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A COMPACT HAUSDORFF SPACE WITHOUT P-POINTS IN WHICH G_δ-SETS HAVE INTERIOR

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ABSTRACT. We construct a compact Hausdorff space which has no P-points and yet in which every nonempty G_δ set has nonempty interior.

Definition 1 (Levy [1]). A space is an *almost P-space* if every nonempty G_δ-set has nonempty interior.

Definition 2. An element x of a topological space X is a *P-point* if x lies in the interior of each G_δ-subset of X which contains it. An element x of a topological space X is a weak P-point if x lies in the closure of no countable subset of $X - \{x\}$.

In 1977 Levy [1] showed that every ordered compact almost P-space contains a P-point, and that every compact almost P-space of weight $\aleph_1$ contains a P-point. He asked whether there is any compact almost P-space which has no P-points. In 1978 Shelah [2] constructed a model of set theory in which ω^* has no P-points. Since ω^* is an almost P-space, this provided a consistent positive solution to Levy's question.

We answer the question affirmatively in ZFC.

Example 1 (ZFC). There is a compact almost P-space which contains no P-points.

Let $\kappa = 2^{\aleph_1}$. Let Fn be the family of all partial functions ρ whose domain is a countably infinite subset of $\kappa \cup (\kappa \times \omega_1)$ so that, if $\lambda \in \kappa$ and $\alpha \in \omega_1$, then $\rho(\lambda)$ is either undefined or an element of 2 and $\rho(\lambda, \alpha)$ is either undefined or an element of $\kappa + 1$. Let φ be a mapping of $\kappa \times \omega_1$ onto Fn such that

$$(\kappa \times \omega_1) \cap \mathrm{dom}(\varphi(\lambda, \alpha)) \subset \kappa \times \alpha.$$

We use the notation $(f, g) \in [\rho]$ to indicate that the total function (f, g) extends the partial function ρ.

Let W be the set of (f, g) in $2^\kappa \times (\kappa + 1)^{\kappa \times \omega_1}$ which satisfies the following

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three conditions:

1. $(\forall\alpha \in \omega_1)|\operatorname{ran}(g \restriction \kappa \times \{\alpha\}) - \{\kappa\}| \leq 1$.
2. $(\forall\alpha \in \omega_1)(\forall\lambda \in \kappa)\kappa \cap \operatorname{dom}(\varphi(\lambda, \alpha)) \subset g(\lambda, \alpha)$.
3. $(\forall\lambda \in \kappa)(\forall\alpha \in \omega_1) g(\lambda, \alpha) \neq \kappa \Rightarrow (f, g) \in [\varphi(\lambda, \alpha)]$.

Let W have the subspace topology induced by the Tychonoff product topology on $2^\kappa \times (\kappa+1)^{\kappa\times\omega_1}$ where $\kappa+1$ has the order topology. Note that since failure of any of the three conditions for membership of some (f, g) in W can be witnessed by finitely many coordinates, we can deduce that the complement of W is open in the compact Hausdorff space $2^\kappa \times (\kappa+1)^{\kappa\times\omega_1}$. Thus, we know that W is a compact Hausdorff space itself.

Lemma 1. *W is an almost P-space.*

Proof. We show that if $\rho \in Fn$ and $[\rho] \cap W \neq \varnothing$, then $[\rho] \cap W$ contains a nonempty open set V in W. This suffices since any nonempty G_δ set in W contains some such nonempty $[\rho] \cap W$.

Let $(\lambda, \alpha) \in \varphi^{-1}(\rho)$. Let $\mu = \sup(\kappa \cap \operatorname{dom}(\rho)) + 1$.

We argue that $[((\lambda, \alpha), \mu)] \cap W \subset [\rho]$. Suppose $(f, g) \in W$ and $g(\lambda, \alpha) = \mu$. Then since $g(\lambda, \alpha) \neq \kappa$, by (3), $(f, g) \in [\varphi(\lambda, \alpha)] = [\rho]$.

We argue that $[((\lambda, \alpha), \mu)] \cap W$ is nonempty. Suppose $(f, g) \in [\rho] \cap W$. Define $g^* \in (\kappa+1)^{\kappa\times\omega_1}$ so that $g^* \supset g \restriction \kappa \times \alpha$ and g^* takes on the value κ elsewhere except at (λ, α) where it is defined to be μ. It suffices to show that $(f, g^*) \in W$. We have constructed g^* so that (1) and (2) hold. So let us check (3) by supposing $g^*(\lambda', \alpha') \neq \kappa$. Now $\alpha' \leq \alpha$, $(f, g) \in [\varphi(\lambda', \alpha')]$ by (3) and $(\kappa \times \omega_1) \cap \operatorname{dom}(\varphi(\lambda', \alpha')) \subset \kappa \times \alpha'$. So $(f, g^*) \in [\varphi(\lambda', \alpha')]$.

Lemma 2. *W can be partitioned into Cantor sets.*

Proof. Suppose $(f, g) \in W$. Let

$$\Lambda = \bigcup\{\kappa \cap \operatorname{dom}(\varphi(\lambda, \alpha)) : \lambda \in \kappa,\ \alpha \in \omega_1,\ g(\lambda, \alpha) \neq \kappa\}.$$

By (2),

$$\sup\Lambda \leq \sup\{g(\lambda, \alpha) : \lambda \in \kappa,\ \alpha \in \omega_1,\ g(\lambda, \alpha) \neq \kappa\}.$$

By (1), since $\operatorname{cf}(\kappa) > \omega_1$, $\sup\Lambda < \kappa$. If $f' \in 2^\kappa$ and $f \restriction \Lambda = f' \restriction \Lambda$, then $(f', g) \in W$ since (3) follows from $\operatorname{dom}(\varphi(\lambda, \alpha)) \subset \Lambda \cup (\kappa \times \omega_1)$ whenever $g(\lambda, \alpha) \neq \kappa$. But $\{(f', g) : f' \restriction \Lambda = f \restriction \Lambda\} \subset W$ is homeomorphic to 2^κ, and the proof is complete.

Corollary 1. *Every point in W is the limit of a nontrivial convergent sequence in W and thus W has no (weak) P-points.*

Note that, by replacing 2^κ by $[0, 1]^\kappa$ in the construction, we get an almost P-space which can be partitioned into closed unit intervals.

Note that we can choose any κ which satisfies the equations $\operatorname{cf}(\kappa) > \omega_1$ and $\kappa^\omega = \kappa$. So $\kappa = 2^{\aleph_0}$ suffices unless $\operatorname{cf}(2^{\aleph_0}) = \aleph_1$. Without any hypothesis, $\kappa = (2^{\aleph_0})^+$ works.

Our original construction used a nonlinear inverse limit in which the bonding maps are the projections from a kind of Alexandroff duplicate (see example 3.1.77 in [3]). The present exposition is a translation of this method into the product of which the inverse limit is a subspace.

We became aware of this question by reading a recent preprint of Williams and Zhou [4]. We thank the referee who suggested a substantial improvement.

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