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Constructing conformal maps of triangulated surfaces

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ABSTRACT

We describe a means of computing the uniformizing conformal map from a triangulated surface whose triangles are realized as Euclidean triangles in R^3 onto a fundamental domain in the unit disc $\mathbb{D}$, plane $\mathbb{C}$, or sphere S^2. Mapping such triangulated surfaces arises in a number of applications, such as conformal brain flattening. We use the circle packing technique of Bowers, Hurdal, Stephenson, et al., to first create a quasiconformal approximation to the conformal map; then we apply a discrete form of conformal welding to reduce the distortion and converge to the conformal map in the limit.

1. Introduction

Applications involving conformal mapping of physical surfaces have gained a great deal of interest in recent years. For example, Bowers, Hurdal, Stephenson, et al., have developed a technique for producing flat maps of the human brain using circle packings [6,17,19,31]. Using MRI data to collect data points representing the surface of the brain, they form the Delaunay triangulation and approximate the brain by a collection of Euclidean triangles. Laying aside the geometry of the triangulated surface, they use the combinatorial pattern of the triangles to create a circle packing sharing the same pattern, thus inducing a quasiconformal map.

We describe how to use the Bowers, Hurdal, Stephenson, et al., method as a starting point for creating a conformal map of any triangulated surface. By using a discrete version of conformal welding to re-create the geometry of the original surface, we can construct the conformal map.

After some initial background in Sections 2 and 3, we describe the conformal welding process in Section 4. Using our discrete analogue of conformal welding, we construct in Section 5 our approximating maps, and finally show they converge in Section 6.

2. Triangulated surfaces

Consider a triangulation $\mathcal{K}$ of orientable topological surface. By a *triangulated surface*, we mean a (non-self-intersecting) realization $|\mathcal{K}|$ of $\mathcal{K}$ in $\mathbb{R}^3$ in which each triangle of $\mathcal{K}$ corresponds to a Euclidean triangle. When we wish to emphasize the purely combinatorial nature of the underlying triangulation $\mathcal{K}$, we will refer to $\mathcal{K}$ as an *abstract triangulation*.

A triangulated surface is then a Riemann surface with a conical complex structure in which triangles are mapped to similar triangles in the plane and power maps are used to glue the Euclidean structures together at the vertices [2]. Such surfaces are used commonly as discretizations of physical surfaces where $\mathcal{K}$ is the Delaunay triangulation of points sampled from the physical surface.

If our triangulated surface is simply connected, the Uniformization Theorem implies it can be mapped conformally onto exactly one of the plane $\mathbb{C}$, the disc $\mathbb{D}$, or the Riemann sphere S^2. Non-simply connected surfaces can be cut along edges

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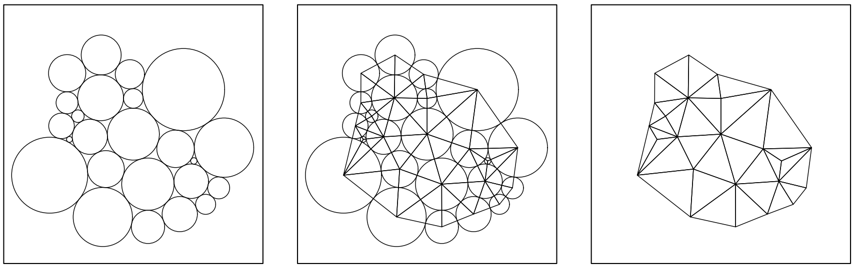

Fig. 1. A circle packing (left) with its carrier (center) and also the carrier alone (right).

of the triangles and lifted conformally to a fundamental region in their universal cover $\mathbb{C}$ or $\mathbb{D}$ (see Beardon and Stephenson's construction [3]). Thus we will focus only on mapping to the three simply connected surfaces. We first consider $\mathbb{D}$, then describe the modifications to the construction involved in mapping onto S^2 or $\mathbb{C}$.

3. Circle packing

3.1. Definitions

The first key tool in our construction is the realization of abstract triangulations via circle packings [29,31,32,37].

Definition 3.1. A *circle packing* is a configuration of circles with a prescribed pattern of tangencies. Typically this pattern is prescribed by means of an abstract triangulation; a circle packing P for an abstract triangulation $\mathcal{K}$ is a collection of circles such that

(1) P contains a circle $\mathcal{C}_v$ for each vertex v in $\mathcal{K}$,
(2) $\mathcal{C}_v$ is externally tangent to $\mathcal{C}_u$ if $[v,u]$ is an edge of $\mathcal{K}$,
(3) $\langle \mathcal{C}_v, \mathcal{C}_u, \mathcal{C}_w\rangle$ forms a positively oriented mutually tangent triple of circles if $\langle v, u, w\rangle$ is a positively oriented face of $\mathcal{K}$.

A packing is called *univalent* if none of its circles overlap, that is, if no pair of circles intersect in more than one point.

Notice that an abstract triangulation $\mathcal{K}$ is a combinatorial, not a geometric, object; however, if we have a packing P for $\mathcal{K}$, we can construct a geometric realization of $\mathcal{K}$ by connecting centers of tangent circles with geodesic segments. This geometric realization of $\mathcal{K}$ is called the *carrier* of P, denoted $\operatorname{carr} P$. If our packing lies in $\mathbb{C}$ or $\mathbb{D}$, the carrier will consist of Euclidean line segments. Bowers, Hurdal, Stephenson, et al. [6,17,32] have effectively used these embeddings created by packings to create quasiconformal maps of triangulated surfaces arising from MRI scans of human brains. See Fig. 1.

3.2. Existence

Koebe, Andreev, and Thurston independently proved the first existence results for finite simply connected packings [1, 16,33]; Beardon and Stephenson's Discrete Uniformization Theorem [3] extends their results to the existence of infinite packings and, as a consequence, packings on Riemann surfaces.

Discrete Uniformization Theorem. *Every abstract triangulation of a simply connected surface can be realized by a packing covering exactly one of S^2, $\mathbb{D}$, or $\mathbb{C}$. This packing is unique up to Möbius transformations.*

3.3. Approximation of conformal maps

The recent interest in circle packings was sparked by Thurston's realization [34] that packings exhibit a rigidity strikingly similar to conformal maps. He conjectured that maps constructed from hex packings (packings in which each interior circle has exactly 6 neighbors) converge to conformal maps under refinement. The conjecture was proved by Rodin and Sullivan [26] and soon extended to more general combinatorial patterns by many other authors [8,12,14,30,36].

Suppose a packing P for an abstract triangulation $\mathcal{K}$ (approximately) fills a simply connected region $\Omega \subsetneq \mathbb{C}$. If another packing $\widetilde{\mathcal{P}}$ for the same triangulation $\mathcal{K}$ is constructed in another region, typically $\mathbb{D}$ [7,28], then the carriers of P and $\widetilde{\mathcal{P}}$ will provide two embeddings of the same triangulation $\mathcal{K}$. We can then construct a piecewise affine *discrete analytic function* [32] sending triangles in the carrier of P to their counterparts in the carrier of $\widetilde{\mathcal{P}}$. Notice that the affine maps defined on adjacent triangles will agree on the common edge, so that the discrete analytic function will be continuous.

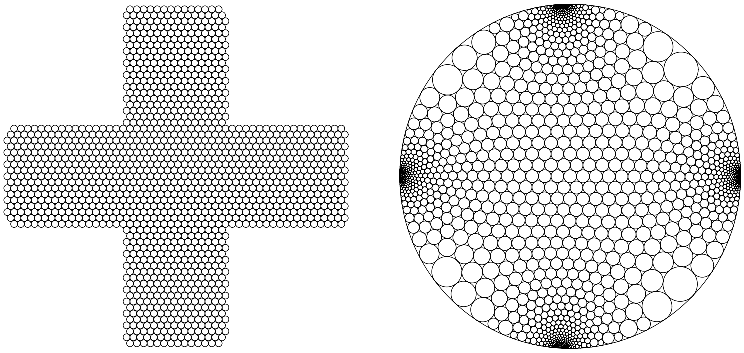

Fig. 2. Two packings with the same underlying triangulation. The induced discrete analytic function between them approximates the classical conformal map from the cross-shaped region (left) to the disc (right).

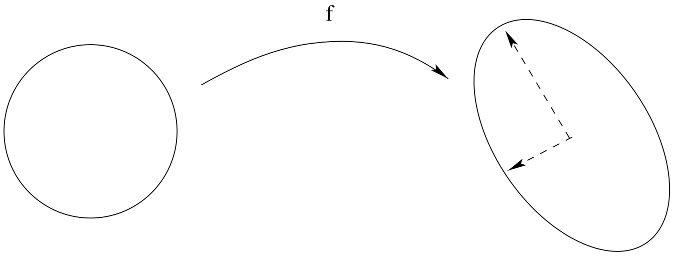

Fig. 3. A quasiconformal map f sends very small circles to very small nearly elliptical curves. The ratio of the lengths of the major and minor axes is the map's dilatation.

If the packings are refined so that they exactly fill Ω in the limit with a global bound on the degree of the triangulations and a suitably normalized sequence $\{f_n\}$ of such maps are constructed, then $\{f_n\}$ will converge locally uniformly to the similarly normalized conformal map from Ω to $\mathbb{D}$. See Fig. 2.

The first step in Rodin and Sullivan's proof was the observation that while each discrete analytic function f_n might not be conformal itself, each will be quasiconformal.

Definition 3.2. A *K-quasiconformal map* f is a homeomorphism whose dilatation

$$D_f(z) = \limsup_{r \to 0^+} \frac{\sup_\theta |f(z+re^{i\theta}) - f(z)|}{\inf_\theta |f(z+re^{i\theta}) - f(z)|}$$

is bounded above by K for all z.

Geometrically, the dilatation measures how far f distorts small circles, or equivalently, how much f distorts angles [20, 21]. A conformal map preserves angles, and is thus a 1-quasiconformal map. See Fig. 3.

Rodin and Sullivan discovered that the combinatorics of the circles force quasiconformal consequences.

Ring Lemma. *Suppose C_1 and C_2 are interior circles of a univalent packing. Then there is a universal bound on the ratio of the radius of C_1 to the radius of C_2, depending only on the number of neighbors of C_1 and C_2.*

The Ring Lemma implies that angles in the carriers of univalent packings cannot be too small or too large. Thus there is a bound, depending only on the degree of the packings, on how much a discrete analytic function can distort angles.

The next ingredient in Rodin and Sullivan's proof was their Length-Area Lemma, a result reminiscent of the classical length-area method of complex analysis, which allowed them to control the size of circles in the range. Together these lemmas imply that a subsequence of discrete analytic functions converge locally uniformly to a quasiconformal map f onto $\mathbb{D}$. Thurston's rigidity observation [34] then allowed them to show that f was indeed conformal, and the entire sequence f_n converges to f.

Hex Packing Lemma. *The angles in interior triangles of the carrier of a hex packing converge uniformly to $\pi/3$ as the number of generations of degree 6 circles surrounding them goes to infinity.*

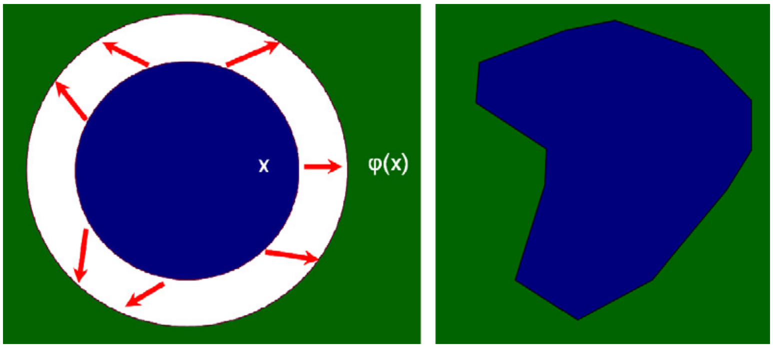

Fig. 4. A conformal welding is created by splitting the plane open along a Jordan curve and using a homeomorphism φ to weld the pieces back together (left). If a welding exists, the curve will be warped into a new shape.

Consequently, the triangles in the domain and range of f_n are converging locally uniformly to equilateral triangles, so the angular distortion disappears in the limit. There are similar results for more general combinatorics [11,13,22,30], but the hex result will be sufficient for our purposes and also has the advantage of being one of the few circle packing results with an estimate on the rate of convergence [24,25].

4. Conformal welding

Suppose $\varphi : \Gamma_1 \to \Gamma_1$ is an orientation preserving homeomorphism of the Jordan curve Γ_1. We say a *conformal welding* exists for φ if there are conformal maps f and g from the inside and outside of Γ_1, respectively, onto complementary Jordan domains so that

$$g = f \circ \phi$$

on Γ_1 [9,10,20,21].

One should envision splitting the plane open along Γ_1 and welding points x on one copy of Γ_1 to the corresponding points $\varphi(x)$ on the other copy. If a welding exists, the inside and outside of Γ_1 will push and pull against each other before settling into conformally compatible positions. The "seam" will be warped into a new Jordan curve Γ_2, whose geometric properties are determined by the analytic properties of φ. See Fig. 4.

The question of the existence of weldings and the geometry that emerges from weldings are very closely tied to quasiconformal maps [4,9,18,20,21].

Definition 4.1. A Jordan curve Γ is a *K-quasicircle* if it is the image of the unit circle under a K-quasiconformal map of $\mathbb{C}$ onto itself.

For example, Euclidean polygons, and triangles in particular, are quasicircles.

Definition 4.2. An orientation preserving homeomorphism $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ is *k-quasisymmetric* or a *k-quasisymmetry* if

$$\frac{1}{k} \leqslant \frac{|\varphi(I)|}{|\varphi(J)|} \leqslant k$$

for any two adjacent intervals (subarcs) I and J of $\partial\mathbb{D}$ having equal length $|I| = |J|$.

The deep connection between quasisymmetries and quasiconformal maps was discovered by Beurling and Ahlfors [4].

Beurling–Ahlfors Extension Theorem. *Every quasisymmetric $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ extends to a quasiconformal self-map of $\mathbb{D}$. Conversely, the boundary values of any quasiconformal self-map of $\mathbb{D}$ are quasisymmetric.*

It follows easily that quasisymmetric homeomorphisms of quasicircles extend quasiconformally inside the quasicircle. A bit deeper consequence is the Conformal Welding Theorem [20,21].

Conformal Welding Theorem. *If $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ is quasisymmetric, then there exists a conformal welding for φ and the resulting "seam" is a quasicircle. Conversely, every quasicircle arises as the "seam" for some quasisymmetry.*

Just as the Beurling–Ahlfors Theorem can be transferred to settings other than the unit disc, conformal weldings have been studied in a large variety of settings besides the unit disc [15,23,27].

5. Construction

5.1. Bowers–Hurdal–Stephenson flattenings

Suppose $|\mathcal{K}|$ is a triangulated surface in $\mathbb{R}^3$. We can assume $\mathcal{K}$ is simply connected; otherwise we could lift to the universal cover. The classical Uniformization Theorem then implies there exists a conformal map F from $|\mathcal{K}|$ onto either S^2, $\mathbb{C}$, or $\mathbb{D}$. If F is suitably normalized, then it will be unique. We will consider the disc case first.

Appealing to the Discrete Uniformization Theorem, $\mathcal{K}$ can be realized as the carrier of a packing P in either S^2, $\mathbb{C}$, or $\mathbb{D}$. If we normalize in the same manner as F, then P will be unique. The Bowers–Hurdal–Stephenson flattening is then easy to construct as the piecewise affine map from triangles in $|\mathcal{K}|$ to triangles in $\operatorname{carr} P \subsetneq \mathbb{D}$.

The Ring Lemma implies that angles in $\operatorname{carr} P$ are bounded away from 0 and π; thus there is a bound on the amount the piecewise affine map can distort angles. Consequently, the Bowers–Hurdal–Stephenson flattening is quasiconformal onto a subset of $\mathbb{D}$.

We seek the *conformal* mapping from $\mathcal{K}$ onto all of $\mathbb{D}$. This will require incorporating the geometry of $|\mathcal{K}|$, not just the combinatorics of $\mathcal{K}$, into our approximation. We need much more than angles bounded away from 0 and π; we need angles to match their counterparts in $|\mathcal{K}|$. It will also be necessary to refine our triangulation so as to force our circles to decrease in size and fill out all of $\mathbb{D}$ in the limit.

5.2. Piecewise conformal maps

The first step in our journey to construct the (normalized) conformal map from $|\mathcal{K}|$ to $\mathbb{D}$ is to observe that there is a conformal map between any two Euclidean triangles. Using the Schwarz–Christoffel formula, the upper half-plane $\mathbb{H}$ can be mapped onto any Euclidean triangle with 0, 1, and ∞ going to the vertices of the triangle. Thus by composing a Schwarz–Christoffel map and the inverse of a Schwarz–Christoffel map, any Euclidean triangle can be mapped to $\mathbb{H}$ then to any other Euclidean triangle with the vertices going to the vertices.

Of course, these piecewise conformal maps will not produce a global conformal map because (unlike piecewise affine maps) they will not agree on the edges of the triangulation. The remainder of the construction will thus be devoted to creating a welding which will permit these piecewise conformal maps to fit together.

5.3. Welding

Consider two adjacent triangles T_s and R_s sharing an edge E_s in the surface $|\mathcal{K}|$. There exist corresponding triangles T_p and R_p sharing an edge E_p in $\operatorname{carr} P \subset \mathbb{D}$ and conformal maps

$$t : T_s \to T_p, \qquad r : R_s \to R_p.$$

Unless both T_p and R_p are scaled copies of T_s and R_s, respectively, t and r will not agree on the common edge. Thus

$$\varphi = t \circ r^{-1}$$

will be a non-trivial homeomorphism of E_p onto itself. Since t and r are both conformal on the interior of T_s and R_s, respectively, and extend smoothly to the common edge except at the endpoints, then φ is C^∞ on E_p, except at the endpoints.

Remark 5.1. Notice that φ is independent of which triangle we chose to label T_p and which we chose to label R_p. Interchanging the roles of T_p and R_p will replace φ by its inverse.

Our goal now is to combinatorially weld T_p to R_p using φ. However, the edge E_p possesses no further combinatorial structure which can be welded together. Thus we hex refine T_p and R_p n times to break E_p into 2^n new edges [5]. These can be embedded as line segments along with the other edges in $\operatorname{carr} P$. Similarly, the original triangles in $|\mathcal{K}|$ can be hex refined and the new edges embedded as Euclidean line segments. All of the new vertices created by hex refinement have exactly 6 neighbors, while the number of neighbors of the original vertices remains unchanged. See Fig. 5.

Since φ need not map the new edges of T_p perfectly onto the new edges of R_p, we take a common refinement of both triangulations so that φ produces a one-to-one correspondence between the new vertices lying on T_p and the new vertices lying on R_p. See Fig. 6.

Since φ is C^∞ except at the endpoints, it is quasisymmetric. Working with quasisymmetric maps of $\partial\mathbb{D}$, we described in [35] a means to construct a lower bound on the size of angles in the refined triangulation and an upper bound on the degree by approximating φ if necessary and applying a standard diagonalization argument. A similar argument applies here. Now if we attach vertices to their images under φ we will have combinatorially welded the two copies of the edge E_p back together.

This process can be repeated for every edge in $\operatorname{carr} P$ to construct a new welded triangulation $\widetilde{\mathcal{K}}_n$. We compute a circle packing $\widetilde{\mathcal{P}}_n$ for $\widetilde{\mathcal{K}}_n$ in $\mathbb{D}$ with the same normalization as P. This “repacking” will now replicate the geometry of the original triangulated surface $\mathcal{K}$.

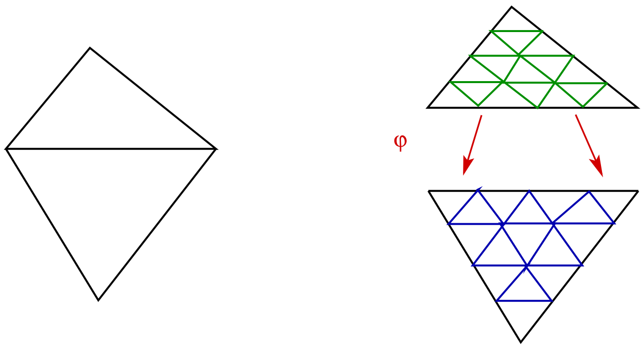

Fig. 5. Two adjacent triangles T_p and R_p (left) with a homeomorphism φ mapping their common edge E onto itself (right). After refining both triangles, we can combinatorially weld them together.

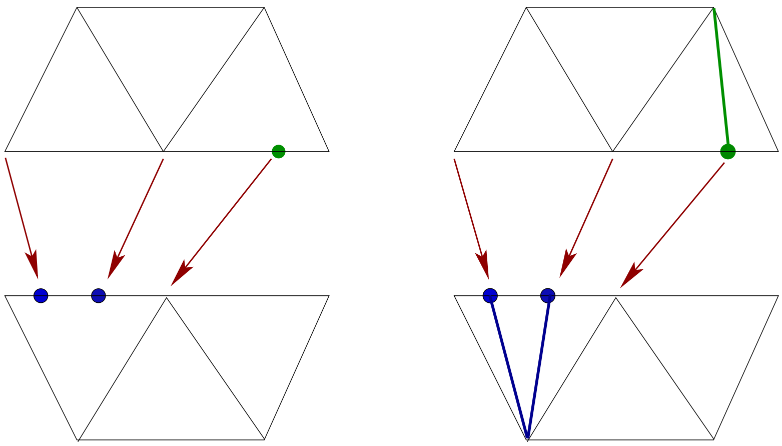

Fig. 6. We would not expect φ to map vertices on T_p exactly to vertices on R_p (left). Thus we may need to add new vertices to R_p and T_p to serve as images and pre-images, respectively, of the existing vertices under φ. We add new edges to complete the triangulation (right).

Moreover, each of the triangles in the refined carrier of P has a corresponding triangle in the carr $\widetilde{\mathcal{P}}$. Thus we define piecewise affine maps f_n and g_n from T_p and R_p to their corresponding regions $\widetilde{T}_p$ and $\widetilde{R}_p$, respectively, in carr $\widetilde{\mathcal{P}}$. Because the edge E_p was cut apart and welded back together, f_n and g_n will not agree on E_p; however, our construction ensures that at the new vertices v we added to E_p during refinement,

$$g_n(v) = f_n \circ \varphi(v) = f_n \circ t \circ r^{-1}(v). \tag{5.1}$$

Hence at the preimages w in $|\mathcal{K}|$ of the vertices v,

$$g_n \circ r(w) = f_n \circ t(w).$$

Thus on the inside of the triangles $T_s, R_s \subset |\mathcal{K}|$ and at the preimages w, the piecewise defined map

$$\begin{aligned} &f_n \circ t(z), \quad z \in T_s, \\ &g_n \circ r(z), \quad z \in R_s \end{aligned} \tag{5.2}$$

will be continuous.

For applications, we need only concern ourselves with convergence on the inside of each triangle and at these vertices. In practice, we can just approximate the images of points in $|\mathcal{K}|$ by the centers of circles in $\widetilde{\mathcal{P}}$; however, to prove convergence analytically, we will appeal to classical quasiconformal mapping results which require continuity throughout the domain. Unfortunately, despite the best efforts of our circles, the piecewise affine maps f_n and g_n interpolate linearly between the new vertices v, while the Schwarz Christoffel maps r and t are decidedly nonlinear. Thus on E_p between the new vertices $g_n \neq f_n \circ t \circ r^{-1}$, and a minor continuity correction will be required. See Fig. 7.

Define our continuity correction map u_n on E_p by

$$u_n = f_n^{-1} \circ g_n \circ r \circ t^{-1}.$$

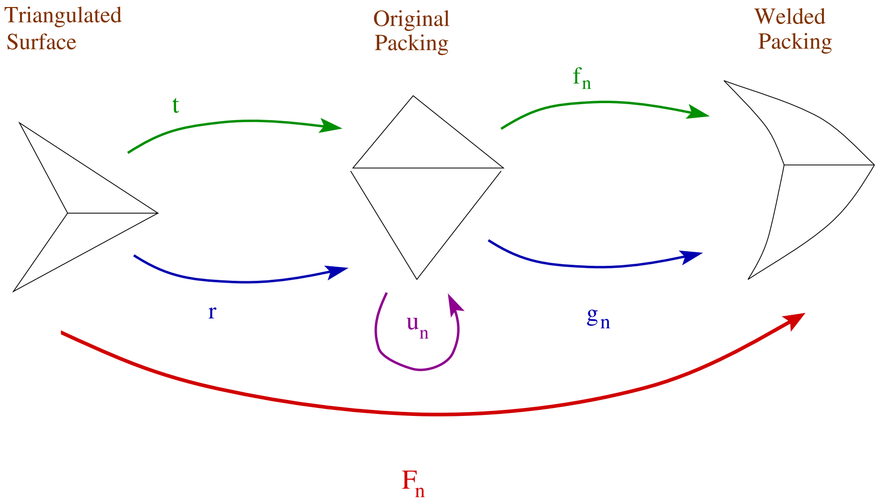

Fig. 7. An overview of the maps used in our construction. As $n \to \infty$, F_n will converge to the normalized conformal map of $|\mathcal{K}|$.

Notice at the vertices v added during refinement, u_n will be the identity map. In between these vertices, u_n will re-arrange the points so that their image under f_n will match their image under g_n. That is, if $z \in E_s \subset |\mathcal{K}|$,

$$f_n \circ u_n \circ t(z) = f_n \circ f_n^{-1} \circ g_n \circ r \circ t^{-1} \circ t(z) = g_n \circ r(z). \tag{5.3}$$

Since f_n and g_n are affine and r and t are C^∞ away from the original vertices of T_p where they act like power maps, u_n is k-quasiconformal, with k depending only on the difference between the corresponding angles of T_s and T_p.

Repeat this procedure throughout the triangulation, so that every for every triangle T_p there is a map u_n defined on all three edges and fixing the vertices. Notice that we can make this continuity correction by working on only one side; that is, if R_p is adjacent to T_p, we can take the correction map for R_p to be the identity on the edge shared with T_p. Since the number of triangles in $|\mathcal{K}|$ is finite, we can choose k large enough so that each u_n is k-quasisymmetric, where k is independent of both n and the triangle on which u_n is defined. Applying the Beurling–Ahlfors Extension Theorem and conjugating with a quasiconformal map [21], each u_n can then be extended to a K-quasiconformal map on the inside of its corresponding triangle, where K is independent of both n and the triangle on which it is defined.

Since each u_n fixes the vertices added during each refinement, at the nth stage there are 2^n evenly spaced fixed points for u_n. As $n \to \infty$, its regularity implies u_n converges locally uniformly to the identity map both on the boundary and, by noting that the Beurling–Ahlfors extension is defined by integration of the boundary map, on the inside as well.

Finally we construct our map from the original triangulated surface $|\mathcal{K}|$ to $\mathbb{D}$ piecewise on each triangle T_s by

$$F_n = f_n \circ u_n \circ t,$$

where t again is the Schwarz–Christoffel map from T_s to its corresponding triangle T_p in carr P.

6. Convergence

With all our careful effort to construct our approximations, we are now ready to show their convergence.

Theorem 6.1. *The sequence F_n converges locally uniformly to the conformal map F from $|\mathcal{K}|$ onto $\mathbb{D}$.*

Proof. First recall that each affine map f_n is quasiconformal, with dilatation depending only on the difference between angles in the domain and range triangles. Angles in the range are uniformly bounded away from 0 and π by the Ring Lemma since the degree of the welded triangulation is uniformly bounded by construction. Similarly, angles in the domain triangles were uniformly bounded away from 0 and π by our careful construction when refining and adding new vertices. Thus each f_n is uniformly quasiconformal, with dilatation independent of both n and, since the number of triangles is finite, the triangle on which it is defined.

Since each Schwarz–Christoffel map t is conformal and each correction map u_n is K-quasiconformal, the maps F_n are uniformly quasiconformal with dilatation independent of n. As we normalized the packings $\widetilde{\mathcal{P}}_n$ to match the normalization of F, we can appeal to standard quasiconformal normal families arguments to show the sequence F_n must have a convergent subsequence [20,21].

Next a standard application of the Rodin–Sullivan Length-Area Lemma [26,35] shows that the carrier of $\widetilde{\mathcal{P}}_n$ converges to all of $\mathbb{D}$ in the sense of Cartheodory as $n \to \infty$. Consequently our limit map will be onto all of $\mathbb{D}$, not just a subset.

Since we used hex refinement to generate our sequence, we can apply the Hex Packing Lemma [26] to show that the dilation of the affine maps f_n must decrease to 1 locally uniformly on the inside of each triangle. Since each Schwarz–Christoffel map t is already conformal, and the correction maps u_n converge to the identity as $n \to \infty$, the limit of our convergent subsequence must be conformal on the inside of each triangle of $|\mathcal{K}|$. The remaining portion of $|\mathcal{K}|$ consists of the one-dimensional edges and has measure 0. It is a well-know result that sets of measure 0 are removable for quasiconformal maps [21]; hence the limit map must be conformal everywhere.

Finally, since the normalized conformal map from $|\mathcal{K}|$ onto $\mathbb{D}$ must be unique, every convergent subsequence must converge to the same limit. As our normal families argument applies to every subsequence of F_n, the entire sequence must converge locally uniformly to the conformal map F. □

Our construction is easily modified to produce mappings onto S^2 if the original surface is spherical. Once we have created our packing P and normalized so that one vertex corresponds to ∞, we stereographically project to $\mathbb{C}$ and form the carrier using Euclidean triangles. The remainder of the welding construction continues as before, except for triangles containing the vertex at infinity, which can be dealt with by conjugating with a rotation of the sphere.

Conformal maps of surfaces consisting of infinitely many triangles as well as maps of (necessarily infinite) surfaces onto $\mathbb{C}$ can be obtained as the limit of conformal mappings of larger and larger finite pieces.

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