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Numerical solution of delay systems containing inverse time by hybrid functions

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Abstract

The hybrid functions consisting of general block-pulse functions and Legendre polynomials are presented to solve delay systems containing inverse time. The direct algorithm for a product of a matrix function and a vector function is given. The general operational matrix is introduced. The delay function and inverse time function are expanded by hybrid functions. The approximate solution of delay systems containing inverse time is derived. Numerical examples illustrate that the algorithms are applicable.

Keywords: Delay systems containing inverse time; Block-pulse functions; Legendre polynomials

1. Introduction

Delay systems containing inverse time are an important class of systems. Delay systems in [1] can be seen as the special cases. Many orthogonal

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functions, such as block-pulse functions [2,3], Walsh [4], Fourier [5], Legendre [6], Chebyshev [7] and Laguerre [8], were also used to derive solutions of some systems. Recently the different kinds of hybrid functions are developing [9–11]. In this article we apply the hybrid functions consisting of general block-pulse functions and Legendre polynomials to solve delay systems containing inverse time. We improve the formula in [1] and present the general formula for expression of a product of a matrix function and a vector function by hybrid functions. We give the general operational matrix and the formula of delay functions and inverse time functions by hybrid functions. Then using the results, the solutions of the linear boundary value problem are evaluated. Numerical examples demonstrate that the algorithms presented are valid.

2. Preliminaries

2.1. Definitions

A set of block-pulse function $\pi_n(t)$, $n = 1,2,\ldots,N$ on the interval $[0, t_f)$ are defined as

$$\pi_n(t) = \begin{cases} 1, & t_{n-1} \leqslant t < t_n, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $t_0 = 0$, $t_N = t_f$ and $[t_{n-1}, t_n) \subset [0, t_f)$, $n = 1,2,\ldots,N$.

The Legendre polynomials $L_m(t)$ on the interval $[-1, 1]$ are given by the following recursive formula

$$\begin{cases} L_0(t) = 1, \ L_1(t) = t, \\ (m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t), \quad m = 1,2,\ldots \end{cases} \tag{2}$$

The hybrid functions $b_{nm}(t)$, $n = 1,2,\ldots,N$; $m = 0,1,\ldots,M-1$ on the interval $[0, t_f)$ are defined as

$$b_{nm}(t) = \pi_n(t)L_m(d_n^{-1}(2t - t_{n-1} - t_n)), \tag{3}$$

so

$$b_{nm}(t) = \begin{cases} L_m(d_n^{-1}(2t - t_{n-1} - t_n)), & t_{n-1} \leqslant t < t_n, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

where $d_n = t_n - t_{n-1}$, $n = 1,2,\ldots,N$.

2.2. Function approximation

A vector or matrix function $F(t)$ on the interval $[0, t_f)$ is expressed as

$$F(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} F_{nm} b_{nm}(t), \tag{5}$$

where

$$F_{nm} = \frac{2m+1}{d_n} \int_{t_{n-1}}^{t_n} F(t) L_m(d_n^{-1}(2t - t_{n-1} - t_n))\,\mathrm{d}t. \tag{6}$$

If $F(t)$ is a vector function, then

$$F(t) \simeq \sum_{n=1}^{N} F_n B_n(t) = FB(t), \tag{7}$$

where

$$F_n = [F_{n0}, \ldots, F_{n,M-1}], \quad F = [F_1, \ldots, F_N], \tag{8}$$

$$B_n(t) = [b_{n0}(t), \ldots, b_{n,M-1}(t)]^{\tau}, \quad B(t) = [B_1^{\tau}(t), \ldots, B_N^{\tau}(t)]^{\tau}. \tag{9}$$

For corresponding F_n and F we denote

$$\hat{F}_n = [F_{n0}^{\tau}, \ldots, F_{n,M-1}^{\tau}]^{\tau}, \quad \hat{F} = [\hat{F}_1^{\tau}, \ldots, \hat{F}_N^{\tau}]^{\tau}, \tag{10}$$

where τ is the transpose.

2.3. Expression of the delay function

The delay function $x(t-\alpha)$ on the interval $[0, t_f)$ is expressed as

$$x(t-\alpha) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} \bar{x}_{nm} b_{nm}(t) = \bar{x} B(t), \tag{11}$$

where $t_{K+n-1} - t_{n-1} = \alpha$, $n = 1, 2, \ldots, N-K$,

$$\bar{x}_{nm} = \frac{2m+1}{d_n} \int_{t_{n-1}}^{t_n} x(t-\alpha) L_m(d_n^{-1}(2t - t_{n-1} - t_n))\,\mathrm{d}t. \tag{12}$$

So

$$\bar{x}_{nm} = \frac{2m+1}{d_n} \int_{t_{n-1}-\alpha}^{t_n-\alpha} x(t) L_m(d_n^{-1}(2t + 2\alpha - t_{n-1} - t_n))\,\mathrm{d}t, \quad n = 1, \ldots, K. \tag{13}$$

Also we may verify that $d_n = d_{n-K}$, $n = K+1,\ldots,N$. Furthermore

$$\int_{t_{n-1}}^{t_n} x(t-\alpha)L_m(d_n^{-1}(2t-t_{n-1}-t_n))\,\mathrm{d}'t$$
$$= \int_{t_{n-1}-\alpha}^{t_n-\alpha} x(t)L_m(d_n^{-1}(2t+2\alpha-t_{n-1}-t_n))\,\mathrm{d}t$$
$$= \int_{t_{n-K-1}}^{t_{n-K}} x(t)L_m(d_{n-K}^{-1}(2t-t_{n-K}-t_{n-K-1}))\,\mathrm{d}t,$$

we have

$$\bar{x}_{nm} = x_{n-K,m}, \quad n = K+1,\ldots,N. \tag{14}$$

Then

$$\hat{\bar{X}}_n = [\bar{x}_{n0}^{\tau},\ldots,\bar{x}_{n,M-1}^{\tau}]^{\tau}, \tag{15}$$

therefore

$$\hat{\bar{X}} = [\hat{\bar{X}}_1^{\tau},\ldots,\hat{\bar{X}}_N^{\tau}]^{\tau} = [\hat{\bar{X}}_1^{\tau},\ldots,\hat{\bar{X}}_K^{\tau},\hat{X}_1^{\tau},\ldots,\hat{X}_{N-K}^{\tau}]^{\tau}$$
$$= \hat{\Phi} + \left(\sum_{i=1}^{N-K} E_{K+i,i}^{(N)} \otimes I_{Mp}\right)\hat{X}, \tag{16}$$

where $E_{ij}^{(m)}$ is the $m\times m$ matrix with 1 at its entry (i,j) and zeros elsewhere, I_k the $k\times k$ identity matrix and $\otimes$ Kronecker product

$$\hat{\Phi} = [\hat{\bar{X}}_1^{\tau},\ldots,\hat{\bar{X}}_K^{\tau},\underbrace{0^{\tau},\ldots,0^{\tau}}_{N-K\text{ terms}}]^{\tau}. \tag{17}$$

2.4. Expression of the inverse time function

The inverse time function $x(t_f - t)$ on the interval $[0,t_f)$ is expressed as

$$x(t_f-t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1} \tilde{x}_{nm}b_{nm}(t) = \tilde{X}B(t), \tag{18}$$

where $t_{n-1} + t_{N-n+1} = t_f$, $n = 1,2,\ldots,N$,

$$\tilde{x}_{nm} = \frac{2m+1}{d_n}\int_{t_{n-1}}^{t_n} x(t_f-t)L_m(d_n^{-1}(2t-t_{n-1}-t_n))\,\mathrm{d}t. \tag{19}$$

It is not difficult to check that $d_n = d_{N-n+1}$, $n = 1,2,\ldots,N$. From

$$\int_{t_{n-1}}^{t_n} x(t_f-t)L_m(d_n^{-1}(2t-t_{n-1}-t_n))\,\mathrm{d}t$$
$$= \int_{t_f-t_{n-1}}^{t_f-t_n} x(t)L_m(d_n^{-1}(2t_f-2t-t_{n-1}-t_n))\,\mathrm{d}t$$
$$= (-1)^m \int_{t_{N-n}}^{t_{N-n+1}} x(t)L_m(d_{N-n+1}^{-1}(2t-t_{N-n}-t_{N-n+1}))\,\mathrm{d}t,$$

we have

$$\tilde{x}_{nm} = (-1)^m x_{N-n+1,m}. \tag{20}$$

Then

$$\begin{aligned}\hat{\tilde{X}}_n &= [\tilde{x}_{n0}^{\tau}, \ldots, \tilde{x}_{n,M-1}^{\tau}]^{\tau} = [x_{N-n+1,0}^{\tau}, \ldots, (-1)^{M-1} x_{N-n+1,M-1}^{\tau}]^{\tau} \\ &= H_{Mp} \hat{X}_{N-n+1},\end{aligned} \tag{21}$$

where

$$H_{Mp} = \mathrm{diag}[1, -1, \ldots, (-1)^{M-1}] \otimes I_p, \tag{22}$$

therefore

$$\hat{\tilde{X}} = [\hat{\tilde{X}}_1^{\tau}, \ldots, \hat{\tilde{X}}_N^{\tau}]^{\tau} = \left(\sum_{k=1}^{N} E_{k,N-k+1}^{(N)} \otimes H_{Mp}\right)\hat{X}. \tag{23}$$

2.5. Expression of the product of a matrix function and a vector function

Let a matrix function $A(t)$ be appropriate to a vector function $x(t)$. We express $A(t)$ and $x(t)$ as

$$A(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} A_{nm} b_{nm}(t), \quad x(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} x_{nm} b_{nm}(t), \tag{24}$$

then

$$A(t)x(t) \simeq \sum_{n=1}^{N} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} A_{ni} x_{nj} b_{ni}(t) b_{nj}(t).$$

From

$$b_{ni}(t) b_{nj}(t) \simeq \sum_{m=0}^{M-1} d_{nm}^{(ij)} b_{nm}(t),$$

where

$$d_{nm}^{(ij)} = \frac{2m+1}{2} \int_{-1}^{1} L_i(t) L_j(t) L_m(t)\,\mathrm{d}t, \tag{25}$$

it can be noted that $d_{nm}^{(ij)} = d_{km}^{(ij)}$. And if the sum of any two number among the three number i, j and k is smaller than the rest number or $i+j+m$ is odd, then $d_{nm}^{(ij)} = 0$. Thus

$$A(t)x(t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1}\left(\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} d_{nm}^{(ij)} A_{ni} x_{nj}\right) b_{nm}(t) = \sum_{n=1}^{N}\sum_{m=0}^{M-1} \tilde{A}_{nm} b_{nm}(t)$$
$$= \sum_{n=1}^{N} \tilde{A}_n B_n(t), \tag{26}$$

where

$$\tilde{A}_{nm} = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} d_{nm}^{(ij)} A_{ni} x_{nj} = \hat{A}_{nm}\hat{X}_n, \quad \hat{X}_n = [x_{n0}^{\tau}, \ldots, x_{n,M-1}^{\tau}]^{\tau}, \tag{27}$$

we also have

$$\hat{\tilde{A}}_n = \hat{A}_n \hat{X}_n, \tag{28}$$

where

$$\hat{\tilde{A}}_n = [\tilde{A}_{n0}^{\tau}, \ldots, \tilde{A}_{n,M-1}^{\tau}]^{\tau}, \quad \hat{A}_n = [\hat{A}_{n0}^{\tau}, \ldots, \hat{A}_{n,M-1}^{\tau}]^{\tau}, \tag{29}$$

$$\hat{A}_{nm} = \left[\sum_{i=0}^{M-1} d_{nm}^{(i1)} A_{ni}, \ldots, \sum_{i=0}^{M-1} d_{nm}^{(i,M-1)} A_{ni}\right]. \tag{30}$$

2.6. The operational matrix

Using Kronecker product we know

$$\int_0^t B(s)\,\mathrm{d}s \simeq PB(t), \tag{31}$$

where

$$P = \operatorname{diag}(d_1, \ldots, d_N) \otimes \frac{1}{2}\left[E_{11}^{(M)} + \sum_{k=1}^{M-1}\left(\frac{1}{2k-1}E_{k,k+1}^{(M)} - \frac{1}{2k+1}E_{k+1,k}^{(M)}\right)\right]$$
$$+ \sum_{k=1}^{N-1}\sum_{i=1}^{N-k} d_i E_{i,i+k}^{(N)} \otimes E_{11}^{(M)}. \tag{32}$$

3. Analysis of systems

Consider the following delay system containing inverse time:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + C(t)x(t-\alpha) + D(t)x(t_f - t) + G(t)u(t), & t \in [0, t_f], \\ x(t_0) = x_0, \quad x(t) = \varphi(t), & t \in [-\alpha, 0], \end{cases} \tag{33}$$

where $x(t)$ is the p-dimensional vector function and $u(t)$ q-dimensional vector function. $A(t)$, $C(t)$, $D(t)$ and $G(t)$ are the matrices of appropriate dimensions. We express $A(t)$, $x(t)$, $C(t)$, $x(t-\alpha)$, $D(t)$, $x(t_f-t)$, $G(t)$ and $u(t)$ respectively. These expressions are similar to Eq. (24). By Eq. (26), we have

$$A(t)x(t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1}\tilde{A}_{nm}b_{nm}(t), \quad C(t)x(t-\alpha) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1}\tilde{C}_{nm}b_{nm}(t), \tag{34}$$

$$D(t)x(t_f-t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1}\tilde{D}_{nm}b_{nm}(t), \quad G(t)u(t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1}\tilde{G}_{nm}b_{nm}(t), \tag{35}$$

then integrating Eq. (33) from 0 to t and combining Eq. (26), (31) and (32) we obtain

$$\begin{aligned}&[x_1,\ldots,x_N]B(t)-[x_{01},\ldots,x_{0N}]B(t)\\ &\quad=\int_0^t\{[\tilde{A}_1,\ldots,\tilde{A}_N]+[\tilde{C}_1,\ldots,\tilde{C}_N]+[\tilde{D}_1,\ldots,\tilde{D}_N]+[\tilde{G}_1,\ldots,\tilde{G}_N]\}B(s)\,\mathrm{d}s,\end{aligned}$$

where

$$x_n=[x_{n0},\ldots,x_{n,M-1}], \quad x_{0n}=[x_0,0,\ldots,0],\ \ n=1,2,\ldots,N.$$

The other signs mean similarly as Eqs. (27)–(30), thus

$$\begin{aligned}[x_1,\ldots,x_N]-[x_{01},\ldots,x_{0N}]=\{&[\tilde{A}_1,\ldots,\tilde{A}_N]+[\tilde{C}_1,\ldots,\tilde{C}_N]+[\tilde{D}_1,\ldots,\tilde{D}_N]\\ &+[\tilde{G}_1,\ldots,\tilde{G}_N]\}P.\end{aligned}$$

Using Kronecker product we rewrite the above equation as

$$\begin{aligned}\hat{X}-X_0=(P^\tau\otimes I_p)\{&[\hat{\tilde{A}}{}_1^\tau,\ldots,\hat{\tilde{A}}{}_N^\tau]^\tau+[\hat{\tilde{C}}{}_1^\tau,\ldots,\hat{\tilde{C}}{}_N^\tau]^\tau+[\hat{\tilde{D}}{}_1^\tau,\ldots,\hat{\tilde{D}}{}_N^\tau]\\ &+[\hat{\tilde{G}}{}_1^\tau,\ldots,\hat{\tilde{G}}{}_N^\tau]\},\end{aligned}$$

where

$$\hat{X}=[\hat{X}_1^\tau,\ldots,\hat{X}_N^\tau]^\tau, \quad \hat{X}_n=[x_{n0}^\tau,\ldots,x_{n,M-1}^\tau]^\tau, \tag{36}$$

$$\hat{\tilde{A}}_n=[\tilde{A}_{n0}^\tau,\ldots,\tilde{A}_{n,M-1}^\tau]^\tau, \quad X_0=[x_{01}^\tau,\ldots,x_{0N}^\tau]^\tau, \tag{37}$$

$\hat{\tilde{C}}_n, \hat{\tilde{D}}_n$ and $\hat{\tilde{G}}_n$ have the similar meaning as $\hat{\tilde{A}}_n$. So

$$\begin{aligned}\hat{X}-X_0&=(P^\tau\otimes I_p)\{[(\hat{A}_1\hat{X}_1)^\tau,\ldots,(\hat{A}_N\hat{X}_N)^\tau]^\tau+[(\hat{C}_1\hat{\bar{X}}_1)^\tau,\ldots,(\hat{C}_N\hat{\bar{X}}_N)^\tau]^\tau\\ &\qquad+[(\hat{D}_1\hat{\tilde{X}}_1)^\tau,\ldots,(\hat{D}_N\hat{\tilde{X}}_N)^\tau]^\tau+[(\hat{G}_1\hat{U}_1)^\tau,\ldots,(\hat{G}_N\hat{U}_N)^\tau]^\tau\}\\ &=(P^\tau\otimes I_p)\left[\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{A}_n)\hat{X}+\sum_{n=1}^{N}(E_{nn}^{(NN)}\otimes\hat{C}_n)\hat{\bar{X}}\right.\end{aligned}$$

$$+\sum_{n=1}^{N}(E_{nn}^{(NN)}\otimes\hat{D}_n)\hat{\hat{X}}+\sum_{n=1}^{N}(E_{nn}^{(NN)}\otimes\hat{G}_n)\hat{U}\Bigg]$$

$$=(P^{\tau}\otimes I_p)\Bigg\{\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{A}_n)\hat{X}+\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{C}_n)$$

$$\times\left[\hat{\Phi}+\left(\sum_{i=1}^{N-K}E_{K+i,i}^{(N)}\otimes I_{Mp}\right)\hat{X}\right]+\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{D}_n)$$

$$\times\left(\sum_{k=1}^{N}E_{k,N-k+1}^{(N)}\otimes H_{Mp}\right)\hat{X}+\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{G}_n)\hat{U}\Bigg\},$$

therefore

$$[I_{MNp}-(P^{\tau}\otimes I_p)\Gamma]\hat{X}=(P^{\tau}\otimes I_p)\Omega+X_0, \tag{38}$$

where

$$\Gamma=\sum_{n=1}^{N}(E_{nn}^{(N)}\otimes\hat{A}_n+E_{n,N-n+1}^{(N)}\otimes\hat{D}_nH_{Mp})+\sum_{i=1}^{N-K}E_{K+i,i}^{(N)}\otimes\hat{C}_{K+i}, \tag{39}$$

$$\Omega=\sum_{n=1}^{N}[(E_{nn}^{(N)}\otimes\hat{C}_n)\hat{\Phi}+(E_{nn}^{(N)}\otimes\hat{G}_n)\hat{U}]. \tag{40}$$

4. Application to boundary value problem

Consider the following boundary value problem:

$$\begin{cases}\dot{y}(t)=R(t)y(t)+Q(t)y(t-\alpha)+S(t)z(t)+a(t),\\ \dot{z}(t)=V(t)y(t)+K(t)z(t+\alpha)+W(t)z(t)+b(t),\\ y(0)=y_0,\quad y(t)=\phi(t),\ \ t\in[-\alpha,0),\\ z(t_f)=z_0,\quad z(t)=\psi(t),\ \ t\in(t_f,t_f+\alpha],\end{cases} \tag{41}$$

where $y(t)$, $z(t)$, $a(t)$ and $b(t)$ are the p-dimensional vectors. $R(t)$, $Q(t)$, $S(t)$, $V(t)$, $K(t)$ and $W(t)$ are the matrices of appropriate dimensions. Let

$$x(t)=[y^{\tau}(t),z^{\tau}(t_f-t)]^{\tau}, \tag{42}$$

then

$$\dot{x}(t)=[\dot{y}^{\tau}(t),-\dot{z}^{\tau}(t_f-t)]^{\tau}$$

satisfies that

$$\begin{cases}\dot{x}(t)=A(t)x(t)+C(t)x(t-\alpha)+D(t)x(t_f-t)+u(t),\\ x(0)=x_0=[y_0^{\tau},z_0^{\tau}]^{\tau},\quad t\in[0,t_f],\end{cases} \tag{43}$$

where

$$A(t) = E_{11}^{(2)} \otimes R(t) - E_{22}^{(2)} \otimes W(t_f - t),$$

$$C(t) = E_{11}^{(2)} \otimes Q(t) - E_{22}^{(2)} \otimes K(t_f - t),$$

$$D(t) = E_{12}^{(2)} \otimes S(t) - E_{21}^{(2)} \otimes V(t_f - t),$$

$$u(t) = [a^\tau(t), -b^\tau(t_f - t)]^\tau.$$

By applying Eq. (38) we have

$$[I_{MN2p} - (P^\tau \otimes I_{2p})\Gamma]\hat{X} = (P^\tau \otimes I_{2p})\left[\sum_{n=1}^{N}(E_{nn}^{(N)} \otimes \hat{C}_n)\hat{\Phi} + \hat{U}\right] + X_0. \tag{44}$$

From Eqs. (42) and (21) we obtain

$$\begin{aligned}\hat{X}_n &= [x_{n0}^\tau, \ldots, x_{n,M-1}^\tau]^\tau = [y_{n0}^\tau, \tilde{z}_{n0}^\tau, \ldots, y_{n,M-1}^\tau, \tilde{z}_{n,M-1}^\tau]^\tau \\ &= [y_{n0}^\tau, z_{N-n+1,0}^\tau, y_{n1}^\tau, -z_{N-n+1,1}^\tau, \ldots, y_{n,M-1}^\tau, (-1)^{M-1} z_{N-n+1,M-1}^\tau]^\tau \\ &= J[\hat{Y}_n^\tau, \hat{Z}_{N-n+1}^\tau]^\tau,\end{aligned}$$

where

$$\hat{Y}_n = [y_{n0}^\tau, \ldots, y_{n,M-1}^\tau]^\tau, \quad \hat{Z}_{N-n+1} = [z_{N-n+1,0}^\tau, \ldots, z_{N-n+1,M-1}^\tau]^\tau,$$

$$J = \sum_{k=1}^{M}[E_{2k-1,k}^{(2M)} + (-1)^{k-1}E_{2k,M+k}^{(2M)}] \otimes I_p, \tag{45}$$

then

$$\begin{aligned}\hat{X} &= [\hat{X}_1^\tau, \ldots, \hat{X}_N^\tau]^\tau = (I_N \otimes J)[\hat{Y}_1^\tau, \hat{Z}_N^\tau, \ldots, \hat{Y}_n^\tau, \hat{Z}_{N-n+1}^\tau, \ldots, \hat{Y}_N^\tau, \hat{Z}_1^\tau]^\tau, \\ &= T[\hat{Y}^\tau, \hat{Z}^\tau]^\tau,\end{aligned}$$

where

$$T = (I_N \otimes J)\left\{\sum_{k=1}^{N}[E_{2k-1,k}^{(2N)} + E_{2k,2N-k+1}^{(2N)}] \otimes I_{Mp}\right\}, \tag{46}$$

$$\hat{Y} = [\hat{Y}_1^\tau, \ldots, \hat{Y}_N^\tau]^\tau, \quad \hat{Z} = [\hat{Z}_1^\tau, \ldots, \hat{Z}_N^\tau]^\tau, \tag{47}$$

therefore

$$\begin{aligned}&[I_{MN2p} - (P^\tau \otimes I_{2p})\Gamma]T[\hat{Y}^\tau, \hat{Z}^\tau]^\tau \\ &\quad = (P^\tau \otimes I_{2p})\left[\sum_{n=1}^{N}(E_{nn}^{(N)} \otimes \hat{C}_n)\hat{\Phi} + \hat{U}\right] + X_0.\end{aligned} \tag{48}$$

5. Numerical examples

5.1. Example 1

Consider the delay system [1] described by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -25 & -5t \end{bmatrix} x\left(t - \frac{1}{4}\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \in \left[-\frac{1}{4}, 0\right].$$

Let $u(t) = 1$. Then by Eq. (38) and choosing $N = 4$, $M = 4$ and $t_n = \frac{n}{N}$, $n = 0,1,\ldots,N$ we have the approximate hybrid solution $x(t) = [x_1(t), x_2(t)]^\tau$:

$$x_1(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \frac{1}{4}; \\ \frac{1}{2}t^2 - \frac{1}{4}t + \frac{1}{32}, & \frac{1}{4} \leqslant t \leqslant \frac{1}{2}; \\ -\frac{5}{12}t^4 + \frac{5}{8}t^3 + \frac{3}{16}t^2 - \frac{19}{96}t + \frac{1}{32}, & \frac{1}{2} \leqslant t \leqslant \frac{3}{4}; \\ \frac{5}{18}t^6 - \frac{85}{96}t^5 - \frac{45}{128}t^4 + \frac{785}{256}t^3 - \frac{1147}{369}t^2 + \frac{674}{443}t - \frac{677}{2301}, & \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

$$x_2(t) = \begin{cases} t, & 0 \leqslant t \leqslant \frac{1}{4}; \\ -\frac{5}{3}t^3 + \frac{5}{8}t^2 + t - \frac{5}{384}, & \frac{1}{4} \leqslant t \leqslant \frac{1}{2}; \\ \frac{5}{3}t^5 - \frac{75}{32}t^4 - \frac{115}{24}t^3 + \frac{1295}{192}t^2 - \frac{17}{8}t + \frac{775}{1536}, & \frac{1}{2} \leqslant t \leqslant \frac{3}{4}; \\ -\frac{25}{21}t^7 + \frac{2125}{576}t^6 + \frac{335}{96}t^5 - \frac{1906}{106}t^4 + \frac{5869}{419}t^3 - \frac{1991}{1022}t^2 - \frac{1051}{1024}t + \frac{355}{533}, & \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

It can be checked that the hybrid solutions are equal to the exact solutions.

5.2. Example 2

Consider the delay system containing inverse time described by

$$\dot{x}(t) = \begin{bmatrix} t^2 + 1 & -t^2 \\ 0 & -9 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ 9 & 0 \end{bmatrix} x\left(t - \frac{1}{3}\right) + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x(1 - t)$$
$$+ \begin{bmatrix} 4t + 3 \\ 8t + 15 \end{bmatrix} u(t), \quad \varphi(t) = \begin{bmatrix} t^2 - 1 \\ t^2 + 1 \end{bmatrix}, \quad -\frac{1}{3} \leqslant t \leqslant 0.$$

Let $u(t) = 1$. Then by Eq. (38) and choosing $N = 3$, $t_n = \frac{n}{N}$, $n = 0, 1, \ldots, N$ and $M = 4$ we have the approximate hybrid solution $x(t) = [x_1(t), x_2(t)]^\tau$:

$$\begin{aligned} x_1(t) = {} & -\frac{26}{27}b_{10}(t) + \frac{1}{18}b_{11}(t) + \frac{1}{54}b_{12}(t) - \frac{20}{27}b_{20}(t) + \frac{1}{6}b_{21}(t) \\ & + \frac{1}{54}b_{22}(t) - \frac{8}{27}b_{30}(t) + \frac{5}{18}b_{31}(t) + \frac{1}{54}b_{32}(t), \\ x_2(t) = {} & \frac{28}{27}b_{10}(t) + \frac{1}{18}b_{11}(t) + \frac{1}{54}b_{12}(t) + \frac{34}{27}b_{20}(t) + \frac{1}{6}b_{21}(t) \\ & + \frac{1}{54}b_{22}(t) + \frac{46}{27}b_{30}(t) + \frac{5}{18}b_{31}(t) + \frac{1}{54}b_{32}(t). \end{aligned}$$

It can be checked that the hybrid solutions are equal to the exact solutions.

5.3. Example 3

Consider the boundary value problem described by

$$\begin{cases} \dot{y}(t) = 16ty(t-\frac{1}{4}) - 16z(t) + 8t^2 + 17t + 16, \\ \dot{z}(t) = 64ty(t) - 64z(t+\frac{1}{4}) + 51t^2 + 76t + 65, \\ y(t) = t^2 - 1, \quad -\frac{1}{4} \leqslant t \leqslant 0, \\ z(t) = t^3 + 1, \quad 1 \leqslant t \leqslant \frac{5}{4}. \end{cases}$$

From Eq. (43) we have

$$\dot{x}(t) = \begin{bmatrix} 16t & 0 \\ 0 & 64 \end{bmatrix} x\left(t - \frac{1}{4}\right) + \begin{bmatrix} 0 & -16 \\ 64t - 64 & 0 \end{bmatrix} x(1-t)$$
$$+ \begin{bmatrix} 8t^2 + 17t + 16 \\ -51t^2 + 178t - 62 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$\varphi(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 + 3t^2 - 3t + 2 \end{bmatrix}, \quad -\frac{1}{4} \leqslant t \leqslant 0.$$

where $x(t) = [y(t), z(1-t)]^{\tau}$. Let $M = 4$, $N = 4$, $t_n = \frac{n}{N}$, $n = 0,1,\ldots,N$ and $K = 1$. By Eq. (48) we have the hybrid expansions of $y(t)$ and $z(t)$:

$$y(t) = -\frac{47}{48}b_{10}(t) + \frac{1}{32}b_{11}(t) + \frac{1}{96}b_{12}(t) - \frac{41}{48}b_{20}(t) + \frac{3}{32}b_{21}(t) + \frac{1}{96}b_{22}(t)$$
$$- \frac{29}{48}b_{30}(t) + \frac{5}{32}b_{31}(t) + \frac{1}{96}b_{32}(t) - \frac{11}{48}b_{40}(t) + \frac{7}{32}b_{41}(t) + \frac{1}{96}b_{42}(t),$$
$$z(t) = \frac{257}{256}b_{10}(t) + \frac{9}{1280}b_{11}(t) + \frac{1}{256}b_{12}(t) + \frac{1}{1280}b_{13}(t)$$
$$+ \frac{271}{256}b_{20}(t) + \frac{69}{1280}b_{21}(t) + \frac{3}{256}b_{22}(t) + \frac{1}{1280}b_{23}(t)$$
$$+ \frac{321}{256}b_{30}(t) + \frac{189}{1280}b_{31}(t) + \frac{5}{256}b_{32}(t) + \frac{1}{1280}b_{33}(t)$$
$$+ \frac{431}{256}b_{40}(t) + \frac{369}{1280}b_{41}(t) + \frac{7}{256}b_{42}(t) + \frac{1}{1280}b_{43}(t).$$

It can be checked that the hybrid solutions are equal to the exact solutions.

6. Conclusion

Using the hybrid function of general block-pulse functions and Legendre polynomials we provide the general algorithms for the delay systems containing inverse time function and reduce it into a set of algebraic equations. It is also shown that the results can be applied to the boundary value problem. The illustrative examples demonstrate that if the solutions are the polynomials then the approximate hybrid solutions may be exact.

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