A REMARK ON CONVEX AND STARLIKE FUNCTIONS

D. R. WILKEN AND J. FENG

Let $K(a)$ and $St(\alpha)$ denote the usual families of convex and starlike functions of order α. In [3] and [4] A. Marx and E. Strohhäcker showed that if $f \in K = K(0)$ then $f \in St(\frac{1}{2})$, that is, $K(0) \subset St(\frac{1}{2})$. In [1] I. S. Jack posed the more general problem: What is the largest number $\beta = \beta(\alpha)$ such that $K(\alpha) \subset St(\beta)$? In [2] T. H. MacGregor determined the exact value of β for each α, $0 \leqslant \alpha < 1$, as the infimum over the disk $\Delta = \{|z| < 1\}$ of the real part of a specific analytic function. It has both been claimed and conjectured (see [1], [2]) that this infimum is attained on the boundary of Δ at $z = -1$ and thus one can obtain a relatively simple formula expressing β in terms of α. To the best of our knowledge no proof of this assertion appears in the literature. It is the purpose of this note to produce a proof of this assertion as a consequence of a somewhat more general result.

If we let

$$G(z) = \begin{cases} \dfrac{(2\alpha-1)z}{(1-z)^{2-2\alpha}[1-(1-z)^{2\alpha-1}]}, & \text{if } \alpha \neq \dfrac{1}{2}, \\[2ex] -\dfrac{z}{(1-z)\log(1-z)}, & \text{if } \alpha = \dfrac{1}{2}, \end{cases}$$

then the assertion is contained in the following theorem.

THEOREM. $\inf\limits_{|z|<1} \operatorname{Re} G(z) = G(-1)$, $0 \leqslant \alpha < 1$.

COROLLARY (MacGregor). *If* $0 \leqslant \alpha < 1$ *and* $f \in K(\alpha)$, *then* $f \in St(\beta)$, *where*

$$\beta = G(-1) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}[1-2^{2\alpha-1}]}, & \text{if } \alpha \neq \dfrac{1}{2}, \\[2ex] \dfrac{1}{2\log 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases}$$

For a proof of the corollary, see [2]. The theorem will follow from the lemmas below.

LEMMA 1. *Let $g(z)$ be analytic and* $\operatorname{Re} g(z) > 0$ *in Δ and let $g(-r)$ be real. Then the following two statements are equivalent.*

(A) $\dfrac{1}{\operatorname{Re} g(z)} \geqslant \dfrac{1}{g(-r)}$ *for* $|z| \leqslant r$.

(B) $|g(z) - g(-r)/2| \leqslant g(-r)/2$ *for* $|z| \leqslant r$.

Received 26 April, 1979.

[J. LONDON MATH. SOC. (2), 21 (1980), 287–290]

Proof.
$$|g(z)-g(-r)/2|^2 \leqslant g^2(-r)/4$$
$$\Leftrightarrow |g(z)|^2 - g(-r)\operatorname{Re} g(z) \leqslant 0$$
$$\Leftrightarrow \frac{1}{\operatorname{Re} g(z)} \geqslant \frac{1}{g(-r)}.$$

Let μ be a positive measure on the unit interval $[0, 1]$. Let $g(t, z)$ be a function analytic in Δ for each $t \in [0, 1]$, and integrable in t for each $z \in \Delta$ and for almost all $t \in [0, 1]$, and suppose that $\operatorname{Re} g(t, z) > 0$ on Δ, $g(t, -r)$ is real and $\operatorname{Re}\dfrac{1}{g(t, z)} \geqslant \dfrac{1}{g(t, -r)}$ for $|z| \leqslant r$ and $t \in [0, 1]$.

LEMMA 2. *Let* $g(z) = \int g(t, z)d\mu(t)$. *Then* $\operatorname{Re}\dfrac{1}{g(z)} \geqslant \dfrac{1}{g(-r)}$ *for* $|z| \leqslant r$.

Proof. It is immediate that $\operatorname{Re} q(z) > 0$, $g(-r)$ is real and, by Lemma 1,

$$\begin{aligned}|g(z)-\tfrac{1}{2}g(-r)| &= \left|\int g(t, z)d\mu(t) - \int \tfrac{1}{2}g(t, -r)d\mu(t)\right| \\ &\leqslant \int |g(t, z)-\tfrac{1}{2}g(t, -r)|d\mu(t) \\ &\leqslant \int \tfrac{1}{2}g(t, -r)d\mu(t) = \tfrac{1}{2}g(-r).\end{aligned}$$

Thus, again by Lemma 1, $\dfrac{1}{\operatorname{Re} g(z)} \geqslant \dfrac{1}{g(-r)}$ for $|z| \leqslant r$.

We wish to apply Lemma 2 to the function $G(z)$ given above, but first we make some simplifications. Let $\gamma = 2\alpha - 1$, $0 < \alpha < 1$, so that $-1 < \gamma < 1$ and let $G(z, \gamma)$ denote the appropriate function corresponding to γ. We can then write

$$G(z, \gamma) = \frac{z}{1-z}\,\frac{\gamma}{\dfrac{1}{(1-z)^\gamma} - 1}, \qquad \gamma \neq 0.$$

A simple computation shows that

$$G(z, \gamma) + \frac{\gamma z}{1-z} = G(z, -\gamma) \quad \text{for } |z| < 1, \quad 0 < \gamma < 1.$$

LEMMA 3. *If the theorem obtains for* $0 < \gamma < 1$, *then it obtains for* $-1 < \gamma < 1$.

Proof. Let $0 < \gamma < 1$. Then

$$\begin{aligned}\operatorname{Re} G(z, -\gamma) &= \operatorname{Re} G(z, \gamma) + \gamma \operatorname{Re}\frac{z}{1-z} \\ &\geqslant G(-1, \gamma) + \gamma(-\tfrac{1}{2}) = G(-1, -\gamma).\end{aligned}$$

The case when $\gamma = 0$ follows since $\lim_{\gamma \to 0} G(z, \gamma) = G(z, 0)$, uniformly in $|z| \leqslant r$, $0 < r < 1$.

Proof of the Theorem for $0 < \gamma < 1$. The theorem is known in the case where $\gamma = -1$ or $\alpha = 0$. Assume then that $0 < \gamma < 1$. The well-known relationship

$$\frac{1}{(1-z)^{\gamma}} = \frac{\Gamma(1)}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{-\gamma} \frac{1}{1-tz}\, dt, \quad 0 < \gamma < 1,$$

yields, after some elementary computations,

$$G(z,\gamma) = \frac{1}{\displaystyle\int_{[0,1]} \frac{1-z}{1-tz}\, d\mu(t)},$$

where

$$d\mu(t) = \frac{1}{\gamma} \frac{\Gamma(1)}{\Gamma(\gamma)\Gamma(1-\gamma)} t^{\gamma}(1-t)^{-\gamma} dt.$$

Let

$$g(t,z) = \frac{1-z}{1-tz} \quad \text{and} \quad g(z) = \int_{[0,1]} g(t,z)\, d\mu(t).$$

Then $\operatorname{Re} g(t,z) > 0, g\,(t, -r)$ is real and

$$\frac{1}{\operatorname{Re} g(t,z)} = \operatorname{Re} \frac{1-tz}{1-z}$$

$$\geqslant \frac{1+tr}{1+r} = \frac{1}{g(t,-r)} \quad \text{for} \quad |z| \leqslant r, 0 < r < 1, 0 \leqslant t \leqslant 1.$$

By Lemma 2 we have

$$\operatorname{Re} G(z,\gamma) \geqslant G(-r,\gamma) \quad \text{for} \quad |z| \leqslant r, 0 < r < 1$$

and thus

$$\operatorname{Re} G(z,\gamma) \geqslant G(-1,\gamma) \quad \text{for} \quad |z| < 1, 0 < \gamma < 1.$$

In closing let us note that the set which supports the measure μ in Lemma 2 is completely arbitrary. The choice of the unit interval was made for obvious reasons. Moreover the application of Lemma 2 in the proof of the theorem suggests the following as one class of functions from which to choose the family $g(t,z)$: those functions whose reciprocal $f(z)$ is convex, $\operatorname{Re} f(z) > 0, f(\bar{z}) = \overline{f(z)}$, and $f'(0) > 0$. Lemma 2 asserts that the reciprocal of the weighted sum of the reciprocals of such convex functions inherits the property that the minimum real part is attained at $z = -r$. It would be interesting to see further applications of Lemma 2 for the same or other classes of functions.

References

1. I. S. Jack, Functions starlike and convex of order α", *J. London Math. Soc.*(2), 3 (1971), 469–474.
2. T. H. MacGregor, "A subordination for convex functions of order α", *J. London Math. Soc.* (2), 9 (1975), 530–536.
3. A. Marx, "Untersuchungen über schlichte Abbildungen", *Math. Ann.*, 107 (1932/33), 40–67.
4. E. Strohhäcker, "Beiträge zur Theorie der schlichten Funktionen", *Math. Z.*, 37 (1933), 356–380.

State University of New York,
Albany,
N.Y. 12222,
U.S.A.