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EXTENDED CESÀRO OPERATORS AND MULTIPLIERS ON BMOA*

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Abstract In this article, we characterize the boundedness and compactness of extended Cesàro operators on the spaces BMOA by the Carleson measures in the unit ball. Meanwhile, we study the pointwise multipliers on BMOA.

Key words Extended Cesàro operator; multiplier; Carleson measure; BMOA.

2000 MR Subject Classification 47B38; 32A37

1 Introduction

Let $\mathbf{B}$ and $\partial\mathbf{B}$ be the open unit ball and the unit sphere in $\mathbf{C}^n$, and let v and σ be the normalized Lebesgue measure on $\mathbf{B}$ and $\partial\mathbf{B}$, respectively. Let $H(\mathbf{B})$ be the family of all holomorphic functions on $\mathbf{B}$. For $0 < p < \infty$, the Hardy space H^p consists of the functions $f \in H(\mathbf{B})$ with

$$\|f\|_p = \sup_{0\le r<1} \left(\int_{\partial\mathbf{B}} |f(r\zeta)|^p \mathrm{d}\sigma(\zeta)\right)^{\frac{1}{p}} < \infty.$$

Denote

$$H^\infty = \left\{ f \in H(\mathbf{B}) : \|f\|_\infty = \sup_{z\in\mathbf{B}} |f(z)| < \infty \right\}.$$

The space BMOA is the set of all $f \in H(\mathbf{B})$ satisfying

$$\|f\|_* = \sup_{a\in\mathbf{B}} \|f \circ \varphi_a - f(a)\|_2 < \infty,$$

where φ_a is the Möbius transformation of $\mathbf{B}$ [1]. It is well known that BMOA is a Banach space under the norm $\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*$. Ouyang [2] proved

$$\|f\|_* \simeq \sup_{a\in\mathbf{B}} \|f \circ \varphi_a - f(a)\|_p$$

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for any $0 < p < \infty$. Here and in the sequel, the expression $A(f) \simeq B(f)$ means that there exists C such that, for all $f \in H(\mathbf{B})$, $C^{-1}A(f) \le B(f) \le CA(f)$, where C stands for a finite positive constant whose value may change from one to another but independent of f. Moreover, Zhu [1] showed that

$$\|f \circ \varphi_a - f(a)\|_2 \simeq \left\{ \int_{\mathbf{B}} P(a,z)|\Re f(z)|^2(1-|z|^2)\mathrm{d}v(z) \right\}^{\frac{1}{2}}, \tag{1.1}$$

where $P(a,z) = \frac{(1-|a|^2)^n}{|1-\langle a,z\rangle|^{2n}}$, $\Re f$ is the radial derivative of f. The space $\mathrm{BMOA}_{\log}$ is the family of all $f \in H(\mathbf{B})$ satisfying

$$\|f\|_{\mathrm{BMOA}_{\log}} = \sup_{a\in\mathbf{B}} \log \frac{2}{1-|a|^2} \|f \circ \varphi_a - f(a)\|_2 < \infty.$$

It is checked that $\mathrm{BMOA}_{\log}$ is a subset of BMOA.

For $g \in H(\mathbf{B})$, the extended Cesàro operator T_g on $H(\mathbf{B})$ is defined by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{\mathrm{d}t}{t}, \quad z \in \mathbf{B}.$$

This operator was discussed by many authors. Hu [3] studied the boundedness and compactness of T_g on the Bergman space. Xiao considered the operator T_g on Bergman spaces and Bloch spaces in [4]. Tang [5] discussed the same problems of T_g on Bloch-type spaces. Fang and Zhou studied T_g on Zygmund spaces in [6].

Let $g \in H(\mathbf{B})$, and let X, Y be two spaces of holomorphic functions on $\mathbf{B}$. Then, g is called a pointwise multiplier from X to Y if $M_g f = gf \in Y$ for every $f \in X$, written as $g \in M(X,Y)$. Denote $g \in M(X)$ simply if $g \in M(X,X)$. Many articles studied there operators on several holomorphic function spaces, including Hardy spaces, Bergman spaces, Dirichlet-type spaces, Bloch-type spaces, $\mathcal{Q}_p$ spaces, VMO_p, and VO spaces [7–11].

Our purpose is to obtain the sufficient and necessary conditions on g for the operator T_g to be bounded or compact on the spaces BMOA by the Carleson measure. Meanwhile, we obtain the properties of pointwise multipliers on BMOA. In Section 2, we discuss the Carleson measure, which not only is used in the proof of main results, but also has its own interest. In Section 3, we study the extended Cesàro operators and pointwise multipliers on BMOA.

2 (Vanishing) p-Logarithmic Carleson Measure

First, we introduce some notations. For $\xi \in \partial\mathbf{B}$ and $t > 0$, we define the Carleson set

$$S(\xi,t) = \{z \in \mathbf{B} : |1 - \langle z,\xi\rangle| < t\}.$$

Given $p \ge 0$, a finite positive Borel measure μ (simply written as $\mu \ge 0$) on $\mathbf{B}$ is called a p-logarithmic Carleson measure if

$$\sup_{\xi\in\partial\mathbf{B},t>0} \frac{\mu(S(\xi,t))\left(\log\frac{2}{t}\right)^p}{t^n} < \infty.$$

If $p=0$, μ is called the classical Carleson measure, and we denote $\|\mu\| = \sup\limits_{\xi\in\partial\mathbf{B},t>0} \frac{\mu(S(\xi,t))}{t^n}$. Moreover, if

$$\lim_{t\to 0}\sup_{\xi\in\partial\mathbf{B}} \frac{\mu(S(\xi,t))\left(\log\frac{2}{t}\right)^p}{t^n}=0,$$

then μ is said to be a vanishing p-logarithmic Carleson measure.

On the unit disk, the p-logarithmic Carleson measure was first introduced in [12], and the vanishing p-logarithmic Carleson measure was studied in [13]. In this section, we exhibit some properties of (vanishing) p-logarithmic Carleson measure, which are useful for the main results.

Lemma 2.1 [1] Let $\mu\geq 0$ on $\mathbf{B}$, $0<p<\infty$. Then, the following statements are equivalent:

(1) The formal identity $i: H^p\to L^p(\mu)$ is bounded;

(2) μ is the classical Carleson measure;

(3) $\sup\limits_{a\in\mathbf{B}}\int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z)<\infty$.

Moreover,

$$\|i\|^p_{H^p\to L^p(\mu)}\simeq\|\mu\|\simeq\sup_{a\in\mathbf{B}}\int_{\mathbf{B}}P(a,z)\mathrm{d}\mu(z).$$

Proposition 2.2 Let $\mu\geq 0$ on $\mathbf{B}$, $0<p<\infty$. Then, the following statements are equivalent:

(1) μ is a p-logarithmic Carleson measure;

(2) μ satisfies

$$\sup_{a\in\mathbf{B}}\left(\log\frac{2}{1-|a|}\right)^p\int_{\mathbf{B}}P(a,z)\mathrm{d}\mu(z)<\infty;$$

(3) There exists some C such that, for any $f\in BMOA$,

$$\sup_{a\in\mathbf{B}}\int_{\mathbf{B}}P(a,z)|f(z)|^p\mathrm{d}\mu(z)\leq C\|f\|_*^p;$$

(4) For $0<q<\infty$, there exists some C such that, for any $f\in$ BMOA and $g\in H^q$,

$$\int_{\mathbf{B}}|f(z)|^p|g(z)|^q\mathrm{d}\mu(z)\leq C\|f\|_*^p\|g\|_q^q.$$

Proof (1)⇒(2) It is trivial if $|a|\leq\frac{3}{4}$. If $|a|>\frac{3}{4}$, set $\xi=\frac{a}{|a|}$, $t_k=2^{k-1}(1-|a|)$, $k=1,2,\cdots,N$, N being the smallest integer with $2^{N-1}(1-|a|)\geq 1$. Then, $2^N(1-|a|)\geq 2$ and $\log_2\frac{2}{1-|a|}\leq N\leq\log_2\frac{2}{1-|a|}+1$. Setting $S(\xi,t_0)=\emptyset$, by a simple calculation, we have

$$P(a,z)\leq\frac{C}{2^{2nk}(1-|a|)^n} \tag{2.1}$$

if $z\in S(\xi,t_k)\backslash S(\xi,t_{k-1})$. As μ is a p-logarithmic Carleson measure, we obtain

$$\begin{aligned}\int_{\mathbf{B}}P(a,z)\mathrm{d}\mu(z)&\leq\frac{C}{(1-|a|)^n}\sum_{k=1}^{N+1}\frac{\mu\left(S(\xi,t_k)\backslash S(\xi,t_{k-1})\right)}{2^{2nk}}\\&\leq\frac{C}{(1-|a|)^n}\sum_{k=1}^{N+1}\frac{1}{2^{2nk}}\cdot\frac{t_k^n}{\left(\log\frac{2}{t_k}\right)^p}\end{aligned}$$

$$\leq C\int_1^{2+\log_2 \frac{2}{1-|a|}} \frac{1}{2^{nt}}\cdot \frac{1}{\left(\log \frac{2}{2^{t-1}(1-|a|)}\right)^p}\mathrm{d}t$$
$$\leq \frac{C}{\left(\log \frac{2}{1-|a|}\right)^p}.$$

This means that (2) holds.

(2)⇒(3) First, for any $a\in \mathbf{B}$, we claim that

$$\int_{\mathbf{B}} |f(a)|^p P(a,z)\mathrm{d}\mu(z)\leq C\|f\|_*^p \tag{2.2}$$

if $f\in \mathrm{BMOA}$ by (2). We have $|f(a)|\leq C\|f\|_*\log\frac{2}{1-|a|}$ because $\|f\|_{\mathcal{B}}\leq C\|f\|_*$. For $f\in \mathrm{BMOA}$, set

$$g_a(z)=\frac{(1-|a|^2)^{\frac{n}{p}}}{(1-\langle z,a\rangle)^{\frac{2n}{p}}}\left(f(z)-f(a)\right),\quad z\in\mathbf{B},$$

then $g_a\in H^p$ and $\|g_a\|_p\leq C\|f\|_*\leq C$ by [1, Corollary 4.4]. By (2), we know that μ is the classical Carleson measure, then Lemma 2.1 yields that

$$\int_{\mathbf{B}} |f(z)-f(a)|^p P(a,z)\mathrm{d}\mu(z)=\int_{\mathbf{B}}|g_a|^p\mathrm{d}\mu(z)\leq C\|g_a\|_p^p\leq C\|f\|_*^p.$$

This, together with (2.2), gives

$$\int_{\mathbf{B}} |f(z)|^p P(a,z)\mathrm{d}\mu(z)\leq C\|f\|_*^p.$$

(3)⇒(1) We assume that $0<t<1$. For $\xi\in\partial\mathbf{B}$, set $a=(1-t)\xi$ and

$$f_a(z)=\log\frac{2}{1-\langle z,a\rangle},\quad z\in\mathbf{B}. \tag{2.3}$$

Then, $f_a\in \mathrm{BMOA}$ and $\|f_a\|_*\leq C$. It is checked that

$$P(a,z)\geq \frac{1}{4^n t^n},\quad |f_a(z)|\geq C\log\frac{2}{t} \tag{2.4}$$

if $z\in S(\xi,t)$. This implies that

$$\frac{\mu(S(\xi,t))\left(\log\frac{2}{t}\right)^p}{t^n}\leq C\int_{S(\xi,t)}P(a,z)|f_a(z)|^p\mathrm{d}\mu(z)$$
$$\leq C\int_{\mathbf{B}}P(a,z)|f_a(z)|^p\mathrm{d}\mu(z)$$
$$\leq C\|f_a\|_*^p\leq C.$$

(3)⇔(4) For $f\in\mathrm{BMOA}$, set $\mathrm{d}\mu_f(z)=\frac{|f(z)|^p}{\|f\|_*^p}\mathrm{d}\mu(z)$. By Lemma 2.1, (3) holds if and only if μ_f is the classical Carleson measure, which is equivalent to

$$\int_{\mathbf{B}}|g(z)|^q\mathrm{d}\mu_f(z)\leq C\|g\|_q^q,\qquad g\in H^q.$$

The proof is completed.

Remark The above result was first proved for the case of unit disk in [12].

Proposition 2.3 Let $\mu \geq 0$ on $\mathbf{B}$, $0 < p < \infty$. Then, the following statements are equivalent:

(1) μ is a vanishing p-logarithmic Carleson measure;

(2) μ satisfies

$$\lim_{|a|\to 1}\left(\log\frac{2}{1-|a|}\right)^p\int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z)=0;$$

(3) For any norm bounded sequence $\{f_k\}\subseteq$ BMOA with $f_k\to 0$ uniformly on any compact subset of $\mathbf{B}$,

$$\lim_{k\to\infty}\sup_{a\in\mathbf{B}}\int_{\mathbf{B}} P(a,z)|f_k(z)|^p\mathrm{d}\mu(z)=0;$$

(4) For $0<q<\infty$, if $\{f_k\}\subseteq$ BMOA is any norm bounded sequence with $f_k\to 0$ uniformly on any compact subset of $\mathbf{B}$, then,

$$\lim_{k\to\infty}\sup_{g\in H^q,\|g\|_q=1}\int_{\mathbf{B}}|f_k(z)|^p|g(z)|^q\mathrm{d}\mu(z)=0.$$

Proof (1)⇒(2) Let μ be a vanishing p-logarithmic Carleson measure. Then, for any $\varepsilon>0$, there exists $\delta\in(0,1)$, such that

$$\frac{\mu(S(\xi,t))\left(\log\frac{2}{t}\right)^p}{t^n}<\varepsilon,\quad t\leq\delta. \tag{2.5}$$

We only need to prove that

$$K_1(a)=\left(\log\frac{2}{1-|a|}\right)^p\int_{S(\xi,\delta)}P(a,z)\mathrm{d}\mu(z)\to 0,$$

and

$$K_2(a)=\left(\log\frac{2}{1-|a|}\right)^p\int_{\mathbf{B}\setminus S(\xi,\delta)}P(a,z)\mathrm{d}\mu(z)\to 0$$

as $|a|\to 1$, where $\xi=\frac{a}{|a|}$. Denote by $S(\xi,t_k)$ the Carleson set with $t_k=\alpha^{k-1}(1-|a|)$, where $1<\alpha<\frac{2}{\delta}$, $k=1,2,\cdots,N$, N being the smallest integer with $\alpha^{N-1}(1-|a|)\geq\delta$. Then, $\log_\alpha\frac{\delta\alpha}{1-|a|}\leq N\leq\log_\alpha\frac{\delta\alpha}{1-|a|}+1$. Setting $S(\xi,t_0)=\emptyset$, if $|a|$ sufficiently close to 1, (2.1) and (2.5) yield

$$\begin{aligned}
K_1(a)\leq{}& C\frac{\left(\log\frac{2}{1-|a|}\right)^p}{(1-|a|)^n}\sum_{k=1}^{N-1}\frac{\mu\left(S\left(\xi,t_k\right)\setminus S\left(\xi,t_{k-1}\right)\right)}{\alpha^{2nk}}\\
&+C\frac{\left(\log\frac{2}{1-|a|}\right)^p}{(1-|a|)^n}\cdot\frac{\mu\left(S\left(\xi,\delta\right)\setminus S\left(\xi,t_{N-1}\right)\right)}{\alpha^{2nN}}\\
\leq{}& C\frac{\left(\log\frac{2}{1-|a|}\right)^p}{(1-|a|)^n}\sum_{k=1}^{N-1}\frac{\varepsilon}{\alpha^{2nk}}\cdot\frac{t_k^n}{\left(\log\frac{2}{t_k}\right)^p}\\
&+C\frac{\left(\log\frac{2}{1-|a|}\right)^p}{(1-|a|)^n}\cdot\frac{\varepsilon}{\alpha^{2nN}}\cdot\frac{\delta^n}{\left(\log\frac{2}{\delta}\right)^p}\\
\leq{}& C\varepsilon\frac{\left(\log\frac{2}{1-|a|}\right)^p}{(1-|a|)^n}\sum_{k=1}^{N}\frac{1}{\alpha^{2nk}}\cdot\frac{t_k^n}{\left(\log\frac{2}{t_k}\right)^p}.
\end{aligned}$$

Similar to the argument in Proposition 2.2, we know that the above sum is bounded by $C\varepsilon$, where C is independent of ε. For $z \in \mathbf{B}\backslash S(\xi,\delta)$, then $|1-\langle z,\xi\rangle| \geq \delta$. Thus,

$$K_2(a) \leq \frac{C(1-|a|)^n}{\delta^{2n}} \left(\log \frac{2}{1-|a|}\right)^p \mu(\mathbf{B}) \to 0$$

as $|a| \to 1$.

(2)⇒(3) Let $\{f_k\} \subseteq$ BMOA be any norm bounded sequence and $f_k \to 0$ uniformly on each compact subset of $\mathbf{B}$. Firstly, we will show that

$$\lim_{k\to\infty} \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} |f_k(a)|^p P(a,z)\mathrm{d}\mu(z) = 0$$

if (2) holds. In fact, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\sup_{a\in\mathbf{B}\backslash\mathbf{B}_\delta} \left(\log \frac{2}{1-|a|}\right)^p \int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z) < \varepsilon,$$

where $\mathbf{B}_\delta = \{z \in \mathbf{B} : |z| \leq \delta\}$. Then,

$$\sup_{a\in\mathbf{B}\backslash\mathbf{B}_\delta} \int_{\mathbf{B}} P(a,z)|f_k(a)|^p \mathrm{d}\mu(z) \leq C\|f_k\|_*^p \sup_{a\in\mathbf{B}\backslash\mathbf{B}_\delta} \left(\log \frac{2}{1-|a|}\right)^p \int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z) < C\varepsilon.$$

As $f_k \to 0$ uniformly on $\mathbf{B}_\delta$, we know, when k is sufficiently large,

$$\sup_{a\in\mathbf{B}_\delta} \int_{\mathbf{B}} P(a,z)|f_k(a)|^p \mathrm{d}\mu(z) \leq \varepsilon \sup_{a\in\mathbf{B}_\delta} \int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z) < C\varepsilon.$$

The constants C above are independent of ε. Secondly, we claim that

$$\lim_{k\to\infty} \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)|f_k(z)-f_k(a)|^p \mathrm{d}\mu(z) = 0. \tag{2.6}$$

On one hand, (2) implies that μ is the classical vanishing Carleson measure, and for any $\varepsilon > 0$, there exists $r \in (0,1)$ such that $\|\mu_r\| < \varepsilon$, where μ_r is the restriction of μ on $\mathbf{B}\backslash\mathbf{B}_r$ (this can be proved similarly as in [14], p.130). For $a \in \mathbf{B}$, set

$$g_{k,a}(z) = \frac{(1-|a|^2)^{\frac{n}{p}}}{(1-\langle z,a\rangle)^{\frac{2n}{p}}} \left(f_k(z) - f_k(a)\right), \quad z \in \mathbf{B}.$$

Then, $\{g_{k,a}\} \subseteq H^p$ and $\|g_{k,a}\|_p \leq C$, where C is independent of k and a. Thus,

$$\sup_{a\in\mathbf{B}} \int_{\mathbf{B}\backslash\mathbf{B}_r} P(a,z)|f_k(z)-f_k(a)|^p \mathrm{d}\mu(z) = \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} |g_{k,a}(z)|^p \mathrm{d}\mu_r(z) \leq C\|\mu_r\| \sup_{a\in\mathbf{B}} \|g_{k,a}\|_p^p < C\varepsilon.$$

On the other hand, when k is sufficiently large, we have

$$\begin{aligned}\sup_{a\in\mathbf{B}} \int_{\mathbf{B}_r} P(a,z)|f_k(z)-f_k(a)|^p \mathrm{d}\mu(z) &\leq C \sup_{a\in\mathbf{B}} \int_{\mathbf{B}_r} P(a,z)\left(|f_k(z)|^p + |f_k(a)|^p\right)\mathrm{d}\mu(z) \\ &\leq \varepsilon C \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)\mathrm{d}\mu(z) \leq C\|\mu\|\varepsilon.\end{aligned}$$

Hence, the estimate (2.6) holds. Therefore,

$$\lim_{k\to\infty} \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} |f_k(z)|^p P(a,z)\mathrm{d}\mu(z) = 0.$$

(3)⇒(1) Suppose that μ is not a vanishing p-logarithmic Carleson measure, then there exists some $\varepsilon_0 > 0$, $\xi_k \in \partial\mathbf{B}$ and $\{t_k\} \subseteq (0,1)$ with $t_k \to 0$ as $k \to \infty$, but for every k,

$$\frac{\mu(S(\xi_k, t_k))\left(\log \frac{2}{t_k}\right)^p}{t_k^n} \geq \varepsilon_0. \tag{2.7}$$

Setting $a_k = (1-t_k)\xi_k$ and

$$f_k(z) = \left(\log \frac{2}{1-|a_k|^2}\right)^{-1}\left(\log \frac{2}{1-\langle z, a_k\rangle}\right)^2, \quad z \in \mathbf{B},$$

then, $\{f_k\} \subseteq$ BMOA is a bounded sequence and $f_k \to 0$ uniformly on any compact subset of $\mathbf{B}$ as $k \to \infty$. Hence,

$$\lim_{k\to\infty} \int_{\mathbf{B}} P(a_k, z)|f_k(z)|^p \mathrm{d}\mu(z) = 0. \tag{2.8}$$

However, for any k, (2.4) and (2.7) yield that

$$\int_{\mathbf{B}} P(a_k, z)|f_k(z)|^p \mathrm{d}\mu(z) \geq \int_{S(\xi_k,t_k)} P(a_k, z)|f_k(z)|^p \mathrm{d}\mu(z) \geq C\frac{\left(\log \frac{2}{t_k}\right)^p}{t_k^n}\int_{S(\xi_k,t_k)} \mathrm{d}\mu(z) \geq C\varepsilon_0.$$

This is a contradiction to (2.8).

(3)⇔(4) Let X_μ^p be the space of $f \in H(\mathbf{B})$ with

$$\|f\|_{X_\mu^p}^p = \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a, z)|f(z)|^p \mathrm{d}\mu(z) < \infty,$$

and for $0 < q < \infty$, let

$$Y_\mu^{p,q} = Y_\mu^p = \left\{ f \in H(\mathbf{B}) : \|f\|_{Y_\mu^p}^p = \sup_{g\in H^q, \|g\|_q=1} \int_{\mathbf{B}} |f(z)|^p |g(z)|^q \mathrm{d}\mu(z) < \infty \right\}.$$

By Lemma 2.1, $f \in X_\mu^p$ if and only if $\mathrm{d}\mu_f(z) = |f(z)|^p\mathrm{d}\mu(z)$ is the classical Carleson measure, which is equivalent to

$$\sup_{g\in H^q, \|g\|_q=1} \int_{\mathbf{B}} |f(z)|^p |g(z)|^q \mathrm{d}\mu(z) \leq C\|g\|_q^q \leq C.$$

Thus, $X_\mu^p = Y_\mu^p$ and $\|f\|_{X_\mu^p} \simeq \|f\|_{Y_\mu^p}$. Consequently, (3) and (4) are equivalent. The proof is completed.

Remark The above result was first proved for the unit disk in [13].

3 The Main Results

Theorem 3.1 For $g \in H(\mathbf{B})$, T_g is bounded on BMOA if and only if $g \in \mathrm{BMOA}_{\log}$. Moreover, $\|T_g\| \simeq \|g\|_{\mathrm{BMOA}_{\log}}$.

Proof First, for $f, g \in H(\mathbf{B})$, by a direct calculation we see

$$\Re(T_g f)(z) = f(z)\Re g(z).$$

For any $f \in$ BMOA, (1.1) yields that

$$\|T_g f\|_*^2 = \sup_{a\in\mathbf{B}} \|(T_g f)\circ\varphi_a - (T_g f)(a)\|_2^2 \simeq \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)|f(z)|^2|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z).$$

Set $\mathrm{d}\mu_g(z) = |\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)$, then, $\mu_g \geq 0$. So, T_g is bounded on BMOA if and only if μ_g is a 2-logarithmic Carleson measure, that is, $g \in \mathrm{BMOA}_{\log}$ by Proposition 2.2. Check the proof of Proposition 2.2 carefully, we obtain $\|T_g\| \simeq \|g\|_{\mathrm{BMOA}_{\log}}$. The proof is completed.

Theorem 3.2 For $g \in H(\mathbf{B})$, T_g is compact on BMOA if and only if

$$\lim_{|a|\to 1} \left(\log\frac{2}{1-|a|}\right)^2 \int_{\mathbf{B}} P(a,z)|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z) = 0.$$

Proof Let $\{f_k\} \subseteq$BMOA be any norm bounded sequence with $f_k \to 0$ uniformly on any compact subset of $\mathbf{B}$ as $k \to \infty$. Then,

$$\|T_g f_k\|_*^2 \simeq \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)|f_k(z)|^2|\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z).$$

Thus, by Proposition 2.3, T_g is compact on BMOA if and only if μ_g is a vanishing 2-logarithmic Carleson measure, where $\mathrm{d}\mu_g(z) = |\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)$. The proof is completed.

Theorem 3.3 For $g \in H(\mathbf{B})$, $g \in M(\mathrm{BMOA})$ if and only if $g \in H^\infty \bigcap \mathrm{BMOA}_{\log}$.

Proof First, a direct calculation shows that

$$\Re(gf)(z) = f(z)\Re g(z) + g(z)\Re f(z)$$

for all $f, g \in H(\mathbf{B})$.

"Necessity" As $g \in H^\infty$, for $f \in$BMOA, we obtain

$$\sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)\left|g(z)\Re f(z)\right|^2 \mathrm{d}v(z) \leq C\|g\|_\infty^2\|f\|_*^2.$$

Hence,

$$\begin{aligned}\|gf\|_*^2 &\simeq \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)\left|g(z)\Re f(z) + f(z)\Re g(z)\right|^2 \mathrm{d}v(z)\\ &\leq 2\|g\|_\infty^2\|f\|_*^2 + 2\sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)|f(z)|^2|\Re g(z)|^2\mathrm{d}v(z).\end{aligned}$$

Setting $\mathrm{d}\mu_g(z) = |\Re g(z)|^2(1-|z|^2)\mathrm{d}v(z)$, if $g \in \mathrm{BMOA}_{\log}$, Proposition 2.2 implies

$$\sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)|f(z)|^2|\Re g(z)|^2\mathrm{d}v(z) = \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)|f(z)|^2\mathrm{d}\mu_g(z) \leq C\|f\|_*^2,$$

which follows that $g \in M(\mathrm{BMOA})$.

"Sufficiently" For any $a \in \mathbf{B}$, set f_a as (2.3), then, $f_a \in$BMOA and $\|f_a\|_* \leq C$. As $g \in M(\mathrm{BMOA})$, $gf_a \in$BMOA, and $\|gf_a\|_* \leq C\|f_a\|_* \leq C$, then, for any $z \in \mathbf{B}$,

$$|g(z)f_a(z)| \leq C\|gf_a\|_* \log\frac{2}{1-|z|} \leq C\log\frac{2}{1-|z|}.$$

Setting $z = a$, then $|g(a)| \leq C$, that means $g \in H^\infty$. Now, we will prove $g \in \mathrm{BMOA}_{\log}$. Notice that $g \in M(\mathrm{BMOA})$, then,

$$\|gf\|_*^2 \simeq \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)\left|f(z)\Re g(z) + g(z)\Re f(z)\right|^2 \mathrm{d}v(z) \leq C\|f\|_*^2$$

for any $f \in$BMOA. As $g \in H^\infty$, we get

$$\sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)\left|g(z)\Re f(z)\right|^2 \mathrm{d}v(z) \le C\|f\|_*^2.$$

Thus,

$$\sup_{a\in\mathbf{B}} \int_{\mathbf{B}} P(a,z)(1-|z|^2)\left|f(z)\Re g(z)\right|^2 \mathrm{d}v(z) \le C\|f\|_*^2.$$

By Proposition 2.2, we obtain $g \in \mathrm{BMOA}_{\log}$. The proof is completed.

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