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ON PROPERTIES OF q-DIFFERENCE EQUATIONS*

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Abstract In this article, we consider some type of q-difference equations, which have meromorphic solutions with Borel exceptional zeros and poles. We also give a precise result in the finite order case and some further results in a particular case where $q_i = q^i$.

Key words q-difference equation; meromorphic solution; Borel exceptional zeros and poles

2000 MR Subject Classification 30D35; 39B32

1 Introduction

Throughout this article, we use standard notations in the Nevanlinna theory (see [1–4]). Let $f(z)$ be a meromorphic function. Here and in the following, the word "meromorphic" means meromorphic in the whole complex plane. Moreover, we use notations $\rho(f)$ and $\mu(f)$ for the order and the lower order of a meromorphic function f, respectively. We also use notations $\lambda(f)$ ($\overline{\lambda}(f)$) and $\lambda(\frac{1}{f})$ ($\overline{\lambda}(\frac{1}{f})$) for the exponent of convergence of zeros (distinct zeros) and the exponent of convergence of poles (distinct poles) of $f(z)$, respectively. For $R(z,f(z)) = \frac{\sum\limits_{i=0}^{m} a_i(z)f(z)^i}{\sum\limits_{j=0}^{n} b_j(z)f(z)^j}$, an irreducible rational function in f, we denote $\deg_f R = \max\{m,n\}$ throughout this article.

Recently, a number of articles (see [5–10]) focused on the complex difference and q-difference. There were also articles (see [5, 11–21]) focusing on the existence and the growth of meromorphic solutions of complex difference equations and q-difference equations.

In [16], Heittokangas et al considered meromorphic solutions with Borel exceptional zeros and poles of some type of difference equations, and obtained the result as follows.

Theorem A Let $c_1, \cdots, c_n \in \mathbb{C}\backslash\{0\}$ and suppose that f is a non-rational meromorphic solution of a difference equation of the form

$$\prod_{i=1}^{n} f(z+c_i) = \frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_t(z)f(z)^t}, \tag{1.1}$$

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with meromorphic coefficients $a_i(z), b_j(z)$ of growth $S(r,f)$ such that $a_p(z)b_t(z) \not\equiv 0$. If

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} < \rho(f),$$

then, (1.1) is of the form

$$\prod_{i=1}^{n} f(z+c_i) = c(z)f(z)^k,$$

where $c(z)$ is meromorphic, $T(r,c) = S(r,f)$, and $k \in \mathbb{Z}$.

Gundersen et al dealt with q-difference equations under a relatively weaker condition than Theorem A and obtained the following result in [14].

Theorem B Suppose that f is a transcendental meromorphic solution of a q-difference equation of the form

$$f(qz) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_t(z)f(z)^t}, \tag{1.2}$$

where $q \in \mathbb{C}$, $|q| > 1$, $b_t(z) \equiv 1$, and meromorphic coefficients $a_i(z), b_j(z)$ are of growth $S(r,f)$. If

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r,f),$$

then, (1.2) is either of the form

$$f(qz) = a_p(z)f(z)^p \quad \text{or} \quad f(qz) = \frac{a_0(z)}{f(z)^t}.$$

In this article, we consider a q-difference equation more general than (1.2) under a condition similar to Theorem A on meromorphic solutions, and obtain the following Theorem 1, which tells us that solutions having Borel exceptional zeros and poles appear in special situations only.

Theorem 1 Suppose that f is a transcendental meromorphic solution of a q-difference equation of the form

$$\prod_{i=1}^{n} f(q_i z) = R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \cdots + b_t(z)f(z)^t}, \tag{1.3}$$

where $q_i \in \mathbb{C}\backslash\{0,1\}, i = 1, \cdots, n$, and $R(z,f)$ is an irreducible rational function in f with meromorphic coefficients $a_i(z)$ $(i = 0, \cdots, p)$ and $b_j(z)$ $(j = 0, \cdots, t)$ of growth $S(r,f)$ such that $b_t(z) \equiv 1$, $a_p(z) \not\equiv 0$. If

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} < \rho(f) = \rho, \tag{1.4}$$

then (1.3) is reduced to the form

$$\prod_{i=1}^{n} f(q_i z) = a_p(z)f(z)^p \tag{1.5}$$

or

$$\prod_{i=1}^{n} f(q_i z) = \frac{a_0(z)}{f(z)^t}. \tag{1.6}$$

We can also improve Theorem 1 in the finite order case and obtain Corollary 1:

Corollary 1 Suppose that f is of finite order ρ, $\sum\limits_{i=1}^{n} q_i^{\rho} + t \neq 0$, $\rho(a_0) < \rho$, and all the other assumptions of Theorem 1 hold. Then, (1.3) is reduced to the form (1.5) only.

In particular, we consider the case $q_i = q^i$, $i = 1, 2, \cdots, n$, in (1.3) and (1.5)–(1.6), and obtain the following results.

First, we consider the problem converse to Theorem 1, following the main idea of [14, Theorem 5.3], and obtain Theorem 2:

Theorem 2 Suppose that f is a transcendental meromorphic solution of a q-difference equation either of the form

$$\prod_{i=1}^{n} f(q^i z) = a_p(z) f(z)^p \quad \text{or} \quad \prod_{i=1}^{n} f(q^i z) = \frac{a_0(z)}{f(z)^t}, \tag{1.7}$$

where $q \in \mathbb{C}$, $|q| \neq 0, 1$, and meromorphic function $a_p(z)(\not\equiv 0)$ or $a_0(z)(\not\equiv 0)$ satisfies $\rho(a_p) < \rho(f)$ or $\rho(a_0) < \rho(f)$ respectively. Then,

$$\max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} < \rho(f).$$

Furthermore, we consider the relation between meromorphic solutions and meromorphic coefficients in some case of (1.7), which partly generalizes [14, Theorem 5.4].

Theorem 3 Suppose that f is a meromorphic function of a q-difference equation of the form

$$\prod_{i=1}^{n} f(q^i z) = \frac{a(z)}{f(z)^t}, \tag{1.8}$$

where $q \in \mathbb{C}, |q| \neq 0, 1$, and $a(z)(\not\equiv 0)$ is a meromorphic function. If $f(z)$ has only finitely many poles, then, $f(z) = H(z)e^{P(z)}$, where $H(z)$ is a rational function and $P(z)$ a polynomial, if and only if $a(z) = S(z)e^{Q(z)}$, where $S(z)$ is a rational function and $Q(z)$ a polynomial. Furthermore, $H(z)$ is a constant if and only if $S(z)$ is a constant.

Note that the proposition converse-negative to Corollary 1 holds, and we omit its statement here. In fact, we can prove a more general result as follows.

Theorem 4 Suppose that f is a meromorphic solution of a q-difference equation of the form

$$Q\left(z, \prod_{i=1}^{n} f(q^i z)\right) = \frac{a(z)}{f(z)^t}, \tag{1.9}$$

where $q \in \mathbb{C}$, $|q| \neq 0, 1$, $a(z)(\not\equiv 0)$ is a rational function and $Q(z, u)$ is a polynomial in u with rational coefficients satisfying $\deg_u Q = m > 0$. If $j \sum\limits_{i=1}^{n} q^{ik} + t \neq 0, j = 0, \cdots, m$ holds for any positive integer k and f has only finitely many poles, then f must be either a rational function or of the form

$$f(z) = r(z)e^{g(z)},$$

where $r(z)(\not\equiv 0)$ is a rational function and $g(z)$ is a transcendental entire function. Thus, the transcendental meromorphic solution of (1.9) with only finitely many poles must be of infinite regular order.

Example 1 The function $f(z) = \cos z$ satisfies

$$f(2z)f(4z) = 16f(z)^6 - 24f(z)^4 + 10f(z)^2 - 1.$$

Clearly, f satisfies $\lambda\left(\frac{1}{f}\right)=0<1=\lambda(f)=\rho(f)$. In contrast, $16\cos^6 z-24\cos^4 z+10\cos^2 z-1$ is a rational function in $\cos z$ not of the form of the right-hand side of (1.5) or (1.6). This example shows that we cannot omit assumption (1.4) in Theorem 1 and that we also cannot replace it by $\min\{\lambda(f),\lambda(\frac{1}{f})\}<\rho(f)$.

Example 2 The function $f(z)=e^{z^2}$ satisfies

$$f(2z)f(4z)=f(z)^{20}$$

and all assumptions in Corollary 1. This example shows that the case (1.5) in Corollary 1 can occur. And it is also an example of Theorem 2, where

$$\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}=\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}=0<2=\rho(f).$$

Example 3 The function $f(z)=e^{z^3}$ satisfies

$$\left(f\left(-\frac{1}{2}z\right)f\left(\frac{1}{4}z\right)\right)^{64}=\frac{1}{f(z)^7},$$

where $m=64,\ q=-\frac{1}{2},\ t=7$ satisfying $m\sum\limits_{i=1}^{2}q^{3i}+t=0$. This example shows that the assumption "$j\sum\limits_{i=1}^{n}q^{ik}+t\neq 0, j=0,\cdots,m$" cannot be omitted in Theorem 4.

2 Lemmas for Proofs of Theorems

Lemma 1 [2] (Valiron-Mohon'ko) Let $f(z)$ be a meromorphic function, then for all irreducible rational functions in f,

$$R(z,f(z))=\frac{\sum\limits_{i=0}^{m}a_i(z)f(z)^i}{\sum\limits_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z),b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),$$

where $d=\max\{m,n\}$ and $\Psi(r)=\max\limits_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Next, the following lemmas can be found in [16, p.37] and [14, p.127] respectively.

Lemma 2 If a meromorphic function f satisfies

$$\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\rho(f),$$

then, f is of regular growth.

Lemma 3 The differential field

$$\mathcal{L}_f=\{g \text{ is meromorphic} \mid T(r,g)=S(r,f)\}$$

is algebraically closed in the field of meromorphic functions in the complex plane. That is, any meromorphic function satisfying an algebraic equation over the field $\mathcal{L}_f$ actually belongs to $\mathcal{L}_f$.

Remark We would also use the observation (see [12, p.249]) that

$$N(r, f(qz)) = N(|q|r, f) + O(1)$$

holds for any meromorphic function f and any non-zero complex constant q. Clearly, we immediately obtain

$$\lambda\left(\frac{1}{f(qz)}\right) = \lambda\left(\frac{1}{f}\right).$$

Similarly, we obtain

$$N\left(r, \frac{1}{f(qz)}\right) = N\left(|q|r, \frac{1}{f}\right) + O(1)$$

and

$$\lambda(f(qz)) = \lambda(f).$$

Lemma 4 Suppose that f is a meromorphic function of finite order $\rho(f) = \rho$ satisfying (1.4), and that $q_i \in \mathbb{C}\backslash\{0\},\ i = 1, \cdots, n$, satisfy

$$\sum_{i=1}^{n} q_i^{\rho} \neq 0, \tag{2.1}$$

then, $H(z) = \prod\limits_{i=1}^{n} f(q_i z)$ satisfies $\rho(H) = \rho$.

Proof By Hadmard Theorem and the assumptions that

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} < \rho(f) = \rho < \infty, \tag{2.2}$$

we can write

$$f(z) = z^m \frac{Q_g(z)}{Q_h(z)} e^{P(z)},$$

where $m \in \mathbb{Z}$, $P(z)$ is a polynomial of degree ρ, and $Q_g(z), Q_h(z)$ are the canonical products formed with nonzero zeros and poles of f respectively. So, we have

$$H(z) = \prod_{i=1}^{n} (q_i z)^m \frac{Q_g(q_i z)}{Q_h(q_i z)} e^{P(q_i z)}.$$

Clearly, by Remark and (2.2), we have

$$\lambda(H) \leq \max_{1 \leq i \leq n} \{\lambda(f(q_i z))\} = \lambda(f) < \rho, \tag{2.3}$$

$$\lambda\left(\frac{1}{H}\right) \leq \max_{1 \leq i \leq n} \left\{\lambda\left(\frac{1}{f(q_i z)}\right)\right\} = \lambda\left(\frac{1}{f}\right) < \rho. \tag{2.4}$$

By denoting

$$G_i(z) = (q_i z)^m \frac{Q_g(q_i z)}{Q_h(q_i z)},$$

we see that $H(z)$ is of the form

$$H(z) = e^{\sum\limits_{i=1}^{n} q_i^{\rho} a_{\rho} z^{\rho} + P_{\rho-1}(z)} \prod_{i=1}^{n} G_i(z),$$

where a_ρ is the leading coefficient of $P(z)$ and $P_{\rho-1}(z)$ is a polynomial of degree $\leq \rho-1$. By (2.3) and (2.4), we know that $\rho(G_i)<\rho$ for all $i=1,\cdots,n$. Hence also $\rho\big(\prod_{i=1}^{n} G_i\big)<\rho$. Then, by (2.1), we obtain $\rho(H)=\rho$.

Lemma 5 [17] If meromorphic coefficients $a_0,\cdots,a_n,Q$ in the q-difference equation of the form

$$\sum_{j=0}^{n} a_j(z)f(q^j z)=Q(z), \tag{2.5}$$

where $q\in\mathbb{C},\ |q|\neq 0,1$, are of finite order $\leq\rho$, then all meromorphic solutions of (2.5) are of finite order $\leq\rho$.

Lemma 6 [3] Let $f_j(z), j=1,\cdots,n$ $(n\geq 2)$, be meromorphic functions, $g_j(z), j=1,\cdots,n$, be entire functions, satisfying:

(i) $\sum\limits_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$;

(ii) when $1\leq j<k\leq n$, $g_j(z)-g_k(z)$ is not a constant;

(iii) when $1\leq j\leq n$, $1\leq h<k\leq n$,

$$T(r,f_j)=o\left(T(r,e^{g_h-g_k})\right),\quad (r\to\infty, r\notin E),$$

where $E\subset(1,\infty)$ is of finite linear measure or finite logarithmic measure.
Then, $f_j(z)\equiv 0, j=1,\cdots,n$.

Lemma 7 [17] Let the coefficients $a_0,\cdots,a_n$ of an equation of the form

$$\sum_{j=0}^{n} a_j f(q^j z)=Q(z),\quad q\in\mathbb{C},|q|\neq 0,1, \tag{2.6}$$

be complex constants and $Q(z)$ be of the reduced form $Q(z)=\frac{P_1(z)}{z^l}$, where $P_1(z)$ is a polynomial of degree d and $l\in\mathbb{N}\cup\{0\}$. Then, all meromorphic solutions $f(z)$ of (2.6) are of the reduced form $f(z)=\frac{P_2(z)}{z^p}$, where $P_2(z)$ is a polynomial and $p\geq l$.

3 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1 We use a reasoning method similar to that of Theorem A, but give a more precise result by further computing. Denote $H(z)=\prod_{i=1}^{n} f(q_i z)$. Fix constants β and γ, such that

$$\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\beta<\gamma<\rho. \tag{3.1}$$

By using (3.1) and the lemma of the logarithmic derivative, we obtain

$$T\left(r,\frac{f'}{f}\right)=m\left(r,\frac{f'}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)=O(r^\beta)+S(r,f). \tag{3.2}$$

Similarly,

$$\begin{aligned}T\left(r,\frac{H'}{H}\right)&=m\left(r,\frac{H'}{H}\right)+\overline{N}(r,H)+\overline{N}\left(r,\frac{1}{H}\right)\\&\leq\sum_{i=1}^{n}\left(\overline{N}(r,f(q_i z))+\overline{N}\left(r,\frac{1}{f(q_i z)}\right)\right)+S(r,H)\end{aligned}$$

$$= \sum_{i=1}^{n} \left(\overline{N}(|q_i|r, f(z)) + \overline{N}\left(|q_i|r, \frac{1}{f(z)} \right) \right) + S(r, H)$$
$$= O(r^{\beta}) + S(r, H).$$

By applying Lemma 1 to (1.3), we obtain

$$T(r, H) = T(r, R(z, f)) = \max\{p, t\} T(r, f) + S(r, f).$$

So,

$$T\left(r, \frac{H'}{H}\right) = O(r^{\beta}) + S(r, f). \tag{3.3}$$

By Lemma 2 and (1.4), there exists $r_0 > 0$ such that $T(r, f) > r^{\gamma}$ for $r \geq r_0$. It follows that

$$O(r^{\beta}) = o(T(r, f)) \tag{3.4}$$

holds for sufficiently large r. Then, by (3.2)–(3.4), we obtain

$$T\left(r, \frac{f'}{f}\right) = S(r, f), \quad T\left(r, \frac{H'}{H}\right) = S(r, f). \tag{3.5}$$

Rewrite (1.3) in the form

$$\frac{b_t(z)}{a_p(z)} H(z) = \frac{P(z, f)}{Q(z, f)} = u(z, f). \tag{3.6}$$

Clearly, $P(z, f)$ and $Q(z, f)$ are monic polynomials in f with coefficients of growth $S(r, f)$. Denote $F = \frac{f'}{f}$, $U = \frac{u'}{u}$. By (3.6), we have

$$U = \frac{a'(z)}{a(z)} + \frac{H'(z)}{H(z)}, \tag{3.7}$$

where $a(z) = \frac{b_t(z)}{a_p(z)}$. So, by (3.5) and (3.7), we have

$$T(r, F) = S(r, f), \quad T(r, U) = S(r, f).$$

Because

$$\frac{P'Q - PQ'}{Q^2} = u' = Uu = \frac{UP}{Q},$$

we have

$$P'Q - PQ' = UPQ. \tag{3.8}$$

Writing $f' = Ff$ in (3.8), and then regarding (3.8) as an algebraic equation in f with coefficients of growth $S(r, f)$, we obtain by Lemma 3

$$(p - t)F = U.$$

Therefore,

$$u(z, f) = \alpha f(z)^{p-t}$$

for some $\alpha \in \mathbb{C}$, and so

$$H(z) = \alpha \frac{a_p(z)}{b_t(z)} f(z)^{p-t} = \alpha a_p(z) f(z)^{p-t}. \tag{3.9}$$

Substituting (3.9) into (1.3), we have

$$\alpha a_p(z)f(z)^{p-t}=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_{t-1}(z)f(z)^{t-1}+f(z)^t}.$$

That is,

$$\begin{aligned}&\alpha a_p(z)f(z)^p+\alpha a_p(z)b_{t-1}(z)f(z)^{p-1}+\cdots\\&+\alpha a_p(z)b_1(z)f(z)^{p-t+1}+\alpha a_p(z)b_0(z)f(z)^{p-t}\\=\,&a_p(z)f(z)^p+a_{p-1}(z)f(z)^{p-1}+\cdots+a_1(z)f(z)+a_0(z)\end{aligned}\tag{3.10}$$

when $p\geq t$, or

$$\begin{aligned}&\alpha a_p(z)f(z)^t+\alpha a_p(z)b_{t-1}(z)f(z)^{t-1}+\cdots+\alpha a_p(z)b_1(z)f(z)+\alpha a_p(z)b_0(z)\\=\,&a_p(z)f(z)^t+a_{p-1}(z)f(z)^{t-1}+\cdots+a_1(z)f(z)^{t-p+1}+a_0(z)f(z)^{t-p}\end{aligned}\tag{3.11}$$

when $p<t$. By regarding (3.10) or (3.11) as an algebraic equation in f with coefficients of growth $S(r,f)$ and by Lemma 3 again, we obtain $\alpha=1$ immediately. Moreover, if $p\neq 0$ and $t\neq 0$, we can deduce a contradiction with the assumption that $R(z,f)$ is irreducible in f. Hence, $p=0$ or $t=0$. Then by (3.9), we obtain the assertion (1.5) or (1.6) finally.

Proof of Corollary 1 As we know by Theorem 1 that (1.3) is either of the form (1.5) or (1.6), it is only left to exclude the case (1.6). By multiplying out the denominator of (1.6), we have

$$f(z)^t\prod_{i=1}^{n}f(q_iz)=a_0(z).\tag{3.12}$$

Rewrite (3.12) in the form

$$\prod_{i=1}^{n+t}f(q_iz)=a_0(z),\tag{3.13}$$

where $q_{n+1}=q_{n+2}=\cdots=q_{n+t}=1$, satisfying $\sum\limits_{i=1}^{n+t}q_i^\rho=\sum\limits_{i=1}^{n}q_i^\rho+t\neq 0$. By Lemma 4, the left-hand side of (3.13) is of order ρ. This contradicts with the assumption in Corollary 1 that $\rho(a_0)<\rho$. So, (1.6) cannot occur, that is, (1.3) is of the form (1.5) only.

4 Proofs of Theorems 2–4

Proof of Theorem 2 Denoting $g=\frac{f'}{f}$, we take logarithmic derivative on both sides of the two forms in (1.7) and have

$$-pg(z)+\sum_{i=1}^{n}q^ig(q^iz)=\frac{a_p'(z)}{a_p(z)},\quad\text{or}\quad tg(z)+\sum_{i=1}^{n}q^ig(q^iz)=\frac{a_0'(z)}{a_0(z)}.$$

By Lemma 5 and the assumption that $\rho(a_p)<\rho(f)$ or $\rho(a_0)<\rho(f)$, we have, respectively,

$$\rho(g)\leq\rho\left(\frac{a_p'}{a_p}\right)<\rho(f)\quad\text{or}\quad\rho(g)\leq\rho\left(\frac{a_0'}{a_0}\right)<\rho(f).$$

Thus,

$$\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}=\lambda\left(\frac{f}{f'}\right)=\lambda\left(\frac{1}{g}\right)\leq\rho(g)<\rho(f).$$

Proof of Theorem 3 (1) Assume first that $f(z)=H(z)e^{P(z)}$, where $H(z)$ is a rational function and $P(z)$ a polynomial. Denoting

$$g(z)=\frac{f'(z)}{f(z)}=\frac{H'(z)}{H(z)}+P'(z),$$

we have by (1.8)

$$tg(z)+\sum_{i=1}^{n}q^ig(q^iz)=\frac{a'(z)}{a(z)}. \tag{4.1}$$

Clearly, $g(z)$ is a rational function. Thus, we see by (4.1) that $\frac{a'(z)}{a(z)}$ must also be a rational function. Because $\frac{a'(z)}{a(z)}$ has at most finitely many poles, $a(z)$ has at most finitely many zeros and poles, then, $a(z)$ must be of the form $a(z)=S(z)e^{Q(z)}$, where $S(z)$ is a rational function and $Q(z)$ an entire function. Consequently, the fact that $\frac{a'(z)}{a(z)}=\frac{S'(z)}{S(z)}+Q'(z)$ is a rational function means that $Q(z)$ is a polynomial.

Suppose next that $a(z)=S(z)e^{Q(z)}$, where $S(z)$ is a rational function and $Q(z)$ a polynomial. Because $f(z)$ has only finitely many poles, we have

$$N\left(r,\frac{1}{f(z)}\right)\le N\left(r,\frac{1}{f(z)^t}\right)=N\left(r,\frac{\prod\limits_{i=1}^{n}f(q^iz)}{a(z)}\right)=O(\log r).$$

That is, $f(z)$ has only finitely many zeros and poles. Thus, there exist a rational function $H(z)$ and an entire function $P(z)$, such that $f(z)=H(z)e^{P(z)}$. Substituting $f(z)=H(z)e^{P(z)}$ and $a(z)=S(z)e^{Q(z)}$ into (1.8), we have

$$H(z)^tH(qz)\cdots H(q^nz)e^{tP(z)+P(qz)+\cdots+P(q^nz)}=S(z)e^{Q(z)}.$$

Consequently, we have

$$tP(z)+P(qz)+\cdots+P(q^nz)\equiv Q(z)+C, \tag{4.2}$$

where C is some complex constant. Otherwise by Lemma 6, we may deduce a contradiction that $S(z)\equiv H(z)\equiv 0$. Because $Q(z)$ is a polynomial, by (4.2) and Lemma 7 we see that $P(z)$ must be rational. Noting that $P(z)$ is an entire function, we deduce that $P(z)$ is a polynomial in the final.

(2) If $H(z)$ is a constant, then $g(z)$ has no poles. Thus, by (4.1), $\frac{a'(z)}{a(z)}$ cannot have any poles, that is, $S(z)$ must be a constant.

Conversely, if $S(z)$ is a constant, then $\frac{a'(z)}{a(z)}$ is a polynomial. By (4.1) and Lemma 7, $g(z)$ is rational and has at most a pole at $z=0$. If $g(z)$ has a pole at $z=0$, then $f(z)$ must have a zero (or a pole) at $z=0$. Clearly, the left-hand side of (1.8) has a zero (or a pole) at $z=0$, while the right-hand side of (1.8) has a pole (or a zero) at $z=0$ because $a(z)$ has neither a zero nor a pole at $z=0$. This is a contradiction. Therefore, $g(z)$ must be a polynomial. Consequently, $H(z)$ must be a constant.

Proof of Theorem 4 Denote

$$Q\left(z,\prod_{i=1}^{n}f(q^iz)\right)=b_m(z)\left(\prod_{i=1}^{n}f(q^iz)\right)^m+\cdots+b_1(z)\left(\prod_{i=1}^{n}f(q^iz)\right)+b_0(z), \tag{4.3}$$

where $b_m(z) \not\equiv 0$. Because $b_i(z), i = 0, 1, \cdots, m$, and $a(z)$ are all rational functions, and $f(z)$ has only finitely many poles, by (1.9) we have

$$N\left(r, \frac{1}{f(z)}\right) \le N\left(r, \frac{1}{f(z)^t}\right) = N\left(r, \frac{Q\left(z, \prod_{i=1}^{n} f(q^i z)\right)}{a(z)}\right) = O(\log r).$$

That is, f has only finitely many zeros and poles. Thus, there exist a rational function $r(z)(\not\equiv 0)$ and an entire function $g(z)$, such that

$$f(z) = r(z)e^{g(z)}. \tag{4.4}$$

If $g(z)$ is a constant, then $f(z)$ is a rational function. If $g(z)$ is not a constant, then we assert that $g(z)$ is transcendental. Assume contrary to the assertion that $g(z)$ is a polynomial of degree d (> 0). Then, for $i = 1, 2, \cdots, n$, we may write

$$g(q^i z) = q^{id} g(z) + g_i(z), \tag{4.5}$$

where $g_i(z)$, $i = 1, 2, \cdots, n$, are polynomials of degrees less than d. Substituting (4.3)–(4.5) into (1.9), we have

$$\sum_{i=0}^{m} \Big(b_i(z)r(z)^t \big(r(qz)\cdots r(q^n z)\big)^i e^{i(g_1(z)+\cdots+g_n(z))}\Big) e^{(i(q^d+\cdots+q^{nd})+t)g(z)} = a(z)e^0. \tag{4.6}$$

Observing that $e^{g_i(z)}$, $i = 1, 2, \cdots, n$, are small relative to $e^{g(z)}$ and that $j \sum_{i=1}^{n} q^{id} + t \neq 0, j = 0, \cdots, m$, $\sum_{i=1}^{n} q^{id} \neq 0$, we may use Lemma 6 to (4.6) and deduce that $a(z) \equiv 0$, which is a contradiction. Thus, g is a transcendental entire function and f is of infinite regular order.

References

[1] Hayman W K. Meromorphic Functions. Oxford: Clarendon Press, 1964

[2] Laine I. Nevanlinna Theory and Complex Differential Equations. Berlin: Walter de Gruyter, 1993

[3] Yang C C, Yi H X. Uniqueness Theory of Meromorphic Functions. Dordrecht: Kluwer Academic Publishers Group, 2003

[4] Yang L. Value Distribution Theory. Berlin: Springer-Verlag, 1993

[5] Barnett D, et al. Nevanlinna Theory for the q-Difference Equations. Proc Roy Soc Edinburgh Sect A, 2007, **137**: 457–474

[6] Bergweiler W, Langley J K. Zeros of Differences of Meromorphic Functions. Math Proc Cambridge Phil Soc, 2007, **142**: 133–147

[7] Chen Z X, Shon K H. On Zeros and Fixed Points of Differences of Meromorphic Functions[J]. J Math Anal Appl, 2008, **344**(1): 373–383

[8] Chen Z X, Huang Z B, Zheng X M. On Properties of Difference Polynomials. Acta Mathematica Scientia, 2011, **31B**(2): 627–633

[9] Halburd R G, Korhonen R J. Nevanlinna Theory for the Difference Operator. Ann Acad Sci Fenn Math, 2006, **31**: 463–478

[10] Laine I, Yang C C. Clunie Theorems for Difference and q-Difference Polynomials. J London Math Soc, 2007, **76**: 556–566

[11] Ablowitz M J, Halburd R G, Herbst B. On the Extension of the Painlevé Property to Difference Equations. Nonlinearity, 2000, **13**: 889–905

[12] Bergweiler W, Ishizaki K, Yanagihara N. Meromorphic Solutions of Some Functonal Equations. Methods Appl Anal, 1998, **5**(3): 248–258 (Correction: Methods Appl Anal, 1999, **6**(4): 617–618)

[13] Chen Z X, Huang Z B, Zhang R R. On Difference Equations Concerning Gamma Functions. Acta Math Scientia, 2011, **31B**(4): 1281–1294

[14] Gundersen G, et al. Meromorphic Solutions of Generalized Schröder Equations. Aequationes Math, 2002, **63**: 110–135

[15] Halburd R G, Korhonen R J. Existence of Finite-Order Meromorphic Solutions as a Detector of Integrability in Difference Equations. Phys D, 2006, **218**: 191–203

[16] Heittokangas J, et al. Complex Difference Equations of Malmquist Type. Comput Methods Funct Theory, 2001, **1**(1): 27–39

[17] Heittokangas J, et al. Meromorphic Solutions of Some Linear Functional Equations. Aequationes Math, 2000, **60**: 148–166

[18] Ishizaki K. Hypertranscendency of Meromorphic Solutions of a Linear Functional Equation. Aequationes Math, 1998, **56**: 271–283

[19] Laine I, Rieppo J, Silvennoinen H. Remarks on Complex Difference Equations. Comput Methods Funct Theory, 2005, **5**(1): 77–88

[20] Zheng X M, Chen Z X. Growth of Meromorphic Solutions of Some Difference Equations. Appl Anal Discrete Math, 2010, **4**(2): 309–321

[21] Zheng X M, Chen Z X. Some Properties of Meromorphic Solutions of q-Difference Equations. J Math Anal Appl, 2010, **361**(2): 472–480