summen sind im ersten Falle für $0 < r < \frac{1}{4}$, *im zweiten Falle für* $0 < r < 1/\sqrt{3}$, "*konvex in der Richtung der imaginären Axe*".

Satz VI und VII sind Beiträge zum schönen und tiefliegenden Satz von Herrn Szegö*, nach welchem sämtliche Partialsummen einer beliebigen, reelle oder komplexe Koeffizienten besitzenden, für $|z| < 1$ regulären und schlichten Potenzreihe, schlicht sind im Kreise $|z| < \frac{1}{4}$; ihre zum Radius r gehörigen Kreisbilder sind weiter noch konvex für $0 < r < \frac{1}{4}$, wenn der Bildbereich der Potenzreihe konvex ist.

Die ausführliche Darstellung dieser und weiterer allgemeiner Sätze, und ihrer Anwendungen, möchte ich später veröffentlichen.

THE "SUM" OF A MEROMORPHIC FUNCTION

J. M. WHITTAKER†

1. The fundamental theorem of finite difference theory, due to Guichard‡, states that, if $f(z)$ is any given integral function, an integral function $g(z)$ exists such that

$$g(z+1)-g(z) = f(z). \tag{1.1}$$

The solution is not, of course, unique, since any function of period unity can be added to $g(z)$. Somewhat later Hurwitz§ showed that, if $f(z)$ is a meromorphic function, a meromorphic function $g(z)$ satisfying (1. 1) can be found.

The main result of a recent paper depended in part on the following extension of Guichard's theorem||.

THEOREM 1. *If* $f(z)$ *is an integral function of order* ρ, *there is an integral function* $g(z)$, *of order less than or equal to* $\max(\rho, 1)$, *satisfying* (1. 1).

A corresponding result for meromorphic functions is proved below.

THEOREM 2. *If* $f(z)$ *is a meromorphic function of order* ρ *whose poles are all on the same side of a line* $\mathfrak{R}z = d$, *there is a meromorphic function* $g(z)$, *of order less than or equal to* $\rho+1$, *satisfying* (1 . 1).

* G. Szegö, "Zur Theorie der schlichten Abbildungen", *Math. Annalen*, 100 (1928), 188–211.

† Received 7 January, 1933; read 19 January, 1933.

‡ Guichard, 2. Nörlund, 5, gives an account of the work of Guichard and his successors.

§ Hurwitz, 3. See also Barnes, 1.

|| Whittaker, 6, Theorem 1.

There is an essential difference between the two cases, due to the fact that the "sum" of a polynomial is a polynomial, whereas in general the "sum" of a rational function, *e.g.* z^{-1}, is a meromorphic function of order greater than or equal to 1.

Guichard and Hurwitz showed that various difference equations, in particular the equation

$$g(z+1) = f(z)\, g(z), \tag{1.2}$$

the general linear equation of the first order, and the linear equation of any order with constant coefficients, can be reduced to (1.1). The order of the solution of these equations can therefore be ascertained with the aid of Theorems 1 and 2. (1.2) has a solution of order less than or equal to $\rho+1$.

2. *Proof of Theorem* 2. Suppose, to start with, that the poles $b_1, b_2, \ldots$ of $f(z)$ are all in $\Re z \leqslant 0$. Define $\rho(k)$ for integral values of k by the equation

$$\max_{|z-k|=k-1} \overset{+}{\log} |f(z)| = k^{\rho(k)}, \tag{2.1}$$

and write

$$\sigma(k) = \max_{l \geqslant k} \rho(l) \geqslant \rho(k), \tag{2.2}$$

so that $\sigma(k)$ is a decreasing function. It will be shown that

$$\sigma = \lim_{k\to\infty} \sigma(k) \leqslant \rho. \tag{2.3}$$

In the z-plane draw the circle $|z| = 2k$ and the line $\Re z = 1$, and denote by C the closed contour consisting of the part of the line intercepted by the circle and the arc of the circle to the right of it. Then, if $\rho < a$,

$$M_k = \max_C |f(z)| < e^{k^a} \quad (k \geqslant k_a). \tag{2.4}$$

For*, if $z = re^{i\theta}$ and $R > r$,

$$\overset{+}{\log} |f(re^{i\theta})| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} |f(Re^{i\phi})| \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\phi-\theta)}\, d\phi + \sum_{|b_s|<R} \log \left| \frac{R^2-\bar{b}_s z}{R(z-\bar{b}_s)} \right|.$$

If z is on C, $\quad |R^2-\bar{b}_s z| \leqslant 2R^2, \quad |z-b_s| \geqslant 1,$

* Nevanlinna, 4, 25.

so that, in the usual Nevanlinna notation,

$$\overset{+}{\log}|f(re^{i\theta})| \leqslant \frac{R+r}{R-r} m(R, \infty)+n(R, \infty)\log 2R.$$

If $\rho<\beta<\alpha$,

$$m(R, \infty), \quad n(R, \infty)<R^{\beta} \quad (R \geqslant R_{\beta}),$$

so that, taking $R=3k$,

$$\overset{+}{\log} M_k \leqslant 4m(3k, \infty)+n(3k, \infty)\log 6k<k^{\alpha} \quad (k \geqslant k_{\alpha}),$$

which is (2.4).

Now the circle $|z-k|=k-1$ is enclosed by C, so that, for all k,

$$k^{\rho(k)} \leqslant \overset{+}{\log} M_k,$$

and hence
$$\rho(k)<\alpha \quad (k \geqslant k_{\alpha}).$$

Thus
$$\sigma=\overline{\lim_{k\to\infty}}\, \rho(k) \leqslant \alpha,$$

and, since α may be any number greater than ρ, this gives (2.3).

Next consider the polynomials

$$P_k(z)=\sum_{n=0}^{p_k}\frac{f^{(n)}(k)}{n!}z^n \quad (k \geqslant 1),$$

where p_k is to be chosen so that

(2.5)
$$|P_k(z)-f(z+k)|<\frac{1}{k^2}, \quad |z| \leqslant \tfrac{1}{3}(k-1).$$

Cauchy's inequality gives, for $n \geqslant 0$,

$$\frac{f^{(n)}(k)}{n!}(k-1)^n \leqslant \max_{|z-k|=k-1}|f(z)| \leqslant \exp\{k^{\sigma(k)}\},$$

so that, if $|z| \leqslant \frac{1}{3}(k-1)$,

$$|P_k(z)-f(z+k)| \leqslant \sum_{n=p_k+1}^{\infty}\frac{|f^{(n)}(k)|}{n!}\left(\frac{k-1}{3}\right)^n$$
$$\leqslant \tfrac{1}{2}3^{-p_k}\exp\{k^{\sigma(k)}\}<k^{-2},$$

provided that*

(2.6)
$$p_k=[k^{\sigma(k)}]+1.$$

(2.5) shows that the series

$$F(z)=\sum_{k=0}^{\infty}\{P_k(z)-f(z+k)\} \quad [P_0(z)=0]$$

converges uniformly in any finite region of the plane, neglecting a finite number of terms at the beginning. $F(z)$ is therefore a meromorphic function. It will be shown that its order does not exceed $\rho+1$.

* There are trivial modifications if $\sigma=0$. Take $p_k=\max\{[k^{\sigma(k)}]+1, 100\log k\}$.

Take a fixed integer l and, corresponding to a given value of $r = |z|$, define an integer $q_r = [3r]+1$. The inequality*

$$m(r, f_1+f_2+\ldots+f_q) \leqslant \sum_1^q m(r, f_s)+\log q$$

gives

$$(2.7) \quad m(r, F) \leqslant m\left\{r, \sum_{k=0}^{l} \{P_k(z)-f(z+k)\}\right\} + \sum_{k=l+1}^{q_r} m\{r, P_k(z)\}$$
$$+\sum_{k=l+1}^{q_r} m\{r, f(z+k)\}+m\left\{r, \sum_{k=q_r+1}^{\infty} \{P_k(z)-f(z+k)\}\right\}$$
$$+\log(2q_r+2).$$

Now, if $l+1 \leqslant k \leqslant q_r$, $r \geqslant 100$,

$$m\{r, P_k(z)\} \leqslant \overset{+}{\log} \sum_{n=0}^{p_k} \frac{|f^{(n)}(k)|}{n!} r^n$$
$$\leqslant \overset{+}{\log} \sum_{n=0}^{p_k} \left(\frac{r}{k-1}\right)^n \exp\{k^{\sigma(l)}\}$$
$$< \log p_k + p_k \log r + k^{\sigma(l)}$$
$$< 2q_r^{\sigma(l)} \log r,$$

and so

$$\sum_{k=l+1}^{q_r} m\{r, P_k(z)\} < 2q_r^{\sigma(l)+1} \log r < 2(4r)^{\sigma(l)+1} \log r.$$

Next, $f(z)$ is of the form $f_1(z)/f_2(z)$, where $f_1(z), f_2(z)$ are integral functions of order less than or equal to ρ. For the same range of k, the inequality

$$m(r, fg) \leqslant m(r, f)+m(r, g)$$

and Nevanlinna's form of Jensen's theorem† give

$$m\{r, f(z+k)\} \leqslant m\{r, f_1(z+k)\}+m\left\{r, \frac{1}{f_2(z+k)}\right\}$$
$$= m\{r, f_1(z+k)\}+m\{r, f_2(z+k)\}-N\left\{r, \frac{1}{f_2(z+k)}\right\}-\log|c_\lambda|$$
$$\leqslant m\{r, f_1(z+k)\}+m\{r, f_2(z+k)\}-\log|c_\lambda|$$
$$\leqslant \log M_1(4r+1)+\log M_2(4r+1)-\log|c_\lambda|$$
$$< r^{\rho+\epsilon} \qquad (r \geqslant r_\epsilon),$$

* Nevanlinna, 4, 14.

† Nevanlinna, 4, 14, 6.

where c_λ is a constant, $M_1(r)$, $M_2(r)$ are the maxima of $|f_1(z)|$, $|f_2(z)|$ for $|z| = r$, and ϵ is a given positive number.

Again, by (2.5),

$$m\left\{r, \sum_{k=q_r+1}^{\infty} \{P_k(z)-f(z+k)\}\right\} < \log \sum_{1}^{\infty} \frac{1}{k^2}.$$

Thus, as $r \to \infty$, the five terms on the right of (2.7) are respectively $O(r^{\rho+\epsilon})$, $O(r^{\sigma(l)+1} \log r)$, $O(r^{\rho+\epsilon+1})$, $O(1)$, $O(\log r)$.

Again, the poles of $F(z)$ are the points $b_n - m$ ($n \geqslant 1$, $m \geqslant 0$), so that

$$n\{r, \infty, F(z)\} \leqslant rn\{r, \infty, f(z)\}$$
$$< r^{\rho+\epsilon+1} \quad (r \geqslant r_\epsilon').$$

Hence
$$T(r, F) = m(r, F) + N(r, F)$$

consists of the five terms enumerated above together with a term $O(r^{\rho+\epsilon+1})$, and so

$$\varlimsup_{r\to\infty} \frac{\log T(r, F)}{\log r} \leqslant \max\{\sigma(l)+1, \rho+\epsilon+1\}.$$

Since $\sigma(l) \to \sigma \leqslant \rho$ and ϵ is arbitrary it follows that the order of $F(z)$ cannot exceed $\rho+1$.

Now, if N is any integer,

$$\Delta F(z) - f(z) = \sum_{k=0}^{N} \{P_k(z+1) - P_k(z)\} - f(z+N+1)$$
$$+ \sum_{k=N+1}^{\infty} \{P_k(z+1) - f(z+k+1)\}$$
$$- \sum_{k=N+1}^{\infty} \{P_k(z) - f(z+k)\},$$

and, if z is confined to any given finite region of the plane, the functions on the right are regular for sufficiently large values of N. Hence

$$\Delta F(z) - f(z) = h(z),$$

an integral function. Evidently the order of $h(z)$ cannot exceed $\rho+1$. Thus, by Theorem 1, there is an integral function $H(z)$ of order less than or equal to $\rho+1$ such that

$$\Delta H(z) = h(z),$$

and so
$$g(z) = F(z) - H(z)$$

is a meromorphic function of order less than or equal to $\rho+1$ satisfying (1.1).

3. If the poles of $f(z)$ are in $\Re z \leqslant d$, the theorem is proved by applying the result of the preceding section to $f(z+d)$. If they are in $\Re z \geqslant d$, the poles of $f(-z)$ are in $\Re z \leqslant -d$, and so there is a function $g_1(z)$, of order less than or equal to $\rho+1$, satisfying

$$g_1(z+1)-g_1(z)=f(-z);$$

and so

$$g(z)=-g_1(-z+1)$$

satisfies (1.1).

Any meromorphic function $f(z)$ can be expressed in the form

$$L_d(z)+R_d(z),$$

where the poles of $L_d(z)$ are in $\Re z \leqslant d$ and those of $R_d(z)$ in $\Re z > d$, by splitting the Mittag-Leffler expansion of $f(z)$ into two parts. Thus, if ρ_1, ρ_2 are the orders* of $L_d(z)$, $R_d(z)$, (1.1) has a solution of order less than or equal to $\max(\rho_1+1, \rho_2+1)$.

4. *The number $\rho+1$ in Theorem 2 is "best possible".* For let $a_1, a_2, \ldots$ be an increasing sequence of positive numbers with exponent of convergence ρ, satisfying the condition that no two a's differ by an integer, and let $f(z)$ be a meromorphic function of order ρ with poles at these points. It is easy to see that $g(z)$ must have poles either at all points a_n-m+1 $(m \geqslant 1)$, or else at all points a_n+m $(m \geqslant 1)$, according as a_n is taken to be a pole of $g(z)$ or of $g(z+1)$. Thus the order of $g(z)$ cannot be less than the exponent of convergence of the double sequence a_n+m $(n, m \geqslant 1)$. Now, if $\lambda > 0$,

$$\sum_{n,m=1}^{\infty}(a_n+m)^{-\lambda} > \sum_{n=1}^{\infty}\int_2^{\infty}(a_n+x)^{-\lambda}\,dx$$

$$> K\sum_{n=1}^{\infty} a_n^{1-\lambda},$$

so that for convergence we must have $\lambda \geqslant \rho+1$.

5. Hurwitz showed how to reduce (1.2) to (1.1), $f(z)$ being a meromorphic function. By making use of his process the following result is obtained.

* Is it always possible to choose d so that $\rho_1, \rho_2 \leqslant \rho$?

THEOREM 3. *If $f(z)$ is a meromorphic function of order ρ, there is a meromorphic function $g(z)$, of order less than or equal to $\rho+1$, satisfying* (1.2).

As before, the number $\rho+1$ is "best possible". $f(z)$ is of the form $f_1(z)f_2(z)$, the poles and zeros of $f_1(z)$ being in $\mathfrak{R}z \leqslant 0$, and those of $f_2(z)$ in $\mathfrak{R}z > 0$.

$f_1'(z)/f_1(z)$ is a meromorphic function, of order less than or equal to ρ, with simple poles in $\mathfrak{R}z \leqslant 0$, the residues at the poles being positive or negative integers. By Theorem 2 there is a function $h(z)$, of order less than or equal to $\rho+1$, such that

$$h(z+1)-h(z) = \frac{f_1'(z)}{f_1(z)}. \tag{5.1}$$

The method of constructing $h(z)$ ensures that its poles are also simple and the residues at them positive or negative integers. Hence, by integration,

$$h(z) = \frac{H'(z)}{H(z)},$$

where
$$H(z) = e^{F(z)} \frac{C_1(z)}{C_2(z)},$$

$F(z)$ being an integral function and $C_1(z)$, $C_2(z)$ canonical products. $C_1(z)$, $C_2(z)$ are of order less than or equal to $\rho+1$. For if one of them was of order greater than $\rho+1$, the exponent of convergence of its zeros, and hence the exponent of convergence of the poles of $h(z)$, would be greater than $\rho+1$. This is impossible. Now

$$f_1(z) = \exp\left\{\int \{h(z+1)-h(z)\}\,dz\right\}$$

$$= \frac{H(z+1)}{H(z)} = e^{F(z+1)-F(z)} \frac{C_1(z+1)\,C_2(z)}{C_1(z)\,C_2(z+1)},$$

and so $F(z+1)-F(z)$ is a polynomial of degree less than or equal to ρ.

Thus
$$F(z+1)-F(z) = P(z+1)-P(z),$$

where $P(z)$ is a polynomial of degree less than or equal to $\rho+1$, and

$$g_1(z) = e^{P(z)} \frac{C_1(z)}{C_2(z)}$$

is a function, of order less than or equal to $\rho+1$, such that

$$\frac{g_1(z+1)}{g_1(z)} = f_1(z).$$

$f_2(z)$ can be treated in the same way, and $g(z)$ is then the product of $g_1(z)$, $g_2(z)$.

6. It was shown by Guichard that the linear difference equation with constant coefficients

$$a_n g(z+n)+a_{n-1} g(z+n-1)+\ldots+a_0 g(z) = f(z) \tag{6.1}$$

can be reduced to the n equations

$$g(z+1)-\lambda_1 g(z) = g_1(z),$$
$$g_1(z+1)-\lambda_2 g_1(z) = g_2(z),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$g_{n-1}(z+1)-\lambda_n g_{n-1}(z) = f(z),$$

where λ_1, λ_2, ... are the roots of the equation

$$a_n \lambda^n + a_{n-1}\lambda^{n-1}+\ldots+a_0 = 0.$$

Moreover, the equation

$$G(z+1)-\lambda G(z) = F(z)$$

can be written $$H(z+1)-H(z) = \lambda^{-z-1} F(z),$$

where $$H(z) = \lambda^{-z} G(z).$$

Hence, if $f(z)$ is an integral function of order ρ, (6.1) has an integral solution $g(z)$ of order less than or equal to $\max(\rho, 1)$.

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