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Bounded functions of vanishing mean oscillation on compact metric spaces ☆

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Abstract

A well-known theorem of Wolff (Duke Math. J 49 (1982) 321) asserts that for every $f \in L^\infty$ on the unit circle T, there is a non-trivial $q \in \mathrm{QA} = \mathrm{VMO} \cap H^\infty$ such that $fq \in \mathrm{QC}$. In this paper we consider the situation where T is replaced by a compact metric space (X, d) equipped with a measure μ satisfying the condition $\mu(B(x, 2r)) \leqslant C\mu(B(x, r))$. We generalize Wolff's theorem to the extent that every function in $L^\infty(X, \mu)$ can be multiplied into $\mathrm{VMO}(X, d, \mu)$ in a non-trivial way by a function in $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$. Wolff's proof relies on the fact that T has a dyadic decomposition. But since this is not available for (X, d) in general, our approach is completely different. Furthermore, we show that the analyticity requirement for the function q in Wolff's theorem must be dropped if T is replaced by S^{2n-1} with $n \geqslant 2$. Move precisely, if $n \geqslant 2$, then there is a $g \in L^\infty(S^{2n-1}, \sigma)$, where σ is the standard spherical measure on S^{2n-1}, such that if $q \in H^\infty(S^{2n-1})$ and if q is not the constant function 0, then gq does not have vanishing mean oscillation on S^{2n-1}. The particular g that we construct also serves to show that a famous factorization theorem of Axler (Ann. of Math. 106 (1977) 567) for L^∞-functions on the unit circle T cannot be generalized to S^{2n-1} when $n \geqslant 2$. We conclude the paper with an index theorem for Toeplitz operators on S^{2n-1}.

Keywords: Mean oscillation; Compact metric space

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1. Introduction

Recall that QC denotes the function algebra $(H^\infty + C(T)) \cap \overline{(H^\infty + C(T))}$ on the unit circle $T = \{z \in \mathbf{C} : |z| = 1\}$. Alternately, it is well known that

$$\mathrm{QC} = \mathrm{VMO} \cap L^\infty.$$

See [7, p. 377]. It is also well known that a function $f \in L^\infty$ belongs to QC if and only if the Hankel operators H_f and $H_{\bar{f}}$ from H^2 to $L^2 \ominus H^2$ are both compact [8]. Of course, the symbol QC stands for "quasi-continuous", which suggests a close connection with $C(T)$. In 1982, Wolff published a theorem which suggests that QC is larger than one would expect in the sense that every function in the L^∞ of the circle can be "multiplied" into QC. More precisely, Wolff proved

Theorem 1 (Wolf [16]). *Suppose $f \in L^\infty$. Then there is an outer function $q \in \mathrm{QA}(= \mathrm{QC} \cap H^\infty)$ such that $qf \in \mathrm{QC}$.*

In particular, it follows from this theorem that every measurable set $E \subset T$ is the zero set of a function in QC, answering a question of Sarason [16, pp. 324–325].

The purpose of this paper is to prove an analogous result on compact metric spaces. We are particularly motivated by the case where the space is the unit sphere S^{2n-1} in $\mathbf{C}^n$ with the metric $d(a,b) = |1 - \langle a, b \rangle|^{1/2}$. Our main reason for doing this is that the method in [16] does not cover the general case of metric spaces. The key step in [16] is the proof of Lemma 1.1, which involves the convergence of a martingale, whose construction requires a stopping-time argument. This argument relies on a special feature of the unit circle T, namely it admits a nice decomposition in terms of *dyadic* arcs. That is, each arc can be neatly subdivided into two non-overlapping halves of equal size. Such a decomposition reflects the fact that T has a flat universal covering space. But when we consider spaces such as S^{2n-1} with $n \geqslant 2$, dyadic decomposition is obviously no longer available. Thus, elegant as Wolff's argument is, it has its limitations in terms of applicability.

Let (X,d) be a compact metric space throughout the paper. As usual, we write $B(x,r) = \{y \in X : d(x,y) < r\}$. Suppose that μ is a positive regular Borel measure on X with a non-zero total mass. Furthermore, it is assumed throughout the paper that there is a constant $0 < C < \infty$ such that

$$\mu(B(x,2r)) \leqslant C\mu(B(x,r)) \tag{1.1}$$

for all $x \in X$ and $r > 0$. For a given $r > 0$, if $k(r) \in \mathbf{N}$ is such that $2^{k(r)} r > \{d(x,y) : x, y \in X\}$, then it follows from (1.1) that

$$\mu(B(x,r)) \geqslant C^{-k(r)} \mu(X) > 0 \quad \text{for every } x \in X. \tag{1.2}$$

As we have already mentioned, our primary motivating example for this general setting is the following:

Example 1.1. On the unit sphere $S^{2n-1} = \{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$, the formula

$$d(a,b) = |1 - \langle a, b \rangle|^{1/2}, \quad a, b \in S^{2n-1}$$

defines a metric [11, p. 66]. This metric is *anisotropic* in the sense that the d-ball

$$\{z \in S^{2n-1} : |1 - \langle z, \zeta \rangle|^{1/2} < \delta\}$$

is close to an ellipsoid of dimension $2n - 1$ when δ is small [11, p. 66]. Let σ be the probability measure on S^{2n-1} which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. According to [11, Proposition 5.1.4], there is a constant $2^{-n} < A_0 < \infty$ such that

$$2^{-n}\delta^{2n} \leqslant \sigma(\{z \in S^{2n-1} : |1 - \langle z, \zeta \rangle|^{1/2} < \delta\}) \leqslant A_0 \delta^{2n}$$

for all $\zeta \in S^{2n-1}$ and $0 < \delta \leqslant \sqrt{2}$. Hence σ satisfies (1.1).

As usual, for any $f \in L^1(X, \mu)$ and Borel set $E \subset X$ with $\mu(E) > 0$, we denote the mean value of f on E by f_E. That is,

$$f_E = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

A function $f \in L^1(X, \mu)$ is said to have *bounded mean oscillation* if the quantity

$$||f||_{\mathrm{BMO}} = \sup_{\substack{x \in X \\ r > 0}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu$$

is finite. A function $f \in L^1(X, \mu)$ is said to have *vanishing mean oscillation* if

$$\lim_{\delta \downarrow 0} \sup_{\substack{x \in X \\ 0 < r \leqslant \delta}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu = 0.$$

Define

$$\mathrm{VMO}(X, d, \mu) = \{f \in L^1(X, \mu) : f \text{ has vanishing mean oscillation}\}.$$

As we have already mentioned, in the case of the unit circle T, we have the equality $\mathrm{VMO} \cap L^\infty = (H^\infty + C(T)) \cap \overline{(H^\infty + C(T))}$. But this fact does not generalize to the unit sphere S^{2n-1} when $n \geqslant 2$ [6, Section 4].

The main result of this paper is that, just as what happens on the unit circle, every $f \in L^\infty(X, \mu)$ can be multiplied into $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$. Hence, just as what happens on the unit circle, every μ-measurable set E in X is the zero set of a function in $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$. Keep in mind that (1.1) is assumed throughout the paper.

Theorem 2. *For any* $f \in L^\infty(X,\mu)$, *there exists a function* $\eta \in \mathrm{VMO}(X,d,\mu)$ *which has the following properties*:

(i) $0 \leqslant \eta \leqslant 1$ *on* X.
(ii) $\log \eta \in \mathrm{VMO}(X,d,\mu)$.
(iii) $\eta f \in \mathrm{VMO}(X,d,\mu) \cap L^\infty(X,\mu)$. *Moreover, if* $q \in \mathrm{VMO}(X,d,\mu) \cap L^\infty(X,\mu)$ *and if the inequality* $|q| \leqslant \eta$ *holds* μ-*a.e. on* X, *then* $qf \in \mathrm{VMO}(X,d,\mu) \cap L^\infty(X,\mu)$.

Note that Theorem 1 is readily recovered from Theorem 2 in the following way. It is easy to see that for a *real-valued* $h \in \mathrm{VMO}$ on the unit circle T, we have $e^{ih} \in \mathrm{QC}$ (see, e.g., [12]). Consider an $\eta \in \mathrm{VMO}$ on T such that $0 \leqslant \eta \leqslant 1$ and $\log \eta \in \mathrm{VMO}$. There is a $g \in \mathrm{VMO} \cap H^2$ such that $\mathrm{Re}(g) = \log \eta$ a.e. on T. Thus the function $q = e^g = \eta e^{i\mathrm{Im}(g)}$ belongs to $\mathrm{VMO} \cap H^\infty = \mathrm{QA}$ and has the property that $|q| = \eta$ a.e. on T.

In Theorem 1, the function f is multiplied into VMO by an *analytic* function $q \in \mathrm{QA} = \mathrm{VMO} \cap H^\infty$. Thus a natural (and obvious) question is whether or not analyticity can also be required in the context of Theorem 2 when $X = S^{2n-1}$. In other words, for an arbitrary $f \in L^\infty(S^{2n-1},\sigma)$, does there exist a non-trivial $q \in \mathrm{VMO}(S^{2n-1},d,\sigma) \cap H^\infty(S^{2n-1})$ such that $fq \in \mathrm{VMO}(S^{2n-1},d,\sigma)$? The answer is negative when $n \geqslant 2$. More precisely, we have

Theorem 3. *Let* $\{S^{2n-1},d,\sigma\}$ *be the same as in Example* 1.1 *and suppose that* $n \geqslant 2$. *Then there is a* $g \in L^\infty(S^{2n-1},\sigma)$ *which has the property that if* $q \in H^\infty(S^{2n-1})$ *and if* q *is not the constant* 0, *then* $gq \notin H^\infty(S^{2n-1}) + \{\mathrm{VMO}(S^{2n-1},d,\sigma) \cap L^\infty(S^{2n-1},\sigma)\}$.

Thus the analyticity requirement of the q in Theorem 1 cannot be generalized even in the sphere case. As it turns out, the same g in Theorem 3 also shows that another famous theorem involving analyticity on the unit circle has no generalization to spheres. Recall that Axler proved the following:

Theorem 4 (Axler [2]). *Let* $g \in L^\infty(T)$. *Then there exists a Blaschke product* b *and a function* $h \in H^\infty + C(T)$ *such that* $g = h/b$.

Although there is no analog of Blaschke product on S^{2n-1} when $n \geqslant 2$, there are plenty of inner functions in $H^\infty(S^{2n-1})$ [1]. But Theorem 3 clearly tells us that, even if b is only required to be a bounded analytic function, Theorem 4 cannot be generalized to S^{2n-1} when $n \geqslant 2$.

Before concluding the section, we would like to explain the idea behind the proof of Theorem 2, which turns out to be extremely simple. Indeed the idea is to exploit the fact that every $f \in L^\infty(X,\mu)$ has a full set of Lebesgue points. That is,

$$\lim_{k\to\infty} \frac{1}{\mu(B(x,1/k))} \int_{B(x,1/k)} |f - f(x)|\, d\mu = 0 \quad \text{for } \mu\text{–a.e. } x \in X$$

(see Proposition 2.2 below). By Egoroff's theorem, there is a sequence $\{K_n\}$ of compact sets in X such that $\lim_{n\to\infty} \mu(U_n) = 0$, where $U_n = X \backslash K_n$, and such that on each K_n the above convergence is uniform. In other words, for each n we have

$$\lim_{k\to\infty} \sup_{x\in K_n} \frac{1}{\mu(B(x,1/k))} \int_{B(x,1/k)} |f - f(x)|\, d\mu = 0. \tag{1.3}$$

If η_n is such that $1/n \leqslant \eta_n \leqslant 1$ on X and $\eta_n = 1/n$ on U_n, and if $q \in \mathrm{VMO}(X,d,\mu)$ satisfies $|q| \leqslant \eta_n$ μ-a.e., then from (1.3) it is easy to see that

$$\lim_{\delta\downarrow 0} \sup\left\{\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |qf - q_{B(x,r)} f_{B(x,r)}|\, d\mu : x \in X, r \leqslant \delta\right\} \leqslant 2\|f\|_\infty / n. \tag{1.4}$$

By requiring $\mu(U_n) \to 0$ to be sufficiently fast, we can find $\eta_n \in \mathrm{VMO}(X,d,\mu)$ such that the product $\eta = \prod_{n=1}^{\infty} \eta_n$ belongs to $\mathrm{VMO}(X,d,\mu)$ and has the property that $\log \eta \in \mathrm{VMO}(X,d,\mu)$. Now if $q \in \mathrm{VMO}(X,d,\mu)$ satisfies $|q| \leqslant \eta$ μ-a.e., then (1.4) holds for every n, which means $qf \in \mathrm{VMO}(X,d,\mu)$. Eq. (1.3) is what allows us to circumvent the dyadic decomposition and the stopping-time argument in the proof of [16, Lemma 1.1]. Our main technical step is Lemma 3.2.

The reader will recognize that this paper is in fact an example of "analysis on metric spaces" discussed in [14].

The rest of the paper is organized as follows. Section 2 contains the necessary preliminaries for the proof of Theorem 2, while the proof itself is given in Section 3. In Section 4 we show that $C(X)$ is dense in $\mathrm{VMO}(X,d,\mu)$ with respect to the norm $\|\cdot\|_{\mathrm{BMO}}$, which is a generalization of a well-known theorem of Sarason [13]. This generalization will be needed in Section 6. We then prove a property of the maximal ideal space of $\mathrm{VMO}(X,d,\mu) \cap L^\infty(X,\mu)$, which generalizes what happens in the case of the unit circle T [16, p. 325]. The proof in [16] for the unit circle case uses the notion of Jensen measure, which is related to the analyticity of the functions in QA. Our proof circumvents Jensen measure and shows that the same result holds even on spaces which have no analytic structure to speak of. The proof of Theorem 3 is given in Section 5, where the main idea is to exploit certain properties of a conditional expectation map. Finally, Section 6 begins with a detailed explanation as to why functions of vanishing mean oscillation in the setting of Example 1.1 are of great interest. In short, $\mathrm{VMO}(S^{2n-1},d,\sigma) \cap L^\infty(S^{2n-1},\sigma)$ is the right generalization of QC on the unit sphere. We conclude the paper with the result that if φ is an invertible element in $\mathrm{VMO}(S^{2n-1},d,\sigma) \cap L^\infty(S^{2n-1},\sigma)$ and if $n \geqslant 2$ then the Toeplitz operator T_φ has index 0.

2. Preliminaries

The properties of maximal functions on $\mathbf{R}^n$ or S^{2n-1} are well known [7,11,15]. In [5], these properties are extended to the more general setting of $\{X,d,\mu\}$.

For any $f \in L^1(X, \mu)$, its *maximal function* Mf is defined by the formula

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu.$$

Under condition (1.1), the maximal operator is of weak type-(1.1). More precisely, we have

Proposition 2.1 (Coifman and Weiss [5, Théorèm III.2.1]). *We have* $\mu(\{x \in X : (Mf)(x) > \lambda\}) \leqslant C^2 \|f\|_1/\lambda$ *for all* $f \in L^1(X, \mu)$ *and* $\lambda > 0$, *where* C *is the constant that appears in* (1.1).

By a standard argument (see [11,15]), Proposition 2.1 implies

Proposition 2.2. *If* $f \in L^1(X, \mu)$, *then*

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)| \, d\mu = 0$$

for μ*-a.e.* $x \in X$.

We will now introduce a function φ which will be fixed for the rest of the section. Let φ be a continuous function on $[0, \infty)$ such that $\varphi = 1$ on [0,1], $\varphi = 0$ on $[2, \infty)$, and $0 \leqslant \varphi \leqslant 1$ on the entire half line $[0, \infty)$. Using this φ, we define a modified maximal function as follows. For each $f \in L^1(X, \mu)$, define

$$(\mathscr{M}f)(x) = \sup_{r>0} \int \varphi(r^{-1}d(x,y))|f(y)| \, d\mu(y) \Big/ \int \varphi(r^{-1}d(x,y)) \, d\mu(y). \tag{2.1}$$

Because $\chi_{B(x,r)}(y) \leqslant \varphi(r^{-1}d(x,y)) \leqslant \chi_{B(x,2r)}(y)$, we have the inequality

$$C^{-1}Mf \leqslant \mathscr{M}f \leqslant CMf, \tag{2.2}$$

where C is the constant that appears in (1.1). But the key here is the assumption that φ be *continuous* on $[0, \infty)$. This continuity "blurs" the boundary of the ball $B(x,r)$ in (2.1), which has the following consequence:

Lemma 2.3. *If* $f \in C(X)$, *then we also have* $\mathscr{M}f \in C(X)$.

Proof. Let $D = \sup_{x,y \in X} d(x,y)$ and suppose that $0 < \delta < D$. By (1.2),

$$F(r,x) = \int \varphi(r^{-1}d(x,y))|f(y)| \, d\mu(y) \Big/ \int \varphi(r^{-1}d(x,y)) \, d\mu(y)$$

is a continuous function on $[\delta, D] \times X$. Note that $\mathcal{M}f = \max\{F_\delta, G_\delta\}$, where

$$F_\delta(x) = \sup_{\delta \leqslant r \leqslant D} F(r, x), \quad G_\delta(x) = \sup_{0 < r \leqslant \delta} F(r, x).$$

By the continuity of F and the compactness of $[\delta, D] \times X$, the limit

$$\lim_{n \to \infty} \max\{F(\delta + (j/n)(D - \delta), x) : j = 0, 1, \ldots, n\} = F_\delta(x)$$

converges *uniformly* in x. Thus $F_\delta \in C(X)$. Suppose now that f is continuous on X. Given an $\varepsilon > 0$, we have $||f| - G_\delta| \leqslant \varepsilon$ on X if δ is sufficiently small. For such a δ, we also have $|\mathcal{M}f - \max\{F_\delta, |f|\}| \leqslant \varepsilon$ on X because $\max\{a, b\} = 2^{-1}(a + b + |a - b|)$. Since $\max\{F_\delta, |f|\}$ is continuous and $\varepsilon > 0$ is arbitrary, this means $\mathcal{M}f \in C(X)$. □

Our next proposition is a well-known result of Coifman and Rochberg [3] reestablished in our setting.

Proposition 2.4. *There is a constant* $0 < C_{2.4} < \infty$ *such that if* $f \in L^\infty(X, \mu)$ *and* $\|f\|_\infty \neq 0$, *then* $\|\log(\mathcal{M}f)\|_{\mathrm{BMO}} \leqslant C_{2.4}$.

Proof. The proof is essentially the same as that of [3, Proposition 2]. We want to find a C_1 such that $(\log(\mathcal{M}f))_B - \inf_B \log(\mathcal{M}f) \leqslant C_1$ for any open ball B in X. Then the constant $C_{2.4} = 2C_1$ will do. Fix a $0 < \delta < 1$. By Jensen's inequality, we only need to find a $C_2 = C_2(\delta)$ such that $((\mathcal{M}f)^\delta)_B \leqslant C_2 \inf_B (\mathcal{M}f)^\delta$. But because of (2.2), it suffices to find a C_3 independent of f, B such that

$$((Mf)^\delta)_B \leqslant C_3 \inf_B (Mf)^\delta. \tag{2.3}$$

To establish (2.3), assume that $B = B(x, r)$ and set $B' = B(x, 2r)$. Write $f = g + h$, where $g = f\chi_{B'}$ and $h = f\chi_{X \setminus B'}$. Then $((Mf)^\delta)_B \leqslant ((Mg)^\delta)_B + ((Mh)^\delta)_B$.

Let $z \in B$ be given. Then $B(x, 4r) \supset B(z, 3r) \supset B'$. Hence $\|g\|_1 = \|f\chi_{B'}\|_1 \leqslant \mu(B(z, 3r))(Mf)(z) \leqslant C^2 \mu(B)(Mf)(z)$. To estimate $((Mg)^\delta)_B$, we recall the identity

$$\frac{1}{\mu(B)} \int_B (Mg)^\delta \, d\mu = \frac{1}{\mu(B)} \int_0^\infty \delta \lambda^{\delta - 1} \mu(\{y \in B : (Mg)(y) > \lambda\}) \, d\lambda$$

(see, e.g., [7, p. 21]). Then write $\int_0^\infty = \int_0^{(Mf)(z)} + \int_{(Mf)(z)}^\infty$ on the right-hand side and apply Proposition 2.1 in the second term. This gives us

$$((Mg)^\delta)_B \leqslant \{(Mf)(z)\}^\delta + \frac{\delta C^2}{1 - \delta} \cdot \frac{\|g\|_1}{\mu(B)} \cdot \{(Mf)(z)\}^{\delta - 1} \leqslant C_4 \{(Mf)(z)\}^\delta,$$

where $C_4 = 1 + \delta C^4 (1 - \delta)^{-1}$. On the other hand, it is straightforward to verify that $(Mh)(y) \leqslant C^3 (Mf)(z)$ for every $y \in B$. Hence $C_3 = C_4 + C^{3\delta}$ will do for (2.3). □

Recall that $\log^+ \lambda = \max\{0, \log \lambda\}$.

Corollary 2.5. *For any* $f \in L^\infty(X,\mu)$ *with* $\|f\|_\infty \neq 0$, *we have* $\|\log^+(\mathcal{M}f)\|_{\mathrm{BMO}} \leqslant 2C_{2.4}$.

Proof. This follows from Proposition 2.4 and the inequality $\|\max\{0,g\}\|_{\mathrm{BMO}} \leqslant 2\|g\|_{\mathrm{BMO}}$ for real valued g, which follows from the identity $\max\{0,a\} = 2^{-1}(|a|+a)$. □

In the above proof we used the fact that if $f \in L^1(X,\mu)$ and $\mu(E) > 0$, then

$$\frac{1}{\mu(E)}\int_E |f - f_E|\, d\mu \leqslant \frac{2}{\mu(E)}\int_E |f-c|\, d\mu \quad \text{for any } c \in \mathbf{C}. \tag{2.4}$$

This inequality will be frequently used in the sequel without reference.

3. Proof of Theorem 2

The proof consists of a number of easy steps based on the facts stated in Section 2.

Lemma 3.1. *Let* $\xi_1, \dots, \xi_n \dots$ *be a sequence of functions in* $\mathrm{VMO}(X,d,\mu)$ *such that* $0 \leqslant \xi_n \leqslant 1$ *on* X *for every* n. *Let* $h = \prod_{n=1}^\infty \xi_n$. *If* $\sum_{n=1}^\infty \|\xi_n\|_{\mathrm{BMO}} < \infty$, *then* $h \in \mathrm{VMO}(X,d,\mu)$. *Moreover*, $\|h\|_{\mathrm{BMO}} \leqslant 2\sum_{n=1}^\infty \|\xi_n\|_{\mathrm{BMO}}$.

Proof. We have $|a_1 \dots a_N - b_1 \dots b_N| \leqslant |a_1 - b_1| + \dots + |a_N - b_N|$ for any $a_1, \dots, a_N, b_1, \dots, b_N \in [0,1]$, which can be proved by an easy induction on N. Let B be an open ball in X. Since $0 \leqslant \xi_n \leqslant 1$ on X and $0 \leqslant (\xi_n)_B \leqslant 1$, we have

$$|\xi_1 \dots \xi_N - (\xi_1)_B \dots (\xi_N)_B| \leqslant |\xi_1 - (\xi_1)_B| + \dots + |\xi_N - (\xi_N)_B|.$$

Letting $N \to \infty$, it follows from the dominated convergence theorem that

$$\frac{1}{\mu(B)}\int_B \left| h - \prod_{n=1}^\infty (\xi_n)_B \right| d\mu \leqslant \sum_{n=1}^{v} \frac{1}{\mu(B)}\int_B |\xi_n - (\xi_n)_B|\, d\mu + \sum_{n=v+1}^\infty \|\xi_n\|_{\mathrm{BMO}}$$

for any $v \in \mathbf{N}$. The conclusions of the lemma are now obvious. □

Lemma 3.2. *Let* U *be an arbitrary open set in* X. *For any* $0 < \varepsilon < 1$, *there is an* $h \in \mathrm{VMO}(X,d,\mu)$ *satisfying the inequality* $0 \leqslant h \leqslant 1$ *on* X *such that the following hold true*:

(i) $h = 0$ *on* U.
(ii) $\mu(\{x \in X : h(x) < 1\}) \leqslant 2C^7 e^{1/\varepsilon}\mu(U)$, *where* C *is the constant in* (1.1).
(iii) $\|h\|_{\mathrm{BMO}} \leqslant 8C_{2.4}\varepsilon$, *where* $C_{2.4}$ *is the constant in Proposition* 2.4.

Proof. It follows from (1.2) that, for each $n \in \mathbf{N}$, the set $E_n = \{x \in U : B(x, 3^{-n}) \subset U\}$ has a subset F_n which is *maximal* with respect to the property that $B(x, 3^{-n}) \cap B(y, 3^{-n}) = \emptyset$ if $x, y \in F_n$ and $x \neq y$. The maximality of F_n implies that $E_n \subset \bigcup_{x \in F_n} B(x, 3^{-n+1})$. Thus, by a classic Covering-Lemma argument, there is a countable collection $\{B_j : j \in J\}$ of pairwise disjoint open balls in U with the following property: If B'_j is the open ball in X whose center coincides with that of B_j and whose radius is three times that of $B_j, j \in J$, then $U \subset \bigcup_{j \in J} B'_j$. Now we consider the following two cases.

(A) $\operatorname{card}(J) < \infty$. We want to show that, for any $0 < \varepsilon < 1$, there is a $\xi \in C(X)$ satisfying the inequality $0 \leqslant \xi \leqslant 1$ on X such that the following hold true:

(a) $\xi = 0$ on U.
(b) $\mu(\{x \in X : \xi(x) < 1\}) \leqslant C^5 e^{1/\varepsilon} \mu(U)$, where C is the constant in (1.1).
(c) $\|\xi\|_{\mathrm{BMO}} \leqslant 2C_{2.4}\varepsilon$, where $C_{2.4}$ is the constant in Proposition 2.4.

Let $\tilde{B}_j$ be the open ball whose center coincides with that of B_j and whose radius is four times that of B_j. For each $j \in J$, there is an $f_j \in C(X)$ such that $0 \leqslant f_j \leqslant 1$ on $X, f_j = 1$ on B'_j, and $f_j = 0$ on $X \backslash \tilde{B}_j$. Define $f = \max\{f_j : j \in J\}$. Then $0 \leqslant f \leqslant 1$ on $X, f = 1$ on U, and $f = 0$ on $X \backslash \tilde{U}$, where $\tilde{U} = \bigcup_{j \in J} \tilde{B}_j$. Of course, the assumption $\operatorname{card}(J) < \infty$ implies $f \in C(X)$. By (1.1), we have $\mu(\tilde{U}) \leqslant \sum_{j \in J} \mu(\tilde{B}_j) \leqslant C^2 \sum_{j \in J} \mu(B_j) \leqslant C^2 \mu(U)$.

Let $g = e^{1/\varepsilon} f$. Since $0 \leqslant g \leqslant e^{1/\varepsilon}$ on X, $g = e^{1/\varepsilon}$ on U, and U is open, we have $\mathscr{M}g = e^{1/\varepsilon}$ on U. Let $\xi = 1 - \varepsilon \log^+(\mathscr{M}g)$. Then $0 \leqslant \xi \leqslant 1$ on X and $\xi = 0$ on U. Lemma 2.3 asserts that $\mathscr{M}g \in C(X)$. Therefore $\xi \in C(X)$. By Corollary 2.5, we have $\|\xi\|_{\mathrm{BMO}} = \varepsilon \|\log^+(\mathscr{M}g)\|_{\mathrm{BMO}} \leqslant 2C_{2.4}\varepsilon$. If $x \in X$ is such that $(\mathscr{M}g)(x) \leqslant 1$, then $\log^+(\mathscr{M}g)(x) = 0$. Thus it follows from (2.2) and Proposition 2.1 that

$$\begin{aligned}\mu(\{x \in X : \xi(x) < 1\}) &\leqslant \mu(\{x \in X : (\mathscr{M}g)(x) > 1\}) \leqslant \mu(\{x \in X : (Mg)(x) > C^{-1}\}) \\ &\leqslant C^3 \|g\|_1 \leqslant C^3 e^{1/\varepsilon} \mu(\tilde{U}) \leqslant C^5 e^{1/\varepsilon} \mu(U).\end{aligned}$$

Hence ξ has the desired properties.

(B) The general case. Let $\varepsilon > 0$ be given. Since $\sum_{j \in J} \mu(B'_j) \leqslant C^2 \sum_{j \in J} \mu(B_j) \leqslant C^2 \mu(U)$, we can write J as the union of *finite* subsets $J_1, \dots, J_n, \dots$ such that

$$\sum_{j \in J_n} \mu(B'_j) \leqslant 2^{-n} e^{(1-2^{n-1})/\varepsilon} C^2 \mu(U) \quad \text{for every } n \geqslant 2.$$

Define $U_n = \bigcup_{j \in J_n} B'_j, n \in \mathbf{N}$. Then $\mu(U_n) \leqslant 2^{-n} e^{(1-2^{n-1})/\varepsilon} C^2 \mu(U)$ when $n \geqslant 2$ and $\mu(U_1) \leqslant C^2 \mu(U)$. Since $\operatorname{card}(J_n) < \infty$, by case (A), there is a $\xi_n \in C(X)$ satisfying the inequality $0 \leqslant \xi_n \leqslant 1$ on X such that

(1) $\xi_n = 0$ on U_n;
(2) $\mu(\{x \in X : \xi_n(x) < 1\}) \leqslant C^5 e^{2^{n-1}/\varepsilon} \mu(U_n)$;
(3) $||\xi_n||_{\mathrm{BMO}} \leqslant 2C_{2.4}(\varepsilon/2^{n-1})$.

Let $h = \prod_{n=1}^{\infty} \xi_n$. Then, of course, $0 \leqslant h \leqslant 1$ on X and $h = 0$ on $\bigcup_{n=1}^{\infty} U_n = \bigcup_{j \in J} B'_j$, which contains U. It follows from (3) and Lemma 3.1 that $h \in \mathrm{VMO}(X, d, \mu)$ and that (iii) holds true. To verify (ii), note that

$$\mu(\{x \in X : h(x) < 1\}) \leqslant \sum_{n=1}^{\infty} \mu(\{x \in X : \xi_n(x) < 1\}) \leqslant \sum_{n=1}^{\infty} C^5 e^{2^{n-1}/\varepsilon} \mu(U_n).$$

Since $\mu(U_1) \leqslant C^2 \mu(U)$ and $\mu(U_n) \leqslant 2^{-n} e^{(1-2^{n-1})/\varepsilon} C^2 \mu(U)$ when $n \geqslant 2$, (ii) follows. □

Lemma 3.3. (a) *Let* $n \in \mathbf{N}$. *If the range of* h *is contained in* $[1/n, 1]$, *then* $||\log h||_{\mathrm{BMO}} \leqslant 2n||h||_{\mathrm{BMO}}$.
(b) *If* $h \in \mathrm{VMO}(X, d, \mu)$ *and if the range of h is contained in* $[1/n, 1]$ *for some* $n \in \mathbf{N}$, *then* $\log h \in \mathrm{VMO}(X, d, \mu)$.

Proof. Note that $|\log u - \log v| \leqslant n|u - v|$ for $u, v \in [1/n, 1]$. Thus,

$$\frac{1}{\mu(B)} \int_B |\log h - \log(h_B)| \, d\mu \leqslant \frac{n}{\mu(B)} \int_B |h - h_B| \, d\mu$$

if the range of h is contained in $[1/n, 1]$. The conclusions are now obvious. □

Proof of Theorem 2. Set $\varepsilon_n = 2^{-n}$ for every $n \in \mathbf{N}$. Let $f \in L^\infty(X, \mu)$ be given. Proposition 2.2 asserts that

$$\lim_{k \to \infty} \frac{1}{\mu(B(x, 1/k))} \int_{B(x,1/k)} |f - f(x)| \, d\mu = 0 \quad \text{for } \mu\text{–a.e. } x \in X.$$

We now apply Egoroff's theorem: For every $n \in \mathbf{N}$, there is a compact set K_n in X such that the above limit converges uniformly on K_n, i.e.,

$$\lim_{k \to \infty} \sup_{x \in K_n} \frac{1}{\mu(B(x, 1/k))} \int_{B(x,1/k)} |f - f(x)| \, d\mu = 0, \tag{3.1}$$

and such that the open set $U_n = X \backslash K_n$ satisfies the inequality

$$\mu(U_n) \leqslant e^{-1/\varepsilon_n} 2^{-n}. \tag{3.2}$$

By Lemma 3.2, for each $n\in\mathbf{N}$, there is an $h_n\in\mathrm{VMO}(X,d,\mu)$ satisfying the inequality $0\leqslant h_n\leqslant 1$ on X such that the following hold:

(α) $h_n = 0$ on U_n.
(β) $\mu(\{x\in X : h_n(x)<1\})\leqslant 2C^7e^{1/\varepsilon_n}\mu(U_n)$.
(γ) $\|h_n\|_{\mathrm{BMO}}\leqslant 8C_{2.4}\varepsilon_n$.

Now define

$$\eta_n = \frac{1}{n} + \frac{n-1}{n}h_n, \quad n\in\mathbf{N}.$$

Then $1/n\leqslant\eta_n\leqslant 1$ on X. Obviously, we have $\eta_n\in\mathrm{VMO}(X,d,\mu)$ and

(α') $\eta_n = 1/n$ on U_n;
(γ') $\|\eta_n\|_{\mathrm{BMO}} = \{(n-1)/n\}\|h_n\|_{\mathrm{BMO}}\leqslant 8C_{2.4}\varepsilon_n$.

Since $\{x\in X : \eta_n(x)<1\}\subset\{x\in X : h_n(x)<1\}$, it follows from ($\beta$) and (3.2) that

(β') $\mu(\{x\in X : \eta_n(x)<1\})\leqslant C^7 2^{-n+1}$.

Now define $\eta = \prod_{n=1}^{\infty}\eta_n$, which obviously satisfies (i). Since $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, it follows from Lemma 3.1 and (γ') that $\eta\in\mathrm{VMO}(X,d,\mu)$.

To verify (ii), note that $\int|\log\eta_n|\,d\mu\leqslant\mu(\{x\in X : \eta_n(x)<1\})\log n$. Thus ($\beta'$) ensures that $\log\eta\in L^1(X,\mu)$. By Lemma 3.3(a) and (γ'), we have

$$\|\log\eta_n\|_{\mathrm{BMO}}\leqslant 2n\|\eta_n\|_{\mathrm{BMO}}\leqslant 16C_{2.4}n2^{-n}.$$

Lemma 3.3(b) asserts that $\log\eta_n\in\mathrm{VMO}(X,d,\mu)$. Thus the function $\log\eta = \sum_{n=1}^{\infty}\log\eta_n$ also belongs to $\mathrm{VMO}(X,d,\mu)$.

Finally, we must verify (iii). Suppose that $q\in\mathrm{VMO}(X,d,\mu)$ and that $|q|\leqslant\eta$ μ-a.e. on X. We must show that $qf\in\mathrm{VMO}(X,d,\mu)$. Given an open ball $B = B(z,r)$, we have

$$\frac{1}{\mu(B)}\int_B|qf - q_Bf_B|\,d\mu\leqslant\frac{1}{\mu(B)}\int_B|q||f - f_B|\,d\mu + \frac{\|f\|_\infty}{\mu(B)}\int_B|q - q_B|\,d\mu. \quad (3.3)$$

Given an $a>0$, pick a $v\in\mathbf{N}$ such that $1/v\leqslant a$. By (3.1), there is a $k_0\in\mathbf{N}$ such that

$$\sup_{x\in K_v}\frac{1}{\mu(B(x,1/k))}\int_{B(x,1/k)}|f - f(x)|\,d\mu\leqslant a \quad \text{if } k\geqslant k_0.$$

There is a $\delta_0>0$ such that $(1/\mu(B))\int_B|q - q_B|\,d\mu\leqslant a$ if $r\leqslant\delta_0$. Set $\delta = \min\{\delta_0, 1/2k_0\}$.

Suppose that $B\cap K_v\neq\emptyset$. If $r\leqslant\delta$, then there is a $k\geqslant k_0$ such that $1/2(k+1)<r\leqslant 1/2k$. Pick an $x\in B\cap K_v$. We have $B(x,1/k)\supset B$ and $B(z,6r)\supset B(x,1/k)$. Therefore

$$\frac{1}{\mu(B)}\int_B|f - f(x)|\,d\mu\leqslant\frac{\mu(B(z,6r))}{\mu(B(z,r))}\cdot\frac{1}{\mu(B(x,1/k))}\int_{B(x,1/k)}|f - f(x)|\,d\mu\leqslant C^3a. \quad (3.4)$$

It now follows from (3.3), (3.4) and (2.4) that

$$\frac{1}{\mu(B)}\int_B |qf - q_B f_B|\, d\mu \leqslant 2C^3 a + \|f\|_\infty a.$$

in the case $B \cap K_v \neq \emptyset$ and $r \leqslant \delta$.

Suppose now that $B \cap K_v = \emptyset$, i.e., $B \subset X \backslash K_v = U_v$. Then we use the fact that $\eta \leqslant \eta_v = 1/v$ on U_v. Since $|q| \leqslant \eta$ μ-a.e., we have

$$\frac{1}{\mu(B)}\int_B |qf - q_B f_B|\, d\mu \leqslant 2\|f\|_\infty / v \leqslant 2\|f\|_\infty a$$

in the case $B \cap K_v = \emptyset$.

Since a is an arbitrary positive number, we conclude from the above two paragraphs that qf has vanishing mean oscillation. □

4. Approximation and maximal ideal space

A well-known theorem of Sarason asserts that, on $\mathbf{R}^n$ and T, $\mathrm{UC} \cap \mathrm{BMO}$ is dense in VMO with respect to the BMO-norm (see [13] or [7, p. 250]). The original proof of this result used convolution, which is made possible by the natural group structure on $\mathbf{R}^n$ and T. We will see that the main idea of Sarason's original proof still works in the general setting $\{X, d, \mu\}$, even though group structure is no longer available.

In this section φ denotes the same function as in Section 2. For each $f \in L^1(X, \mu)$ and each $r > 0$, define

$$f_r(x) = \int \varphi(r^{-1} d(x,y)) f(y)\, d\mu(y) \Big/ \int \varphi(r^{-1} d(x,y))\, d\mu(y).$$

The continuity of φ guarantees that $f_r \in C(X)$ for all such f and r.

Proposition 4.1. *Let f be a function of bounded mean oscillation on X and define*

$$\|f\|_{\mathrm{LMO}} = \lim_{\delta \downarrow 0} \sup_{\substack{x \in X \\ 0 < r \leqslant \delta}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu.$$

Then,

$$\limsup_{r \downarrow 0} \|f - f_r\|_{\mathrm{BMO}} \leqslant 2C^2(1 + C + C^3)\|f\|_{\mathrm{LMO}},$$

where C is the constant in (1.1).

Proof. Let $\varepsilon > 0$ be given. Then by definition there is a $\delta_0 > 0$ such that

$$\sup_{\substack{x \in X \\ 0 < r \leqslant \delta_0}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leqslant \|f\|_{\mathrm{LMO}} + \varepsilon.$$

Suppose that $0<r\leqslant\delta_0/4$ and that $u\in X$. If $z\in B(u,2r)$, then $B(z,2r)\subset B(u,4r)\subset B(z,6r)$. Hence for each $z\in B(u,2r)$ we have

$$\begin{aligned}|f_r(z)-f_{B(u,4r)}| &= \left|\int \varphi(r^{-1}d(z,y))(f(y)-f_{B(u,4r)})\,d\mu(y)\right| \bigg/ \int \varphi(r^{-1}d(z,y))\,d\mu(y)\\ &\leqslant \frac{1}{\mu(B(z,r))}\int_{B(z,2r)}|f-f_{B(u,4r)}|\,d\mu\\ &\leqslant \frac{\mu(B(z,6r))}{\mu(B(z,r))}\cdot\frac{1}{\mu(B(u,4r))}\int_{B(u,4r)}|f-f_{B(u,4r)}|\,d\mu\\ &\leqslant C^3(\|f\|_{\mathrm{LMO}}+\varepsilon),\end{aligned}\tag{4.1}$$

where the last $\leqslant$ follows from (1.1) and the condition $4r\leqslant\delta_0$. Also,

$$|f_{B(u,2r)}-f_{B(u,4r)}|\leqslant\frac{1}{\mu(B(u,2r))}\int_{B(u,2r)}|f-f_{B(u,4r)}|\,d\mu\leqslant C(\|f\|_{\mathrm{LMO}}+\varepsilon).\tag{4.2}$$

Combining (4.1) and (4.2) and using the condition $2r\leqslant\delta_0$, we have

$$\begin{aligned}\frac{1}{\mu(B(u,2r))}\int_{B(u,2r)}|f-f_r|\,d\mu &\leqslant \frac{1}{\mu(B(u,2r))}\int_{B(u,2r)}|f-f_{B(u,2r)}|\,d\mu\\ &\quad+|f_{B(u,2r)}-f_{B(u,4r)}|+\frac{1}{\mu(B(u,2r))}\int_{B(u,2r)}|f_r-f_{B(u,4r)}|\,d\mu\\ &\leqslant(1+C+C^3)(\|f\|_{\mathrm{LMO}}+\varepsilon).\end{aligned}\tag{4.3}$$

Since $\varepsilon>0$ is arbitrary, it suffices to show that $\|f-f_r\|_{\mathrm{BMO}}\leqslant 2C^2(1+C+C^3)(\|f\|_{\mathrm{LMO}}+\varepsilon)$ when $0<r\leqslant\delta_0/4$. Let such an r be given. With (2.4) in mind, we only need to estimate

$$\inf_{c\in\mathbf{C}}\frac{1}{\mu(B(\xi,\rho))}\int_{B(\xi,\rho)}|f-f_r-c|\,d\mu$$

for all $\xi\in X$ and $\rho>0$. Given such a pair of ξ,ρ, let us write $B=B(\xi,\rho)$. We divide the required estimate into two cases.

(i) Suppose that $\rho>r$. By (1.2), B has a finite subset F which is *maximal* with respect the property that $B(u,r)\cap B(v,r)=\emptyset$ if $u,v\in F$ and $u\neq v$. The maximality of F means, of course, that $B(z,r)\cap\{\bigcup_{u\in F}B(u,r)\}\neq\emptyset$ for every $z\in B$. Thus $B\subset\bigcup_{u\in F}B(u,2r)$. By (4.3), we have

$$\begin{aligned}\frac{1}{\mu(B)}\int_B|f-f_r|\,d\mu &\leqslant \sum_{u\in F}\frac{\mu(B(u,2r))}{\mu(B)}\cdot\frac{1}{\mu(B(u,2r))}\int_{B(u,2r)}|f-f_r|\,d\mu\\ &\leqslant \sum_{u\in F}\frac{\mu(B(u,r))}{\mu(B)}\cdot\frac{\mu(B(u,2r))}{\mu(B(u,r))}(1+C+C^3)(\|f\|_{\mathrm{LMO}}+\varepsilon).\end{aligned}$$

But $\{B(u,r) : u \in F\}$ are pairwise disjoint balls contained in $B(\xi, \rho + r)$. Hence,

$$\inf_{c\in\mathbf{C}} \frac{1}{\mu(B)} \int_B |f - f_r - c|\, d\mu \leqslant \frac{1}{\mu(B)} \int_B |f - f_r|\, d\mu \leqslant C^2(1 + C + C^3)(\|f\|_{\mathrm{LMO}} + \varepsilon)$$

in the case $\rho > r$.

(ii) Suppose that $0 < \rho \leqslant r$. In this case it follows from (4.1) that $|f_r(z) - f_{B(\xi,4r)}| \leqslant C^3(\|f\|_{\mathrm{LMO}} + \varepsilon)$ whenever $z \in B = B(\xi, \rho)$. Hence,

$$\begin{aligned}\inf_{c\in\mathbf{C}} \frac{1}{\mu(B)} \int_B |f - f_r - c|\, d\mu &\leqslant \frac{1}{\mu(B)} \int_B |f - f_r - f_B + f_{B(\xi,4r)}|\, d\mu \\ &\leqslant \frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu + \frac{1}{\mu(B)} \int_B |f_r - f_{B(\xi,4r)}|\, d\mu \leqslant (1 + C^3)(\|f\|_{\mathrm{LMO}} + \varepsilon)\end{aligned}$$

in this case. □

Corollary 4.2. *Let w be a unimodular function in* $\mathrm{VMO}(X, d, \mu)$. *Then there exists a sequence $\{v_k\}$ of unimodular functions in $C(X)$ such that* $\lim_{k\to\infty} \|w - v_k\|_{\mathrm{BMO}} = 0$.

Proof. Since $w \in \mathrm{VMO}(X, d, \mu)$, Proposition 4.1 tells us that $\lim_{r\downarrow 0} \|w - w_r\|_{\mathrm{BMO}} = 0$. But $w_r \in C(X)$ and we can write $w - (w_r/|w_r|) = (w - w_r) + (w_r/|w_r|)(|w_r| - 1)$ if w_r does not vanish on X. Hence the corollary will follow if we can show that

$$\lim_{r\downarrow 0} \||w_r| - 1\|_\infty = 0. \tag{4.4}$$

To prove (4.4), let $\varepsilon > 0$ be given. By (4.1), there is a $\delta_0 > 0$ such that

$$|w_r(u) - w_{B(u,4r)}| \leqslant C^3\varepsilon \quad \text{for every } u \in X \text{ if } 0 < r \leqslant \delta_0/4. \tag{4.5}$$

Because $w\bar{w} = 1$ on X, for any non-empty open ball B we have

$$\begin{aligned}|1 - |w_B|| \leqslant |1 - |w_B|^2| &= \frac{1}{\mu(B)} \int_B |1 - |w_B|^2|\, d\mu \\ &= \frac{1}{\mu(B)} \int_B |(w - w_B)\bar{w} + w_B(\bar{w} - \bar{w}_B)|\, d\mu \leqslant \frac{2}{\mu(B)} \int_B |w - w_B|\, d\mu.\end{aligned} \tag{4.6}$$

Clearly, (4.4) follows from (4.5), (4.6) and the assumption $w \in \mathrm{VMO}(X, d, \mu)$. □

The rest of the section concerns the maximal ideal space of $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$. For convenience let us denote $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$ by $\mathcal{Q}$, which is a C^*-subalgebra of $L^\infty(X, \mu)$. Let Y be the maximal ideal space of $\mathcal{Q}$. For each $g \in \mathcal{Q}$, let $\hat{g}$ denote its Gelfand transform, which is an element in $C(Y)$. By the Riesz representation theorem, there is a regular Borel measure $\hat{\mu}$ on Y such that

$$\int_X g\, d\mu = \int_Y \hat{g}\, d\hat{\mu}, \quad g \in \mathcal{Q}.$$

Clearly, if V is a non-empty open set in Y, then $\hat{\mu}(V) > 0$. In the special case where X is the unit circle, Wolff showed in [16] that a closed set F in Y has the property that $\hat{\mu}(F) = 0$ if and only if F is nowhere dense. The same result holds in the setting of any $\{X, d, \mu\}$ satisfying (1.1).

Proposition 4.3. *If F is a closed set in Y, then $\hat{\mu}(F) = 0$ if and only if F is nowhere dense in Y.*

Proof. Let F be a nowhere dense closed set in Y. If $\hat{\mu}(F) > 0$, then the measure $\hat{\mu}_F(\Delta) = \hat{\mu}(\Delta \cap F)$ would have a non-empty compact support K. That is, $F \supset K, \hat{\mu}(K) > 0$, and if U is open in Y such that $U \cap K \neq \emptyset$, then $\hat{\mu}(U \cap K) = \hat{\mu}_F(U) > 0$. We will complete the proof by deducing a contradiction from this. Of course, K is also nowhere dense.

Since $|\int_K \hat{g}\, d\hat{\mu}| \leqslant \int_Y |\hat{g}| d\hat{\mu} = \int_X |g|\, d\mu$ for every $g \in \mathcal{Q}$ and since $\mathcal{Q}$ is dense in $L^1(X, \mu)$, there is a $\chi \in L^\infty(X, \mu)$ such that

$$\int_X g\chi\, d\mu = \int_Y \hat{g}\chi_K\, d\hat{\mu}, \quad g \in \mathcal{Q}.$$

By Theorem 2, there is an $\eta \in \mathcal{Q}$ such that $0 \leqslant \eta \leqslant 1$ on $X, \eta\chi \in \mathcal{Q}$, and $\log \eta \in \mathrm{VMO}(X, d, \mu)$. For any $\delta > 0$, we have $\log(\eta + \delta) \in \mathcal{Q}$. Since the function $t \mapsto \log(t + \delta)$ is continuous on [0,1], $\log(\hat{\eta} + \delta)$ is the Gelfand transform of $\log(\eta + \delta)$. Thus

$$\int_X \log(\eta + \delta)\, d\mu = \int_Y \log(\hat{\eta} + \delta)\, d\hat{\mu}.$$

Let $Z = \{z \in Y : \hat{\eta}(z) = 0\}$. Since $\log \eta \in L^1(X, \mu)$, the above implies $\hat{\mu}(Z) = 0$. We have

$$\int_Y \hat{g}\hat{\eta}\chi_K\, d\hat{\mu} = \int_X g\eta\chi\, d\mu = \int_Y \hat{g}\widehat{\eta\chi}\, d\hat{\mu}$$

for every $g \in \mathcal{Q}$. Therefore, there is a Borel set $N \subset Y$ with $\hat{\mu}(N) = 0$ such that $\hat{\eta}\chi_K = \widehat{\eta\chi}$ on $Y \backslash N$. In particular, $\hat{\eta}\chi_K$ is continuous on $Y \backslash N$.

Since $\hat{\mu}(Z) = 0$, $Y \backslash Z = \bigcup_{n=1}^\infty \{y \in Y : \hat{\eta}(y) > 2^{-n}\}$, and since $\hat{\mu}(K) > 0$, there exists a $k \in \mathbf{N}$ such that the set $W = \{y \in Y : \hat{\eta}(y) > 2^{-k}\}$ has the property that $\hat{\mu}(W \cap K) > 0$. Therefore $\hat{\mu}((W \cap K) \backslash N) > 0$. Let $y \in (W \cap K) \backslash N$ and let V be an arbitrary open neighborhood of y. Now $W \cap V$ is open and $W \cap V \cap K \neq \emptyset$. Hence $\hat{\mu}((W \cap V \cap K) \backslash N) = \hat{\mu}(W \cap V \cap K) > 0$. Since K is nowhere dense, we also have $(W \cap V) \backslash K \neq \emptyset$, which gives us $\hat{\mu}(((W \cap V) \backslash K) \backslash N) = \hat{\mu}((W \cap V) \backslash K) > 0$. In particular, both $(W \cap V \cap K) \backslash N$ and $((W \cap V) \backslash K) \backslash N$ are non-empty. On the other hand, $\hat{\eta}\chi_K \geqslant 2^{-k}$ on $(W \cap V \cap K) \backslash N$ and $\hat{\eta}\chi_K = 0$ on $((W \cap V) \backslash K) \backslash N$. Since V is an arbitrary neighborhood of y and since $y \in Y \backslash N$, this contradicts the conclusion of the preceding paragraph. $\square$

5. A conditional expectation

We will now focus on the setting in Example 1.1. For the rest of the paper, n will be an integer greater than or equal to 2, and d will denote the metric on the unit sphere $S^{2n-1} = \{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$ defined by the formula

$$d(a,b) = |1 - \langle a,b \rangle|^{1/2}, \quad a,b \in S^{2n-1}. \tag{5.1}$$

Recall that σ denotes the probability measure on S^{2n-1} which is invariant under the orthogonal group $O(2n)$. For the rest of the paper, the space $L^p(S^{2n-1}, \sigma)$ will simply be denoted by $L^p(S^{2n-1}), 1 \leqslant p \leqslant \infty$.

Let T^n be the natural representation of the n-dimensional torus in $\mathbf{C}^n$. That is,

$$T^n = \{(\tau_1, \dots, \tau_n) \in \mathbf{C}^n : |\tau_1| = \cdots = |\tau_n| = 1\}.$$

Recall that, with the usual componentwise multiplication, T^n is a compact topological group. Let m_n be the Haar measure on T^n normalized so that $m_n(T^n) = 1$. For each $\tau = (\tau_1, \dots, \tau_n) \in T^n$, define the unitary transformation U_τ on $\mathbf{C}^n$ by the formula

$$U_\tau(z_1, \dots, z_n) = (\tau_1 z_1, \dots, \tau_n z_n), \quad (z_1, \dots, z_n) \in \mathbf{C}^n.$$

Since $C(S^{2n-1})$ is dense in $L^1(S^{2n-1})$, it is easy to see that, for any $\varphi \in L^\infty(S^{2n-1})$ and $f \in L^1(S^{2n-1})$, the function

$$\tau \mapsto \int f(z)\varphi(U_\tau z) d\sigma(z)$$

is continuous on T^n. Thus the right-hand side of the above can be integrated over T^n against dm_n, and it is obvious that the absolute value of this integral is dominated by $||\varphi||_\infty ||f||_1$. Therefore, by the duality between L^∞ and L^1, for each $\varphi \in L^\infty(S^{2n-1})$, there exists a unique $E(\varphi) \in L^\infty(S^{2n-1})$ such that

$$\int fE(\varphi)\, d\sigma = \int \left\{ \int f(z)\varphi(U_\tau z)\, d\sigma(z) \right\} dm_n(\tau), \quad f \in L^1(S^{2n-1}). \tag{5.2}$$

The function $E(\varphi)$ should be thought of as a conditional expectation of φ, and we obviously have $||E(\varphi)||_\infty \leqslant ||\varphi||_\infty$. Our proof of Theorem 3 is based on certain properties of this conditional expectation, which we will now establish.

Lemma 5.1. *If φ belongs to* $\mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$, *then so does $E(\varphi)$.*

Proof. Let $\zeta \in S^{2n-1}$ and $r > 0$ be given and let B denote $B(\zeta, r)$. It follows from (5.2) and the unitary invariance of σ that

$$E(\varphi)_B = \int \left\{ \frac{1}{\sigma(B)} \int_B \varphi \circ U_\tau d\sigma \right\} dm_n(\tau) = \int \varphi_{U_\tau B}\, dm_n(\tau)$$

for any $\varphi \in L^\infty(S^{2n-1})$. For such a pair of φ and B, there is an f with $|f| = \chi_B/\sigma(B)$ such that $(E(\varphi) - E(\varphi)_B)f = |E(\varphi) - E(\varphi)_B|\chi_B/\sigma(B)$. Hence,

$$\frac{1}{\sigma(B)}\int_B |E(\varphi) - E(\varphi)_B| d\sigma = \int (E(\varphi) - E(\varphi)_B) f d\sigma$$
$$\leqslant \int \left\{ \frac{1}{\sigma(U_\tau B)} \int_{U_\tau B} |\varphi - \varphi_{U_\tau B}| d\sigma \right\} dm_n(\tau). \quad (5.3)$$

By (5.1), if U is a unitary transformation on $\mathbf{C}^n$, then $UB = B(U\zeta, r)$. Thus the conclusion of the lemma follows from (5.3). □

Let $q \in H^\infty(S^{2n-1})$. Then $\lim_{r\uparrow 1} q(rw) = q(w)$ for σ-a.e. $w \in S^{2n-1}$ [11, Theorem 5.6.4]. Hence the Cauchy integral formula (see [11, p. 39])

$$q(z) = \int \frac{q(w)}{(1 - \langle z, w\rangle)^n} d\sigma(w), \quad |z| < 1,$$

holds. Now $(1-u)^{-n} = \sum_{m=0}^\infty \binom{m+n-1}{m} u^m$ if $|u| < 1$. Thus the Taylor expansion for q can be expressed as

$$q(z) = \sum_{m=0}^\infty \binom{m+n-1}{m} \int (\langle z, w\rangle)^m q(w) d\sigma(w). \quad (5.4)$$

For each tuple $k = (k_1, \ldots, k_n)$ of non-negative integers, define the analytic function

$$e_k(z_1, \ldots, z_n) = z_1^{k_1} \ldots z_n^{k_n}.$$

Write $|k| = k_1 + \cdots + k_n$ for such a tuple as usual. Then we can rewrite (5.4) as

$$q(z) = \sum_{m=0}^\infty \left\{ \sum_{|k|=m} a_k e_k(z) \right\}, \quad (5.5)$$

where the a_k's are the Taylor coefficients for q. Form (5.4) we see that

$$\left| \sum_{|k|=m} a_k e_k(z) \right| \leqslant \binom{m+n-1}{m} |z|^m \|q\|_\infty. \quad (5.6)$$

Lemma 5.2. *Let $q \in H^\infty(S^{2n-1})$. Then for any tuple $k = (k_1, \ldots, k_n)$ of non-negative integers we have*

$$E(q\bar{e}_k) = a_k |e_k|^2,$$

where a_k is the Taylor coefficient of q in (5.5). In particular, $E(q\bar{e}_k) \in C(S^{2n-1})$ whenever $q \in H^\infty(S^{2n-1})$.

Proof. Let $k' = (k'_1, \dots, k'_n)$ also be an n-tuple of non-negative integers. Then

$$e_{k'}(U_\tau z)\bar{e}_k(U_\tau z) = \tau_1^{k'_1 - k_1} \cdots \tau_n^{k'_n - k_n} e_{k'}(z)\bar{e}_k(z), \quad \tau = (\tau_1, \dots, \tau_n).$$

Hence $E(e_{k'}\bar{e}_k) = 0$ if $k' \neq k$ and $E(e_k\bar{e}_k) = |e_k|^2$. Combining this with (5.5) and (5.6), it is clear that if we define $q_r(z) = q(rz)$ for any given $0 < r < 1$, then $E(q_r\bar{e}_k) = a_k r^{|k|}|e_k|^2$. Because $q_r \to q$ σ-a.e. as $r \uparrow 1$ [11, Theorem 5.6.4], applying the dominated convergence theorem twice, we have $E(q_r\bar{e}_k) \to E(q\bar{e}_k)$ in the weak-* topology on $L^\infty(S^{2n-1})$. On the other hand, $a_k r^{|k|}|e_k|^2 \to a_k|e_k|^2$ as $r \uparrow 1$. Hence $E(q\bar{e}_k) = a_k|e_k|^2$. $\square$

For the function g promised in Theorem 3, we simply define

$$g(z_1, \dots, z_n) = \chi_{[0,(2n^n)^{-1}]}(|z_1|^2 \cdots |z_n|^2), \quad (z_1, \dots, z_n) \in S^{2n-1}. \tag{5.7}$$

Thus g has only two values on S^{2n-1}, 0 and 1. Clearly, $g^{-1}\{1\}$ contains a neighborhood of $(0, (n-1)^{-1/2}, \dots, (n-1)^{-1/2})$ and $g^{-1}\{0\}$ contains a neighborhood of $(n^{-1/2}, \dots, n^{-1/2})$. The point is that $\sigma(g^{-1}\{1\}) \neq 0$ and $\sigma(g^{-1}\{0\}) \neq 0$. Thus g is a *non-trivial projection* in C^*-algebraic terms. It is fairly easy to see directly that $g \notin \mathrm{VMO}(S^{2n-1}, d, \sigma)$. But we will state this conclusion as a consequence of the following general proposition, which, surprisingly, does not appear to be available in the literature.

Proposition 5.3. *Suppose that the compact metric space (X, d) is connected. Then the C^*-algebra $\mathrm{VMO}(X, d, \mu) \cap L^\infty(X, \mu)$ contains no non-trivial projections.*

Proof. Suppose that there were a Borel set G in X with $\mu(G) > 0$ and $\mu(X \backslash G) > 0$ such that $\chi_G \in \mathrm{VMO}(X, d, \mu)$. By (4.1), there would be a $\delta_1 > 0$ such that

$$|(\chi_G)_{r/4}(u) - (\chi_G)_{B(u,r)}| \leqslant 1/10 \quad \text{for all } u \in X \text{ and } 0 < r \leqslant \delta_1. \tag{5.8}$$

(See the beginning of Section 4 for the definition of f_r.) We will complete the proof by showing that this leads to a contradiction.

Let L be the set of Lebesgue points for χ_G. Proposition 2.2 asserts that $\mu(X \backslash L) = 0$. Since $\mu(G) > 0$ and $\mu(X \backslash G) > 0$, we can pick a $w \in L \cap G$ and a $z \in L \backslash G$. That is,

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(w,r))} \int_{B(w,r)} |\chi_G - 1| \, d\mu = 0 = \lim_{r \downarrow 0} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} |\chi_G - 0| \, d\mu.$$

Thus there exists a $\delta_2 > 0$ such that

$$(\chi_G)_{B(w,r)} \geqslant 9/10 \quad \text{and} \quad (\chi_G)_{B(z,r)} \leqslant 1/10 \quad \text{if } 0 < r \leqslant \delta_2. \tag{5.9}$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Combining (5.8) and (5.9), we find that $(\chi_G)_{r/4}(X) \cap [4/5, 1] \neq \emptyset$ and $(\chi_G)_{r/4}(X) \cap [0, 1/5] \neq \emptyset$ if $0 < r \leqslant \delta$. But $(\chi_G)_{r/4}$ is a continuous function on X for any $r>0$. Because X is connected, we conclude that

$$(\chi_G)_{r/4}(X) \supset [1/5, 4/5] \quad \text{if } 0<r\leqslant\delta.$$

Thus for each $0<r\leqslant\delta$, there is an $x(r)\in X$ such that $(\chi_G)_{r/4}(x(r)) = 1/2$. By (5.8), we have $|(1/2) - (\chi_G)_{B(x(r),r)}|\leqslant 1/10$. Hence

$$\frac{1}{\mu(B(x(r),r))}\int_{B(x(r),r)} |\chi_G - (\chi_G)_{B(x(r),r)}|\, d\mu \geqslant \frac{1}{\mu(B(x(r),r))}\int_{B(x(r),r)} \left\{\left|\chi_G - \frac{1}{2}\right| - \left|\frac{1}{2} - (\chi_G)_{B(x(r),r)}\right|\right\} d\mu \geqslant \frac{1}{2} - \frac{1}{10}$$

for all $0<r\leqslant\delta$. This contradicts the assumption that $\chi_G\in \mathrm{VMO}(X,d,\mu)$. $\square$

Let η be a *non-negative* continuous function on $[0,\infty)$ such that $\eta = 1$ on $[0,(4n^n)^{-1}]$ and $\eta = 0$ on $[(3n^n)^{-1},\infty)$. Define

$$\psi(z_1,\dots,z_n) = \eta(|z_1|^2\dots|z_n|^2), \quad (z_1,\dots,z_n)\in S^{2n-1}.$$

By design, $\eta\chi_{[0,(2n^n)^{-1}]} = \eta$ on $[0,\infty)$. Thus $\psi g = \psi\in C(S^{2n-1})$. Clearly,

$$|e_k(z_1,\dots,z_n)|^2 \geqslant (|z_1|^2\dots,|z_n|^2)^{|k|}, \quad (z_1,\dots,z_n)\in S^{2n-1},$$

for any given tuple k of non-negative integers. Therefore

$$\inf_{z\in S^{2n-1}} (\psi(z) + |e_k(z)|^2) \geqslant \inf_{0\leqslant x\leqslant 1} (\eta(x) + x^{|k|}) > 0. \tag{5.10}$$

Proof of Theorem 3. Let g be the function defined by (5.7) and let $q\in H^\infty(S^{2n-1})$ be any function which is not the constant 0. We must show that $gq\notin H^\infty(S^{2n-1}) + \{\mathrm{VMO}(S^{2n-1},d,\sigma)\cap L^\infty(S^{2n-1})\}$. Assuming the contrary, there would be an $h\in H^\infty(S^{2n-1})$ and an $f\in \mathrm{VMO}(S^{2n-1},d,\sigma)\cap L^\infty(S^{2n-1})$ such that $gq = h+f$. We will complete the proof by showing that this leads to the contradiction $g\in \mathrm{VMO}(S^{2n-1},d,\sigma)$.

From (5.7) it is clear that $g(U_\tau z) = g(z)$ for all $\tau\in T^n$ and $z\in S^{2n-1}$. From this invariance property of g and the definition of the conditional expectation E it is easy to see that $E(g\varphi) = gE(\varphi)$ for every $\varphi\in L^\infty(S^{2n-1})$.

The assumption that q is not the constant function 0 implies that there is at least one non-zero coefficient in the Taylor expansion (5.5) of q. Let a_k denote such a non-zero coefficient. By Lemma 5.2, we have

$$a_k g|e_k|^2 = gE(q\bar{e}_k) = E(gq\bar{e}_k) = E((h+f)\bar{e}_k) = E(h\bar{e}_k) + E(f\bar{e}_k).$$

Lemma 5.2 also asserts that $E(h\bar{e}_k)\in C(S^{2n-1})$. Because $f\bar{e}_k\in \mathrm{VMO}(S^{2n-1},d,\sigma)\cap L^\infty(S^{2n-1})$, Lemma 5.1 tells us that $E(f\bar{e}_k)\in \mathrm{VMO}(S^{2n-1},d,\sigma)\cap L^\infty(S^{2n-1})$. Hence

$|e_k|^2 g = a_k^{-1}(E(h\bar{e}_k) + E(f\bar{e}_k)) \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$. As we mentioned in the paragraph preceding this proof, $\psi g = \psi \in C(S^{2n-1})$. Therefore

$$(\psi + |e_k|^2)g = \psi + |e_k|^2 g \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1}).$$

But, according to (5.10), $\psi + |e_k|^2$ is an invertible element in $C(S^{2n-1})$, which leads to the contradiction $g \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$ promised earlier. □

As we mentioned in the Introduction, Theorem 3 tells us that, even if one only requires b to be bounded and analytic, the relation $g = h/b$ in Theorem 4 cannot be generalized to S^{2n-1}. But since Blaschke products are unimodular, the equation $g = h/b$ in Theorem 4 can be alternately written as $g = h\bar{b}$. This raises the following natural question: If we only require b to be bounded and analytic, does this alternate version of Theorem 4 have a generalization to S^{2n-1}? More precisely, given a $g \in L^\infty(S^{2n-1})$, is it always possible to express it in the form of $g = h\bar{q}$ with $q \in H^\infty(S^{2n-1})$ and $h \in H^\infty(S^{2n-1}) + C(S^{2n-1})$? The answer is still negative. This fact can be deduced from [9] in the following way.

Let $\mathscr{L}$ be the C^*-subalgebra of $L^\infty(S^{2n-1})$ generated by $H^\infty(S^{2n-1})$. Because $\mathscr{L}$ contains all the analytic polynomials, it follows from the Stone–Weierstrass approximation theorem that $\mathscr{L} \supset C(S^{2n-1})$. Hence $\mathscr{L}$ also equals the C^*-subalgebra of $L^\infty(S^{2n-1})$ generated by $H^\infty(S^{2n-1}) + C(S^{2n-1})$. But it is well known that $\mathscr{L} \neq L^\infty(S^{2n-1})$ under the assumption $n \geqslant 2$ [9, Section 3]. If $g \in L^\infty(S^{2n-1}) \backslash \mathscr{L}$, then it cannot be expressed in the form of $h\bar{q}$ with $q \in H^\infty(S^{2n-1})$ and $h \in H^\infty(S^{2n-1}) + C(S^{2n-1})$.

6. Vanishing Mean Oscillation on S^{2n-1}

As usual, let P denote the orthogonal projection from $L^2(S^{2n-1})$ to the Hardy space $H^2(S^{2n-1})$. Recall that, with the normalization $\sigma(S^{2n-1}) = 1$,

$$k_z(\zeta) = (1 - |z|^2)^{n/2}/(1 - \langle \zeta, z \rangle)^n, \quad |z| < 1, |\zeta| = 1,$$

provides the normalized reproducing kernel for $H^2(S^{2n-1})$. That is, $\langle g, k_z \rangle = (1 - |z|^2)^{n/2} g(z)$ for every $g \in H^2(S^{2n-1})$ and $||k_z||^2 = 1$ [11, Section 3.2]. Because of the denominator of k_z, it is the metric d defined by (5.1), not the usual Euclidian metric, that is closely associated with the theory of Toeplitz operators and Hankel operators on S^{2n-1}. Of course, in the special case of $n = 1$, d^2 reverts to the Euclidian metric on the unit circle T. But when $n \geqslant 2$, the metric d is considerably different from the Euclidian metric in that the d-balls in S^{2n-1} are anisotropic when viewed from the center.

Recall that the well-known Calderón–Zygmund lemma also holds true on the unit sphere [11, Lemma 6.2.1]. Therefore the John–Nirenberg theorem (see [10] or [7, p. 230]) also holds for functions of bounded mean oscillation on S^{2n-1}.

Let φ_a be the Möbius transform defined by (2) on page 25 of [11], $|a|<1$. Then $\varphi_a \circ \varphi_a = id$ and we have the substitution formula

$$\int |k_a|^2 f \circ \varphi_a d\sigma = \int f d\sigma, \quad f \in L^1(S^{2n-1}), \tag{6.1}$$

for integration. See pages 26 and 45 in [11]. For $f \in L^2(S^{2n-1})$, define

$$\mathrm{Var}(f; z) = \int |k_z|^2 \left| f - \int |k_z|^2 f d\sigma \right|^2 d\sigma = ||fk_z||^2 - |\langle fk_z, k_z \rangle|^2.$$

For any $|z|<1$, the formula

$$(U_z g)(\zeta) = k_z(\zeta) f(\varphi_z(\zeta)), \quad f \in L^2(S^{2n-1}),$$

defines a unitary operator on $L^2(S^{2n-1})$. From (6.1) and the identity $\varphi_z \circ \varphi_z = id$ it is easy to deduce that $U_z = U_z^{-1}$. Obviously, $H^2(S^{2n-1})$ is invariant under U_z. Hence $[P, U_z] = 0$. Thus, if $f \in L^2(S^{2n-1})$, then

$$||(1-P)(f \circ \varphi_z)|| = ||U_z(1-P)(f \circ \varphi_z)|| = ||(1-P)U_z(f \circ \varphi_z)|| = ||(1-P)(fk_z)||. \tag{6.2}$$

This identity is in fact well known, so is the following:

$$||(1-P)f||^2 \leqslant \mathrm{Var}(f; 0) \leqslant ||(1-P)f||^2 + ||(1-P)\bar{f}||^2. \tag{6.3}$$

Indeed $\mathrm{Var}(f; 0) = ||(1-P)f||^2 + (||Pf||^2 - |\langle f, 1 \rangle|^2)$. Let $\bar{P}$ (resp. P_0) be the orthogonal projection from $L^2(S^{2n-1})$ to the subspace $\{\bar{g} : g \in H^2(S^{2n-1})\}$ (resp. $\mathbf{C}$). We have $0 \leqslant ||Pf||^2 - |\langle f, 1 \rangle|^2 = ||\bar{P}\bar{f}||^2 - |\langle \bar{f}, 1 \rangle|^2 = ||(\bar{P} - P_0)\bar{f}||^2 \leqslant ||(1-P)\bar{f}||^2$.

Recall that the Hankel operator $H_f : H^2(S^{2n-1}) \to L^2(S^{2n-1}) \ominus H^2(S^{2n-1})$ is defined by the formula $H_f g = (1-P)(fg)$. Note that this formula defines the so-called "big" Hankel operator, not to be confused with the "little" Hankel operator K_b considered in [3, Section 4]. By (6.1) and the identity $\varphi_z \circ \varphi_z = id$, we have $\mathrm{Var}(f \circ \varphi_z; 0) = \mathrm{Var}(f; z) = \mathrm{Var}(\bar{f}; z)$. Thus it follows from (6.2) and (6.3) that

$$\max\{||H_f k_z||^2, ||H_{\bar{f}} k_z||^2\} \leqslant \mathrm{Var}(f; z) \leqslant ||H_f k_z||^2 + ||H_{\bar{f}} k_z||^2, \quad |z|<1. \tag{6.4}$$

Zheng proved the following result on the compactness of H_f:

Theorem 5 (Zheng [17, Theorem 5]). *If $f \in \mathrm{BMO}(S^{2n-1}, d, \sigma)$, then the Hankel operator H_f is compact if and only if $\lim_{|z|\uparrow 1} ||H_f k_z|| = 0$.*

Combining Zheng's theorem with (6.4), we see that for any $f \in L^\infty(S^{2n-1})$, the Hankel operators H_f and $H_{\bar{f}}$ are both compact if and only if

$$\lim_{|z|\uparrow 1} \mathrm{Var}(f;z) = 0. \tag{6.5}$$

Approximating $|k_z|^2$ by characteristic functions of d-balls in S^{2n-1}, it follows from the John–Nirenberg theorem and a standard argument that (6.5) is equivalent to the condition $f \in \mathrm{VMO}(S^{2n-1}, d, \sigma)$. Hence H_f and $H_{\bar{f}}$ are both compact if and only if $f \in \mathrm{VMO}(S^{2n-1}, d, \sigma)$. We know that this statement characterizes the function algebra QC on T in the case $n = 1$. In this sense the function algebra $\mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$ is the right analog of QC on the sphere. This was what initially motivated our investigation of bounded functions of vanishing mean oscillation on spaces for which dyadic decomposition is not available. We will conclude the paper with a result about the Fredholm index of Toeplitz operators on $H^2(S^{2n-1})$. We remind the reader that we have been assuming $n \geqslant 2$ since the beginning of Section 5.

As usual, for any $f \in L^\infty(S^{2n-1})$, we let T_f denote the Toeplitz operator defined by the formula $T_f g = P(fg)$, $g \in H^2(S^{2n-1})$. Let us recall some well-known facts. Since S^{2n-1} is simply connected, every continuous map from S^{2n-1} to T lifts to a continuous map from S^{2n-1} to $\mathbf{R}$, the universal covering of T. Therefore, if φ is an invertible element in $C(S^{2n-1})$, then T_φ is a Fredholm operator of index 0. Furthermore, Davie and Jewell showed that if φ is an invertible element in $H^\infty(S^{2n-1}) + C(S^{2n-1})$, then T_φ is also a Fredholm operator of index 0 [6, Theorem 3.1]. On the other hand, Davie and Jewell also showed that $H^\infty(S^{2n-1}) + C(S^{2n-1})$ does not contain $\mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$ [6, Section 4]. The final result of the paper is

Theorem 6. *Suppose that $n \geqslant 2$. If φ is an invertible element in the C^*-algebra $\mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$, then T_φ is a Fredholm operator of index* 0.

If $f, g \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$, then $T_{fg} - T_f T_g = H_{\bar{f}}^* H_g$ is compact (see Theorem 5). Moreover, it is known that the essential spectrum of T_f contains the essential range of f [6, p. 359]. Hence if $\mathscr{A}$ is the C^*-algebra generated by the Toeplitz operators $\{T_f : f \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})\}$, then

$$\mathscr{A} = \{T_f + K : f \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1}),\ K \text{ is compact on } H^2(S^{2n-1})\}.$$

Thus Theorem 6 in fact says that if A is a Fredholm operator contained in $\mathscr{A}$, then $\mathrm{index}(A) = 0$. There are two main ingredients in the proof of Theorem 6: Corollary 4.2 and the following:

Lemma 6.1. *Given any $n \geqslant 2$, there is a constant $0 < C_{6.1} < \infty$ which depends only on n such that $\|H_f\| \leqslant C_{6.1} \|f\|_{\mathrm{BMO}}$ for every $f \in L^\infty(S^{2n-1})$.*

Because $H_f = [M_f, P]P$, Lemma 6.1 follows from [4, Theorem I]. Although the proof of [4, Theorem I] was carried out for commutators with kernel functions on $\mathbf{R}^n$, the authors of [4] pointed out at the beginning of Section 5 (pp. 630–631) that the same result is valid in our spherical setting. Also, the norm bound given in Lemma 6.1 was implicitly contained in Zheng's paper [17]. In fact one can even derive the norm bound in Lemma 6.1 from the compactness result in Theorem 5! The point is that if the desired $C_{6.1}$ did not exist, then there would be a sequence $\{f_k\}$ in $L^\infty(S^{2n-1})$ such that $\lim_{k\to\infty} \|f_k\|_{\mathrm{BMO}} = 0$ and such that $\|H_{f_k}\| = 1$ for every k. By Proposition 4.1, one can require the sequence $\{f_k\}$ to be contained in $C(S^{2n-1})$. From such a sequence one can construct an $f \in \mathrm{VMO}(S^{2n-1}, d, \sigma)$ for which H_f fails to be compact, contradicting Theorem 5. We leave this construction as an exercise to the interested reader. But here is a hint: $\text{s-lim}_{k\to\infty} H_{f_k} = 0$.

Proof of Theorem 6. Given an invertible $\varphi \in \mathrm{VMO}(S^{2n-1}, d, \sigma) \cap L^\infty(S^{2n-1})$, we have $\varphi = w|\varphi|$, where $w = \varphi/|\varphi|$ is a unimodular function in $\mathrm{VMO}(S^{2n-1}, d, \sigma)$. Since $T_\varphi = T_w T_{|\varphi|} + K$, where K is compact, and since $\mathrm{index}(T_{|\varphi|}) = 0$, it suffices to show that $\mathrm{index}(T_w) = 0$. By Corollary 4.2, there is a sequence $\{v_k\}$ of unimodular functions in $C(S^{2n-1})$ such that $\lim_{k\to\infty} \|w - v_k\|_{\mathrm{BMO}} = 0$. Lemma 6.1 tells us that

$$\lim_{k\to\infty} \|H_w - H_{v_k}\| = 0 = \lim_{k\to\infty} \|H_{\bar{w}} - H_{\bar{v}_k}\|. \tag{6.6}$$

Denote $A = T_w$ and $A_k = T_{v_k}$. We claim that

$$\lim_{k\to\infty} \|A_k^* A_k - A^* A\| = 0 = \lim_{k\to\infty} \|A_k A_k^* - AA^*\|. \tag{6.7}$$

Indeed because $|w|^2 = 1 = |v_k|^2$, we have

$$T_{v_k}^* T_{v_k} - T_w^* T_w = (T_{|w|^2} - T_{\bar{w}} T_w) - (T_{|v_k|^2} - T_{\bar{v}_k} T_{v_k}) = H_w^* H_w - H_{v_k}^* H_{v_k}.$$

Thus the first equality in (6.7) follows from the first equality in (6.6). The second identity in (6.7) follows similarly if we consider $\bar{w}$ and $\bar{v}_k$ in place of w and v_k.

As we have already mentioned, $\mathrm{index}(A_k) = 0$ by virtue of the fact that v_k has a logarithm in $C(S^{2n-1})$. To prove that $\mathrm{index}(A) = 0$, it suffices to show that $\mathrm{index}(A) = \mathrm{index}(A_k)$ when k is sufficiently large.

Since A is Fredholm, $0 \notin \mathrm{sp}_{\mathrm{ess}}(A^*A)$. Thus 0 is an isolated point in $\mathrm{sp}(A^*A)$, if it belongs to the spectrum of A^*A at all. The same is also true if we replace A^*A by AA^*. Therefore there is an $\varepsilon > 0$ such that $(0, 5\varepsilon) \cap \{\mathrm{sp}(A^*A) \cup \mathrm{sp}(AA^*)\} = \emptyset$. By (6.7), there is an $N > 0$ such that $[\varepsilon, 4\varepsilon] \cap \{\mathrm{sp}(A_k^*A_k) \cup \mathrm{sp}(A_kA_k^*)\} = \emptyset$ for all $k \geqslant N$.

Let η be a continuous function on $[0, \infty)$ such that $\eta = 1$ on $[0, 2\varepsilon]$ and $\eta = 0$ on $[3\varepsilon, \infty)$. Let $k \geqslant N$. Then $\eta(A_k^*A_k) = E_k[0, 2\varepsilon]$ and $\eta(A_kA_k^*) = F_k[0, 2\varepsilon]$, where E_k and F_k are the spectral resolution of $A_k^*A_k$ and $A_kA_k^*$ respectively. But, by the polar decomposition of $A_k, E_k(0, 2\varepsilon]$ and $F_k(0, 2\varepsilon]$ are unitarily equivalent. Therefore $\mathrm{index}(A_k) = \mathrm{rank}(\eta(A_k^*A_k)) - \mathrm{rank}(\eta(A_kA_k^*))$. Similarly, $\eta(A^*A)$ and $\eta(AA^*)$ are also orthogonal projections and $\mathrm{index}(A) = \mathrm{rank}(\eta(A^*A)) - \mathrm{rank}(\eta(AA^*))$. It follows from the continuity of η and (6.7) that $\lim_{k\to\infty}||\eta(A^*A)-\eta(A_k^*A_k)||=0=\lim_{k\to\infty}||\eta(AA^*)-\eta(A_kA_k^*)||$. Therefore $\text{index}(A)=\text{index}(A_k)$ when k is sufficiently large. □

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