Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2012,**32B**(2):710–716
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

ILL-POSEDNESS OF MODIFIED KAWAHARA EQUATION AND KAUP-KUPERSHMIDT EQUATION*

Yan Wei (闫威) *Li Yongsheng* (李用声)
Department of Mathematics, South China University of Technology, Guangzhou 510640, China
E-mail: yanwei.scut@yahoo.com.cn; yshli@scut.edu.cn

Abstract In this article, we consider the Cauchy problems for the modified Kawahara equation

$$\partial_t u + \mu\partial_x(u^3) + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u = 0$$

and the Kaup-Kupershmidt equation

$$\partial_t u + \mu u\partial_x^2 u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u = 0.$$

Using the general well-posedness principle introduced by I. Bejenaru and T. Tao, we prove that the modified Kawahara equation is ill-posed for the initial data in $H^s(\mathbf{R})$ with $s < -\frac{1}{4}$ and that the Kaup-Kupershmidt equation is ill-posed for the initial data in $H^s(\mathbf{R})$ with $s < 0$.

Key words Ill-posedness; modified Kawahara equation; Kaup-Kupershmidt equation; general well-posedness principle

2000 MR Subject Classification 35E15; 35Q53

1 Introduction

This article is devoted to the ill-posedness of the initial value problems of the modified Kawahara equation

$$\partial_t u + \mu\partial_x(u^3) + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u = 0, \quad x, t \in \mathbf{R}, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

and the Kaup-Kupershmidt equation

$$\partial_t u + \mu u\partial_x^2 u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u = 0, \quad x, t \in \mathbf{R}, \tag{1.3}$$

*Received March 20, 2010. This work is supported by NNSFC under grant numbers 10771074 and 11171116; Yan is also supported in part by NNSFC under grant number 10801055 and the Doctoral Program of NEM of China under grant number 200805611026; Li is also supported in part by the Fundamental Research Funds for the Central Universities under the grant number 2012ZZ0072.

$$u(x,0) = u_0(x), \tag{1.2}$$

where α, β, and γ are real constants and $\alpha \neq 0$, μ is a complex number. (1.1) is the modified Kawahara equation, and (1.3) is called the Kaup-Kupershmidt equation, which was first proposed by Kaup and Kupershmidt in [12, 16]. The modified Kawahara equation appears in the study of water waves with surface tension, in which the Bona number takes on the critical value, where the Bona number represents a dismension-less magnitude of surface tension in the shallow water region, see [3, 11, 15]. The model of Kaup-Kupershmidt equation arose in the study of the capillary-gravity waves, see [1, 8]. In [6], using the $[k;Z]$-multiplier norm method proposed by [17] and Bourgain spaces introduced in [4] and developed in [13, 14], the authors proved that (1.1) is locally well-posed for the initial data in $H^s(\mathbf{R})$ with $s \geq -\frac{1}{4}$. In [10], using the Fourier restriction norm method, the authors established the local well-posedness of (1.1)–(1.2) for the initial data u_0 in $H^s(\mathbf{R})$ with $s \geq -\frac{1}{4}$ and of (1.3)–(1.2) for the initial data u_0 in $H^s(\mathbf{R})$ with $s > 0$. In [19], using the I-method, the authors proved that (1.1)–(1.2) is globally well-posed for the initial data u_0 in $H^s(\mathbf{R})$ with $s > -\frac{3}{22}$.

In this article, inspired by [2, 5, 7, 10, 18, 20], we prove that (1.1) is ill-posed for the initial data u_0 in $H^s(\mathbf{R})$ with $s < -\frac{1}{4}$ and that (1.3) is ill-posed for the initial data u_0 in $H^s(\mathbf{R})$ with $s < 0$. The result implies that $s = 0$ is critical for the local well-posedness of IVP of (1.3)–(1.2).

We give some notations and definitions before stating the main result. $\mathcal{F}_x u$ is the Fourier transform of u with respect to its space variable. $\mathcal{S}'(\mathbf{R})$ is the space of tempered distributions in $\mathbf{R}_x$. We define

$$H^s(\mathbf{R}) = \{u_0 \in \mathcal{S}'(\mathbf{R}) : \|u_0\|^2_{H^s(\mathbf{R})} = \int_{\mathbf{R}} \langle\xi\rangle^{2s} |\mathcal{F}_x u_0(\xi)|^2 \mathrm{d}\xi < \infty\},$$

where $s \in \mathbf{R}$. We define

$$W(t)u_0 = C \int_{\mathbf{R}} e^{\mathrm{i}x\xi} e^{\mathrm{i}t\phi(\xi)} \mathcal{F}_x u_0(\xi) \mathrm{d}\xi$$

where $\phi(\xi) = -\alpha\xi^5 + \beta\xi^3 - \gamma\xi$. C is a generic constant which may vary from line to line. We use $X \sim Y$ to denote $C_1|X| \leq |Y| \leq C_2|X|$, where C_1, C_2 are positive constants. We use $X \gg Y$ to denote $|X| > C_3|Y|$, where C_3 is some large positive constant.

The main results of this article are as follows.

Theorem 1.1 Let $s < -\frac{1}{4}$. Then, (1.1)–(1.2) is ill-posed in $H^s(\mathbf{R})$ in the sense that the solution map S_t of the Cauchy problem for (1.1) is not Lipschitz continuous at zero. More precisely, for any $T > 0$, the solution map

$$S_t : \quad u_0 \in H^s(\mathbf{R}) \longrightarrow u \in C([0,T]; H^s(\mathbf{R}))$$

is not Lipschitz continuous at zero.

Theorem 1.2 Let $s < 0$. Then, (1.3)–(1.2) is ill-posed in $H^s(\mathbf{R})$ in the sense that the solution map S_t of the Cauchy problem for (1.3) is not C^2 at zero. More precisely, for any $T > 0$, the solution map

$$S_t : \quad u_0 \in H^s(\mathbf{R}) \longrightarrow u \in C([0,T]; H^s(\mathbf{R}))$$

is not C^2 at zero.

The remainder of this article is organized as follows. Using the general well-posedness principle proposed by [2], we prove Theorem 1.1 and Theorem 1.2 in Section 2 and Section 3 respectively.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. On the contrary, we assume that the solution map of (1.1)–(1.2)

$$S_t: \quad u_0 \in H^s(\mathbf{R}) \longrightarrow u \in C([0,T]; H^s(\mathbf{R}))$$

is Lipschitz continuous at zero with $s < -\frac{1}{4}$. Then, from the general well-posedness principle of [2], we must have

$$\sup_{t\in[0,T]} \|B_3(u_0)\|_{H^s} \leq C\|u_0\|_{H^s}^3, \quad \text{for all } u_0 \in H^s(\mathbf{R}), \tag{2.1}$$

where

$$\begin{aligned} B_3(u_0)(x,t) &= \int_0^t W(t-\tau)\partial_x(B_1(u_0)B_1(u_0)B_1(u_0))(\tau)\mathrm{d}\tau, \\ B_1(u_0)(x,t) &= W(t)u_0. \end{aligned}$$

We consider the initial data

$$u_0(x) = r^{-1/2}N^{-s}\left\{e^{-\mathrm{i}Nx}\left(\int_0^r e^{\mathrm{i}x\xi}\mathrm{d}\xi\right) + e^{\mathrm{i}Nx}\left(\int_r^{2r} e^{\mathrm{i}x\xi}\mathrm{d}\xi\right)\right\},$$

which can be seen in [18], where $r^2N^3 = O(1)$ and

$$N \gg 16a = 16\max\left\{1, \left(2\left|\frac{3\beta}{5\alpha}\right|\right)^{1/2}, \left(\left|\frac{3\beta}{10\alpha}\right| + \left|\frac{2\gamma}{5\alpha} - \frac{1}{2}\left(\frac{3\beta}{5\alpha}\right)^2\right|^{1/2}\right)^{1/2}\right\},$$

where a is defined as in [10]. Thus, we have

$$\mathcal{F}_x u_0(\xi) = Cr^{-1/2}N^{-s}\left\{\chi_{[-N,-N+r]}(\xi) + \chi_{[N+r,N+2r]}(\xi)\right\},$$

where χ_I denotes the characteristic function of a set $I \subset \mathbf{R}$. It is checked that $\|u_0\|_{H^s} \sim 1$. Let $I_1 = [-N, -N+r]$ and $I_2 = [N+r, N+2r]$ and $\Omega_1 = I_1 \bigcup I_2$. We have

$$\mathcal{F}_x B_1(u_0)(\xi) = Ce^{\mathrm{i}t\phi(\xi)}\mathcal{F}_x u_0(\xi)$$

and thus,

$$B_1(u_0)(x,t) = Cr^{-1/2}N^{-s}\int_{\xi\in\Omega_1} e^{\mathrm{i}x\xi}e^{\mathrm{i}t\phi(\xi)}\mathrm{d}\xi,$$

$$B_3(u_0)(x,t) = Cf,$$

where

$$f = r^{-3/2}N^{-3s}\int_{\xi_1\in\Omega_1}\int_{\xi_2\in\Omega_1}\int_{\xi_3\in\Omega_1}(\xi_1+\xi_2+\xi_3)e^{\mathrm{i}x(\xi_1+\xi_2+\xi_3)}Q\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{2.2}$$

where

$$Q = \frac{e^{\mathrm{i}t(\phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3))}-e^{\mathrm{i}t\phi(\xi_1+\xi_2+\xi_3)}}{\phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3)-\phi(\xi_1+\xi_2+\xi_3)}.$$

As $\phi(\xi) = -\alpha\xi^5+\beta\xi^3-\gamma\xi$, we define

$$\begin{aligned}\theta_1 :&= \phi(\xi_1)+\phi(\xi_2)+\phi(\xi_3)-\phi(\xi_1+\xi_2+\xi_3)\\ &= 5\alpha(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)\left((\xi_2+\xi_3)(\xi_1+\xi_3)+\xi_1^2+\xi_2^2-\frac{3\beta}{5\alpha}\right).\end{aligned}$$

To estimate $\|f\|_{H^s}$, we consider the following three cases.

$$\begin{aligned}&\text{Case (1)}: \ \xi_j(j=1,2,3)\in I_1,\\ &\text{Case (2)}: \ \xi_j(j=1,2,3)\in I_2,\\ &\text{Case (3)}: \ \xi_j(j=1,2)\in I_1, \xi_3\in I_2; \text{or } \ \xi_1\in I_1, \xi_j(j=2,3)\in I_2;\\ &\qquad\qquad \text{or } \ \xi_j(j=1,2)\in I_2, \xi_3\in I_1; \text{ or } \ \xi_1\in I_2, \xi_j(j=2,3)\in I_1.\end{aligned}$$

The the integrals in (2.2) corresponding to cases (1), (2), (3) are denoted by f_1, f_2, f_3 respectively.

Case (1). In this case, $|\theta_1|\sim N^5$ and $|\xi_1+\xi_2+\xi_3|\sim N$. As $r^2N^3=O(1)$, we have

$$\|f_1\|_{H^s}\le Cr^{-3/2}N^{-3s}N^s r^{5/2}N^{-4}\sim N^{-2s-\frac{11}{2}}.$$

Case (2). In this case, $|\theta_1|\sim N^5$ and $|\xi_1+\xi_2+\xi_3|\sim N$. As $r^2N^3=O(1)$, we have

$$\|f_2\|_{H^s}\le Cr^{-3/2}N^{-3s}N^s r^{5/2}N^{-4}\sim N^{-2s-\frac{11}{2}}.$$

Case (3). In this case, $|\theta_1|\sim r^2N^3=O(1)$, $|Q|\ge$ const and $|\xi_1+\xi_2+\xi_3|\sim N$. As $r^2N^3=O(1)$, we have

$$\|f_3\|_{H^s}\ge Cr^{-3/2}N^{-3s}N^s r^{5/2}N\sim N^{-2s-\frac{1}{2}}.$$

From (2.1), we have

$$\begin{aligned}C\sim C\|u_0\|_{H^s}^3 &\ge \|f_1+f_2+f_3\|_{H^s}\ge \|f_3+f_2\|_{H^s}-\|f_1\|_{H^s}\\ &\ge \|f_3\|_{H^s}-\|f_2\|_{H^s}-\|f_1\|_{H^s}\\ &\ge C(N^{-2s-\frac{1}{2}}-N^{-2s-\frac{11}{2}}),\end{aligned}$$

which yields

$$N^{-2s-\frac{1}{2}}\le C(1+N^{-2s-\frac{11}{2}}). \tag{2.3}$$

When $s<-\frac{11}{4}$, from (2.3), we have $N^{-2s-\frac{1}{2}}\le CN^{-2s-\frac{11}{2}}$, which yields $N^5\le C$. We obtain a contradiction since $N\gg1$. When $-\frac{11}{4}\le s<-\frac{1}{4}$, from (2.3), we have $N^{-2s-\frac{1}{2}}\le C$. We obtain a contradiction since $N\gg 1$.

Thus, we complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. On the contrary, we assume that the solution map of (1.3)–(1.2)

$$S_t: \quad u_0 \in H^s(\mathbf{R}) \longrightarrow u \in C([0,T]; H^s(\mathbf{R}))$$

is C^2 at zero with $s<0$. Following the general well-posedness principle of [2], we have

$$\sup_{t\in[0,T]} \|D_2(u_0)\|_{H^s} \le C\|u_0\|_{H^s}^2, \quad \text{for all } u_0 \in H^s(\mathbf{R}), \tag{3.1}$$

where

$$\begin{aligned} D_2(u_0)(x,t) &= \int_0^t W(t-\tau)(D_1(u_0)\partial_x^2 D_1(u_0))(\tau)\mathrm{d}\tau, \\ D_1(u_0)(x,t) &= W(t)u_0. \end{aligned}$$

We consider the initial data

$$u_0(x) = r^{-1/2}N^{-s}\left\{e^{-\mathrm{i}Nx}\left(\int_0^r e^{\mathrm{i}x\xi}\mathrm{d}\xi\right) + e^{\mathrm{i}Nx}\left(\int_r^{2r} e^{\mathrm{i}x\xi}\mathrm{d}\xi\right)\right\},$$

which can be seen in [18], where $rN^4 = O(1)$ and

$$N \gg 16a = 16\max\left\{1, \left(2\left|\frac{3\beta}{5\alpha}\right|\right)^{1/2}, \left(\left|\frac{3\beta}{10\alpha}\right| + \left|\frac{2\gamma}{5\alpha} - \frac{1}{2}\left(\frac{3\beta}{5\alpha}\right)^2\right|^{1/2}\right)^{1/2}\right\},$$

where a is defined as in [10]. Thus, we have

$$\mathcal{F}_x u_0(\xi) = Cr^{-1/2}N^{-s}\left\{\chi_{[-N,-N+r]}(\xi) + \chi_{[N+r,N+2r]}(\xi)\right\},$$

where χ_I denotes the characteristic function of a set $I \subset \mathbf{R}$. Clearly, $\|u_0\|_{H^s} \sim 1$. Let $I_1 = [-N,-N+r]$ and $I_2 = [N+r, N+2r]$ and $\Omega_2 = I_1 \bigcup I_2$. As

$$\mathcal{F}_x D_1(u_0)(\xi) = Ce^{\mathrm{i}t\phi(\xi)}\mathcal{F}_x u_0(\xi),$$

we have

$$D_1(u_0)(x,t) = Cr^{-1/2}N^{-s}\int_{\xi\in\Omega_2} e^{\mathrm{i}x\xi}e^{\mathrm{i}t\phi(\xi)}\mathrm{d}\xi.$$

Thus, we have

$$D_2(u_0)(x,t) = Ch,$$

where

$$h = r^{-1}N^{-2s}\int_{\xi_1\in\Omega_2}\int_{\xi_2\in\Omega_2} \xi_2^2 e^{\mathrm{i}x(\xi_1+\xi_2)}\frac{e^{\mathrm{i}t(\phi(\xi_1)+\phi(\xi_2))} - e^{\mathrm{i}t\phi(\xi_1+\xi_2)}}{\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)}\mathrm{d}\xi_1\mathrm{d}\xi_2. \tag{3.2}$$

As $\phi(\xi) = -\alpha\xi^5 + \beta\xi^3 - \gamma\xi$, we may define

$$\theta_2 := \phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2) = 5\alpha(\xi_1+\xi_2)\xi_1\xi_2\left(\xi_1^2 + \xi_1\xi_2 + \xi_2^2 - \frac{3\beta}{5\alpha}\right).$$

To deal with $\|h\|_{H^s}$, we consider the following three cases.

$$\begin{aligned}&\text{Case (1)}: \ \xi_j(j=1,2)\in I_1,\\ &\text{Case (2)}: \ \xi_j(j=1,2)\in I_2,\\ &\text{Case (3)}: \ \xi_1\in I_1,\xi_2\in I_2 \ \text{ or } \ \xi_1\in I_2,\xi_2\in I_1.\end{aligned}$$

The integrals in (3.2) corresponding to cases (1), (2), and (3) are denoted as h_1, h_2, and h_3 respectively.

Case (1): $\xi_j(j=1,2)\in I_1$. In this case, $|\xi_1+\xi_2|\sim N$ and $|\theta_2|\sim N^5$. As $rN^4=O(1)$, we have

$$\|h_1\|_{H^s}\leq Cr^{-1}N^{-2s}N^s r^{3/2}N^{-3}\sim N^{-s-5}.$$

Case (2): $\xi_j(j=1,2)\in I_2$. In this case, $|\xi_1+\xi_2|\sim N$, and $|\theta_2|\sim N^5$. As $rN^4=O(1)$, we have

$$\|h_2\|_{H^s}\leq Cr^{-1}N^{-2s}N^s r^{3/2}N^{-3}\sim N^{-s-5}.$$

Case (3): $\xi_1\in I_1,\xi_2\in I_2$ or $\xi_1\in I_2,\xi_2\in I_1$. In this case, $|\xi_1+\xi_2|\sim r$, $|\theta_2|=rN^4=O(1)$, and $\langle\xi\rangle^s\geq 2^s$. As $rN^4=O(1)$ and

$$\left|\frac{e^{\mathrm{i}t(\phi(\xi_1)+\phi(\xi_2))}-e^{\mathrm{i}t\phi(\xi_1+\xi_2)}}{\phi(\xi_1)+\phi(\xi_2)-\phi(\xi_1+\xi_2)}\right|\geq \text{const},$$

we have

$$\|h_3\|_{H^s}\geq Cr^{-1}N^{-2s}r^{\frac{3}{2}}N^2 2^s\sim N^{-2s}2^s.$$

From (3.1), we have

$$\begin{aligned}C\sim C\|u_0\|_{H^s}^2&\geq \|h_3+h_2+h_1\|_{H^s}\geq \|h_3+h_2\|_{H^s}-\|h_1\|_{H^s}\\ &\geq \|h_3\|_{H^s}-\|h_2\|_{H^s}-\|h_1\|_{H^s}\geq C(N^{-2s}2^s-N^{-s-5}),\end{aligned}$$

which yields

$$2^sN^{-2s}\leq C(1+N^{-s-5}). \tag{3.3}$$

When $s<-5$, from (3.3), we have $2^sN^{-2s}=(\frac{N}{2})^{-s}N^{-s}\leq CN^{-s-5}$, which yields that $(\frac{N}{2})^{-s}N^5\leq C$, a contradiction because $N\gg 1$. When $-5\leq s<0$, from (3.3), we have $2^sN^{-2s}=(\frac{N}{2})^{-s}N^{-s}\leq C$, again a contradiction because $N\gg 1$. Therefore, we complete the proof of Theorem 1.2.

References

[1] Benilov E S, Grimshaw R, Kuznetsova E P. The generationof radiating waves in a singularly-perturbed Koreteweg-de Vries equation. Phys D, 1993, **69**: 270–278

[2] Bejennaru I, Tao T. Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation. J Funct Analysis, 2006, **233**: 228–259

[3] Bona J L, Smith R S. A model for the two-ways propagation of water waves in a channel. Math Proc Cambridge Philos Soc, 1976, **79**: 167–182

[4] Bourgain J. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part I: Schrodinger equation, Part II: The KdV equation. Geom Funct Analysis, 1993, **3**: 107–156; 209–262

[5] Bourgain J. Periodic Korteweg de Vries equation with measures as initial data. Selecta Math (NS), 1997, **3**: 115–159

[6] Chen W G, Li J F, Miao C X, Wu J H. Low regularity solutions of two fifth-order KdV type equations. J D'Anal Math, 2009, **107**: 221–238

[7] Chen W G, Liu Z P. Well-posedness and ill-posedness for a fifth-order shallow water wave equation. Nonlinear Analysis, 2010, **72**: 2412–2420

[8] Hunter J K, Scheurle J. Existence of perturbed solitary wave solutions to a model equation for water waves. Phys D, 1988, **32**: 253–268

[9] Huo Z H. The Cauchy problem for the fifth-order shallow water equation. Acta Math Sin Appl Engl Ser, 2005, **21**(3): 441–454

[10] Jia Y L, Huo Z H. Well-posedness for the fifth-order shallow water equations. J Diff Eqns, 2009, **246**: 2448–2467

[11] Kawahara T. Oscillatory solitary waves in dispersive media. J Phys Soc Japan, 1972, **33**: 260–264

[12] Kaup D J. On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$. Stud Appl Math, 1980, **62**(3): 189–216

[13] Kenig C E, Ponce G, Vega L. The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices. J Duke Math, 1993, **71**: 1–21

[14] Kenig C E, Ponce G, Vega L. A bilinear estimate with applications to the KdV equation. J Amer Math Soc, 1996, **9**: 573–603

[15] Kichenassamy S, Olver P J. Existence and nonexistence of solitary wave solutions to higher-order model evolution equations. SIAM J Math Anal, 1992, **23**: 1141–1166

[16] Kupershmidt B A. A super Korteweg-de Vries equation: An integrable system. Phys Lett A, 1984, **102**(5/6): 213–215

[17] Tao T. Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations. Amer J Math, 2001, **123**: 839–908

[18] Tzvetkov N. Remark on the local ill-posedness for KdV equation. C R Acad Sci Paris, 1999, **329**: 1043–1047

[19] Yan W, Li Y S, Yang X Y. The Cauchy problem for the modified Kawahara equation in Sobolev spaces with low regularity. Mathematical and Computer Modelling, 2011, **54**: 1252–1261

[20] Shen C X, Guo B L. III-posedness for the nonlinear davey-stewartson equation. Acta Mathematica Scientia, 2008, **28B**(1): 117–127