is known that (see, for example, [12]; see also [21])

$$\mathcal{K}\subset\mathcal{S}^*\subset\mathcal{S}\subset\mathcal{A}.$$

Nasr and Aouf [17] (see also [18]) and Wiatrowski [23] extended the classes $\mathcal{S}^*$ and $\mathcal{K}$ by introducing the analytic function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}^*(\gamma)$, respectively, as follows.

**Definition 1.** A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{S}^*(\gamma)$ of starlike functions of complex order $\gamma$ if it is satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right)>0 \quad (z\in\mathbb{U};\ \gamma\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\}). \tag{4}$$

**Definition 2.** A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{C}^*(\gamma)$ of convex functions of complex order $\gamma$ if it is satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right)>0 \quad (z\in\mathbb{U};\ \gamma\in\mathbb{C}^*). \tag{5}$$

Obviously, we have

$$\mathcal{S}^*(1)=\mathcal{S}^* \quad \text{and} \quad \mathcal{C}^*(1)=\mathcal{K}.$$

**Remark 1.** The above-defined function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}^*(\gamma)$ have recently been investigated rather extensively by (for example) Altıntaş et al. (see [1–8]), Deng [11], Murugusundaramoorthy et al. ([15,16]), Srivastava et al. [22], and others (see also [12,21]). Moreover, these function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}^*(\gamma)$ stem essentially from the familiar classes of starlike and convex functions of order $\alpha$ $(0\leqq alpha<1)$ in $\mathbb{U}$, which were introduced by Robertson [19] (see also [12,21]).

**Definition 3.** (see, for example, [14]; see also [20]). For two given functions $f$ and $g$, analytic in $\mathbb{U}$, the function $f$ is said to be *subordinate* to the function $g$ if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0)=0 \quad \text{and} \quad |\mathfrak{w}(z)|<1 \quad (z\in\mathbb{U}),$$

such that

$$f(z)=g\big(\mathfrak{w}(z)\big) \quad (z\in\mathbb{U}).$$

This subordination will be denoted here by

$$f\prec g \quad (z\in\mathbb{U})$$

or, more precisely, by

$$f(z)\prec g(z) \quad (z\in\mathbb{U}).$$

In particular, when the function $g$ is univalent in $\mathbb{U}$, we have

$$f\prec g \quad (z\in\mathbb{U}) \iff f(0)=g(0) \quad \text{and} \quad f(\mathbb{U})\subset g(\mathbb{U}) \quad (z\in\mathbb{U}). \tag{6}$$

Definitions 4 and 5 below are due to Srivastava et al. [20].

**Definition 4.** (see, for details, [20]). The analytic function class $\mathcal{S}(\lambda,\gamma,A,B)$ is defined by

$$\mathcal{S}(\lambda,\gamma,A,B):=\left\{f: f\in\mathcal{A} \quad \text{and} \quad 1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\prec\frac{1+Az}{1+Bz}\right\} \tag{7}$$

$$(z\in\mathbb{U};\ 0\leqq\lambda\leqq1;\ \gamma\in\mathbb{C}^*;\ -1\leqq B<A\leqq1).$$

Obviously, we have

$$\mathcal{S}(0,\gamma,1,-1)=\mathcal{S}^*(\gamma) \quad \text{and} \quad \mathcal{S}(1,\gamma,1,-1)=\mathcal{C}^*(\gamma).$$

**Definition 5** (*see, for details, [20]*). A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\lambda, \gamma, A, B, m; \mu)$ if it satisfies the following *non-homogeneous Cauchy–Euler type differential equation of order m*:

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\mu + m - 1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1}(\mu + j) = g(z)\prod_{j=0}^{m-1}(\mu + j + 1) \tag{8}$$

$$\Big(w = f(z) \in \mathcal{A};\ g(z) \in \mathcal{S}(\lambda, \gamma, A, B);\ \mu \in \mathbb{R} \setminus (-\infty, -1];\ m \in \mathbb{N} \setminus \{1\}\Big).$$

The *special* function class $\mathcal{S}(\lambda, 1, 1 - 2\alpha, -1)$ was introduced and studied earlier by Altıntaş (see [1,13]), who also introduced and studied the special cases of the function class $\mathcal{K}(\lambda, \gamma, A, B, m; \mu)$ when $m = 2$ and $m = 3$.

The following results due to Srivastava et al. [20] provide the coefficient bounds for the general function classes $\mathcal{S}(\lambda, \gamma, A, B)$ and $\mathcal{K}(\lambda, \gamma, A, B, m; \mu)$ given by Definitions 4 and 5, respectively.

**Theorem 1.** (see [20]). *Let the function $f(z)$ be given by* (1). *If $f(z) \in \mathcal{S}(\lambda, \gamma, A, B)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}\left(j + \frac{2|\gamma|(A-B)}{1-B}\right)}{(n-1)![1 + \lambda(n-1)]} \quad (n \in \mathbb{N} \setminus \{1\}) \tag{9}$$

$$(0 \leqq \lambda \leqq 1;\ \gamma \in \mathbb{C}^*;\ -1 \leqq B < A \leqq 1).$$

**Theorem 2.** (see [20]). *Let the function $f(z)$ be given by* (1). *If $f(z) \in \mathcal{K}(\lambda, \gamma, A, B, m; \mu)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}\left(j + \frac{2|\gamma|(A-B)}{1-B}\right)\prod_{j=0}^{m-1}(\mu + j + 1)}{(n-1)![1 + \lambda(n-1)]\prod_{j=0}^{m-1}(\mu + j + n)} \quad (m, n \in \mathbb{N} \setminus \{1\}) \tag{10}$$

$$(0 \leqq \lambda \leqq 1;\ \gamma \in \mathbb{C}^*;\ -1 \leqq B < A \leqq 1;\ \mu \in \mathbb{R} \setminus (-\infty, -1]).$$

Ever since the proof of the celebrated Bieberbach conjecture [9] (that is, the de Branges theorem [10]) on the sharp coefficient bounds for the univalent function class $\mathcal{S}$, many authors obtained interesting results for various subclasses of analytic functions. We choose to recall here the recent results given by (for example) Xu et al. (see [24]), Altıntaş et al. (see [1–8]) and other authors (see [15,11,13]). In our present sequel to many of the above-cited investigations, we propose to derive several interesting properties (including, for example, the sharp coefficient bounds) for functions belonging to the general function classes $\mathcal{S}(\lambda, \gamma, A, B)$ and $\mathcal{K}(\lambda, \gamma, A, B, m; \mu)$ of complex order $\gamma \in \mathbb{C}^*$. Our results generalize as well as improve the corresponding results (see Theorems 1 and 2 above) which were obtained earlier by Srivastava et al. [20]. In conclusion, we also pose some closely-related open problems.

## 2. A set of sharp coefficient bounds

The following lemma will be needed in order to prove the desired results on sharp coefficient bounds for functions in the classes

$$\mathcal{S}(\lambda, \gamma, A, B) \quad \text{and} \quad \mathcal{K}(\lambda, \gamma, A, B, m; \mu).$$

**Lemma 1.** *Let the parameters $A, B, \lambda$, and $\gamma$, as well as the integer m, constrained by*

$$-1 \leqq B < A \leqq 1, \quad 0 \leqq \lambda \leqq 1, \quad \gamma \in \mathbb{C}^* \quad \text{and} \quad m \in \mathbb{N} \setminus \{1\},$$

*be fixed. Suppose also that*

$$|\gamma(A - B) - B(m - 2)| \geqq qm - 2.$$

*Then*

$$|\gamma|^2(A - B)^2 + \sum_{k=2}^{m-1} \frac{\left||\gamma(A - B) - B(k - 1)|^2 - (k - 1)^2\right|}{[(k - 1)!]^2} \cdot \prod_{j=0}^{k-2}|\gamma(A - B) - Bj|^2 = \frac{\prod_{j=0}^{m-2}|\gamma(A - B) - Bj|^2}{[(m - 2)!]^2}. \tag{11}$$

**Proof.** We make use of the principle of mathematical induction on $m \in \mathbb{N} \setminus \{1\}$. Indeed, for $m = 2$, the assertion (11) of the above Lemma holds true trivially. We now assume that

$$|\gamma|^2(A-B)^2+\sum_{k=2}^{m-2}\frac{\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|}{[(k-1)!]^2}\cdot\prod_{j=0}^{k-2}|\gamma(A-B)-Bj|^2=\frac{\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2}{[(m-3)!]^2}$$

for some fixed positive integer $m\in\mathbb{N}\setminus\{1,2\}$. We then readily observe that

$$\begin{aligned}
&|\gamma|^2(A-B)^2+\sum_{k=2}^{m-1}\frac{\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|}{[(k-1)!]^2}\cdot\prod_{j=0}^{k-2}|\gamma(A-B)-Bj|^2\\
&=|\gamma|^2(A-B)^2+\sum_{k=2}^{m-2}\frac{\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|}{[(k-1)!]^2}\cdot\prod_{j=0}^{k-2}|\gamma(A-B)-Bj|^2\\
&+\frac{\left||\gamma(A-B)-B(m-2)|^2-(m-2)^2\right|}{[(m-2)!]^2}\cdot\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2\\
&=\frac{\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2}{[(m-3)!]^2}+\frac{|\gamma(A-B)-B(m-2)|^2-(m-2)^2}{[(m-2)!]^2}\cdot\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2\\
&=\frac{\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2}{[(m-2)!]^2}\cdot\left[(m-2)^2+|\gamma(A-B)-B(m-2)|^2-(m-2)^2\right]\\
&=\frac{\prod_{j=0}^{m-3}|\gamma(A-B)-Bj|^2}{[(m-2)!]^2}\cdot|\gamma(A-B)-B(m-2)|^2=\frac{\prod_{j=0}^{m-2}|\gamma(A-B)-Bj|^2}{[(m-2)!]^2},
\end{aligned}$$

which evidently completes the proof of the assertion (11) of the Lemma by the principle of mathematical induction on $m\in\mathbb{N}\setminus\{1\}$. □

Our main results in this paper are given by Theorems 3 and 4 below. We first state and prove Theorem 3 for functions in the general class $\mathcal{S}(\lambda,\gamma,A,B)$.

**Theorem 3.** *Let the parameters $A,B,\lambda$, and $\gamma$, as well as the integer $n$, constrained by*

$$-1\leqq B<A\leqq 1,\quad 0\leqq\lambda\leqq 1,\quad \gamma\in\mathbb{C}^*\quad\text{and}\quad n=\mathbb{N}\setminus\{1\},$$

*be fixed. Suppose also that*

$$|\gamma(A-B)-B(n-2)|\geqq n-2 \tag{12}$$

*and that the function $f(z)$ is given by (1). If $f\in\mathcal{S}(\lambda,\gamma,A,B)$, then*

$$|a_n|\leqq\frac{\prod_{j=0}^{n-2}|\gamma(A-B)-Bj|}{(n-1)![1+\lambda(n-1)]}\quad(n=\mathbb{N}\setminus\{1\}). \tag{13}$$

*The estimates in (13) are sharp.*

**Proof.** Since $f\in\mathcal{S}(\lambda,\gamma,A,B)$, there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0)=0\quad\text{and}\quad|\mathfrak{w}(z)|<1\quad(z\in\mathbb{U}),$$

such that

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)=\frac{1+A\mathfrak{w}(z)}{1+B\mathfrak{w}(z)}$$

$$(-1\leqq B<A\leqq 1;\ 0\leqq\lambda\leqq 1;\ \gamma\in\mathbb{C}^*),$$

that is,

$$1+\frac{1}{\gamma}\left(\frac{\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]a_kz^k}{z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]a_kz^k}\right)=\frac{1+A\mathfrak{w}(z)}{1+B\mathfrak{w}(z)}.$$

Hence we have

$$\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]a_kz^k=\left(\gamma(A-B)z+\sum_{k=2}^{\infty}[\gamma(A-B)-B(k-1)][1+\lambda(k-1)]a_kz^k\right)\mathfrak{w}(z),$$

which may be rewritten as follows:

$$\sum_{k=2}^{n}(k-1)[1+\lambda(k-1)]a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k$$
$$= \left(\gamma(A-B)z + \sum_{k=2}^{n-1}[\gamma(A-B)-B(k-1)][1+\lambda(k-1)]a_k z^k\right)\cdot \sum_{k=1}^{\infty}\mathfrak{c}_k z^k,$$

where

$$\mathfrak{w}(z) = \sum_{k=1}^{\infty}\mathfrak{c}_k z^k \quad (z\in\mathbb{U})$$

and the coefficients $b_k$ are given by the following series:

$$\sum_{k=n+1}^{\infty} b_k z^k = \sum_{k=n+1}^{\infty}(k-1)[1+\lambda(k-1)]a_k z^k - \left(\sum_{k=n}^{\infty}[\gamma(A-B)-B(k-1)][1+\lambda(k-1)]a_k z^k\right)\cdot\sum_{k=1}^{\infty}\mathfrak{c}_k z^k,$$

which obviously is convergent in $\mathbb{U}$. We note that

$$|\mathfrak{w}(z)|<1 \quad (z\in\mathbb{U}).$$

Thus, by Parseval's Theorem, we obtain

$$\sum_{k=2}^{n}(k-1)^2[1+\lambda(k-1)]^2|a_k|^2 \leqq |\gamma|^2(A-B)^2 + \sum_{k=2}^{n-1}|\gamma(A-B)-B(k-1)|^2\cdot[1+\lambda(k-1)]^2|a_k|^2$$

or, equivalently,

$$(n-1)^2[1+\lambda(n-1)]^2|a_n|^2 \leqq \sum_{k=1}^{n-1}\left[|\gamma(A-B)-B(k-1)|^2-(k-1)^2\right]\cdot[1+\lambda(k-1)]^2|a_k|^2 \qquad (a_1:=1). \tag{14}$$

We now apply the principle of mathematical induction on $n\in\mathbb{N}\setminus\{1\}$. For $n=2$, it follows from (14) that

$$|a_2| \leqq \frac{|\gamma|(A-B)}{1+\lambda},$$

which is equivalent to (13). Suppose now that (13) holds true for $k=n-1$, for some fixed $n$, that is, the following inequality holds true:

$$|a_k| \leqq \frac{\prod_{j=0}^{k-2}|\gamma(A-B)-Bj|}{(k-1)![1+\lambda(k-1)]}. \tag{15}$$

Then, by using (14), (15) and the Lemma, we deduce for $k=n$ that

$$|a_n|^2 \leqq \frac{1}{(n-1)^2[1+\lambda(n-1)]^2}\sum_{k=1}^{n-1}\left[|\gamma(A-B)-B(k-1)|^2-(k-1)^2\right]$$
$$\cdot[1+\lambda(k-1)]^2|a_k|^2 \leqq \frac{1}{(n-1)^2[1+\lambda(n-1)]^2}$$
$$\times\left(|\gamma|^2(A-B)^2+\sum_{k=2}^{n-1}\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|\cdot[1+\lambda(k-1)]^2|a_k|^2\right)$$
$$\leqq \frac{1}{(n-1)^2[1+\lambda(n-1)]^2}\left(|\gamma|^2(A-B)^2+\sum_{k=2}^{n-1}\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|\cdot[1+\lambda(k-1)]^2\frac{\prod_{j=0}^{k-2}|\gamma(A-B)-Bj|^2}{[(k-1)!]^2[1+\lambda(k-1)]^2}\right)$$
$$=\frac{\prod_{j=0}^{n-2}|\gamma(A-B)-Bj|^2}{[(n-1)!]^2[1+\lambda(n-1)]^2}.$$

This establishes the inequality (13) asserted by Theorem 3.

In order to see that the coefficient estimates in (13) asserted by Theorem 3 are sharp, it suffices to consider the following function:

$$f(z)=\begin{cases} z^{\frac{\lambda-1}{\lambda}}\left(\int_0^z \frac{t^{\frac{1-\lambda}{\lambda}}}{\lambda(1+Bt)^{(\frac{B-A}{B})\gamma}}dt\right) & (B\neq 0;\ \lambda\neq 0)\\ \frac{z}{(1+Bz)^{(\frac{B-A}{B})\gamma}} & (B\neq 0;\ \lambda=0)\\ \frac{1}{\lambda}z^{\frac{\lambda-1}{\lambda}}\left(\int_0^z t^{\frac{1-\lambda}{\lambda}}\exp(A\gamma t)dt\right) & (B=0;\ \lambda\neq 0)\\ z\exp(A\gamma z) & (B=0;\ \lambda=0). \end{cases}$$

This completes the proof of Theorem 3. □

**Theorem 4.** *Let the parameters $A, B, \lambda$ and $\gamma$, as well as the integers $m$ and $n$, constrained by*

$$-1 \leqq B < A \leqq 1, \quad 0 \leqq \lambda \leqq 1, \quad \gamma \in \mathbb{C}^*, \quad \mu \in \mathbb{R} \setminus (-\infty, -1] \quad \text{and} \quad m, n \in \mathbb{N} \setminus \{1\},$$

*be fixed. Suppose also that*

$$|\gamma(A - B) - B(n-2)| \geqq n - 2$$

*and that the function $f(z)$ is given by (1). If $f \in \mathcal{K}(\lambda, \gamma, A, B, m; \mu)$, then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2} |\gamma(A-B) - Bj| \prod_{j=0}^{m-1} (\mu + j + 1)}{(n-1)![1 + \lambda(n-1)] \prod_{j=0}^{m-1} (\mu + j + n)} \quad (m, n \in \mathbb{N} \setminus \{1\}). \tag{16}$$

*The estimates in (16) are sharp.*

**Proof.** Suppose that the function $f \in \mathcal{A}$ is given by (1). Also let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}(\lambda, \gamma, A, B).$$

We then deduce from (8) that

$$a_n = \left( \frac{\prod_{j=0}^{m-1} (\mu + j + 1)}{\prod_{j=0}^{m-1} (\mu + j + n)} \right) b_n \quad \Big(m, n \in \mathbb{N} \setminus \{1\};\ \mu \in \mathbb{R} \setminus (-\infty, -1]\Big). \tag{17}$$

Now, by using Theorem 3 in conjunction with (17), we arrive at the assertion (16) of Theorem 4 and the estimates in (16) are easily seen to be sharp. This completes the proof of Theorem 4. □

## 3. Corollaries and consequences

In this section, we apply our main results (Theorems 3 and 4 of Section 2) in order to deduce each of the following corollaries and consequences.

**Corollary 1.** *Let the function $f(z) \in \mathcal{A}$ be given by (1) and let $0 \leqq \beta < 1$. If*

$$f \in \mathcal{S}(\lambda, \gamma, 1 - 2\beta, -1) \equiv \mathcal{SC}(\gamma, \lambda, \beta)$$

*and*

$$|2\gamma(1 - \beta) + n - 2| \geqq n - 2,$$

*then*

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2} |2\gamma(1-\beta) + j|}{(n-1)![1 + \lambda(n-1)]} \quad (n \in \mathbb{N} \setminus \{1\}).$$

*These estimates are sharp for the function $f(z)$ defined by*

$$f(z) = \begin{cases} z^{\frac{\lambda-1}{\lambda}} \left( \int_0^z \frac{t^{\frac{1-\lambda}{\lambda}}}{\lambda(1-t)^{2(1-\beta)\gamma}} dt \right) & (\lambda \neq 0) \\ \dfrac{z}{(1-z)^{2(1-\beta)\gamma}} & (\lambda = 0). \end{cases}$$

**Corollary 2.** *Let the function $f(z) \in \mathcal{A}$ be given by (1) and let $0 \leqq \beta < 1$. If*

$$f \in \mathcal{K}(\lambda, \gamma, 1 - 2\beta, -1, 2; \mu) \equiv \mathcal{B}(\gamma, \lambda, \beta; \mu),$$

*then*

$$|a_n| \leqq \frac{(\mu+1)(\mu+2) \prod_{j=0}^{n-2} |2\gamma(1-\beta) + j|}{(n-1)![1 + \lambda(n-1)](\mu+n)(\mu+n+1)} \quad (n \in \mathbb{N} \setminus \{1\}).$$

*These estimates are sharp.*

**Remark 2.** Unlike the results (see Theorems 1 and 2 of Section 1) which were obtained by Srivastava et al. [20] for the *special* function classes $\mathcal{SC}(\gamma,\lambda,\beta)$ and $\mathcal{B}(\gamma,\lambda,\beta;\mu)$ considered in [20], the bounds asserted here by Corollaries 1 and 2 are sharp.

**Remark 3.** In case the condition (12) in Theorem 3 is dropped, we do not know whether or not the assertion of Theorem 3 will still hold true. Consequently, we pose the following two *open problems*.

**Open Problem 1.** Let the parameters $A, B, \lambda$ and $\gamma$, as well as the integer $n$, constrained by

$$-1 \leqq B < A \leqq 1, \quad 0 \leqq \lambda \leqq 1, \quad \gamma \in \mathbb{C}^* \quad \text{and} \quad n \in \mathbb{N} \setminus \{1\},$$

be fixed. Suppose also that the function $f(z) \in \mathcal{A}$ is given by (1). If $f \in \mathcal{S}(\lambda,\gamma,A,B)$, then prove or disprove that

$$|a_n| \leqq \frac{\prod_{j=0}^{n-2}|\gamma(A-B)-Bj|}{(n-1)![1+\lambda(n-1)]} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{18}$$

**Open Problem 2.** If the coefficient estimates in 18 do hold true, prove or disprove that these estimates are sharp.

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