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## A COMPACT HAUSDORFF SPACE WITHOUT P-POINTS IN WHICH $G_\delta$ -SETS HAVE INTERIOR

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(Communicated by Franklin D. Tall)

**ABSTRACT.** We construct a compact Hausdorff space which has no P-points and yet in which every nonempty  $G_\delta$  set has nonempty interior.

**Definition 1** (Levy [1]). A space is an *almost P-space* if every nonempty  $G_\delta$ -set has nonempty interior.

**Definition 2.** An element  $x$  of a topological space  $X$  is a *P-point* if  $x$  lies in the interior of each  $G_\delta$ -subset of  $X$  which contains it. An element  $x$  of a topological space  $X$  is a *weak P-point* if  $x$  lies in the closure of no countable subset of  $X - \{x\}$ .

In 1977 Levy [1] showed that every ordered compact almost P-space contains a P-point, and that every compact almost P-space of weight  $\aleph_1$  contains a P-point. He asked whether there is any compact almost P-space which has no P-points. In 1978 Shelah [2] constructed a model of set theory in which  $\omega^*$  has no P-points. Since  $\omega^*$  is an almost P-space, this provided a consistent positive solution to Levy's question.

We answer the question affirmatively in ZFC.

**Example 1** (ZFC). There is a compact almost P-space which contains no P-points.

Let  $\kappa = 2^{\aleph_1}$ . Let  $Fn$  be the family of all partial functions  $\rho$  whose domain is a countably infinite subset of  $\kappa \cup (\kappa \times \omega_1)$  so that, if  $\lambda \in \kappa$  and  $\alpha \in \omega_1$ , then  $\rho(\lambda)$  is either undefined or an element of 2 and  $\rho(\lambda, \alpha)$  is either undefined or an element of  $\kappa + 1$ . Let  $\varphi$  be a mapping of  $\kappa \times \omega_1$  onto  $Fn$  such that

$$(\kappa \times \omega_1) \cap \text{dom}(\varphi(\lambda, \alpha)) \subset \kappa \times \alpha.$$

We use the notation  $(f, g) \in [\rho]$  to indicate that the total function  $(f, g)$  extends the partial function  $\rho$ .

Let  $W$  be the set of  $(f, g)$  in  $2^\kappa \times (\kappa + 1)^{\kappa \times \omega_1}$  which satisfies the following

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three conditions:

1.  $(\forall \alpha \in \omega_1) |\text{ran}(g \upharpoonright \kappa \times \{\alpha\}) - \{\kappa\}| \leq 1$ .
2.  $(\forall \alpha \in \omega_1)(\forall \lambda \in \kappa) \kappa \cap \text{dom}(\varphi(\lambda, \alpha)) \subset g(\lambda, \alpha)$ .
3.  $(\forall \lambda \in \kappa)(\forall \alpha \in \omega_1) g(\lambda, \alpha) \neq \kappa \Rightarrow (f, g) \in [\varphi(\lambda, \alpha)]$ .

Let  $W$  have the subspace topology induced by the Tychonoff product topology on  $2^\kappa \times (\kappa + 1)^{\kappa \times \omega_1}$  where  $\kappa + 1$  has the order topology. Note that since failure of any of the three conditions for membership of some  $(f, g)$  in  $W$  can be witnessed by finitely many coordinates, we can deduce that the complement of  $W$  is open in the compact Hausdorff space  $2^\kappa \times (\kappa + 1)^{\kappa \times \omega_1}$ . Thus, we know that  $W$  is a compact Hausdorff space itself.

**Lemma 1.**  $W$  is an almost  $P$ -space.

*Proof.* We show that if  $\rho \in Fn$  and  $[\rho] \cap W \neq \emptyset$ , then  $[\rho] \cap W$  contains a nonempty open set  $V$  in  $W$ . This suffices since any nonempty  $G_\delta$  set in  $W$  contains some such nonempty  $[\rho] \cap W$ .

Let  $(\lambda, \alpha) \in \varphi^{-1}(\rho)$ . Let  $\mu = \sup(\kappa \cap \text{dom}(\rho)) + 1$ .

We argue that  $[(\lambda, \alpha), \mu] \cap W \subset [\rho]$ . Suppose  $(f, g) \in W$  and  $g(\lambda, \alpha) = \mu$ . Then since  $g(\lambda, \alpha) \neq \kappa$ , by (3),  $(f, g) \in [\varphi(\lambda, \alpha)] = [\rho]$ .

We argue that  $[(\lambda, \alpha), \mu] \cap W$  is nonempty. Suppose  $(f, g) \in [\rho] \cap W$ . Define  $g^* \in (\kappa + 1)^{\kappa \times \omega_1}$  so that  $g^* \supset g \upharpoonright \kappa \times \alpha$  and  $g^*$  takes on the value  $\kappa$  elsewhere except at  $(\lambda, \alpha)$  where it is defined to be  $\mu$ . It suffices to show that  $(f, g^*) \in W$ . We have constructed  $g^*$  so that (1) and (2) hold. So let us check (3) by supposing  $g^*(\lambda', \alpha') \neq \kappa$ . Now  $\alpha' \leq \alpha$ ,  $(f, g) \in [\varphi(\lambda', \alpha')]$  by (3) and  $(\kappa \times \omega_1) \cap \text{dom}(\varphi(\lambda', \alpha')) \subset \kappa \times \alpha'$ . So  $(f, g^*) \in [\varphi(\lambda', \alpha')]$ .

**Lemma 2.**  $W$  can be partitioned into Cantor sets.

*Proof.* Suppose  $(f, g) \in W$ . Let

$$\Lambda = \bigcup \{ \kappa \cap \text{dom}(\varphi(\lambda, \alpha)) : \lambda \in \kappa, \alpha \in \omega_1, g(\lambda, \alpha) \neq \kappa \}.$$

By (2),

$$\sup \Lambda \leq \sup \{ g(\lambda, \alpha) : \lambda \in \kappa, \alpha \in \omega_1, g(\lambda, \alpha) \neq \kappa \}.$$

By (1), since  $\text{cf}(\kappa) > \omega_1$ ,  $\sup \Lambda < \kappa$ . If  $f' \in 2^\kappa$  and  $f \upharpoonright \Lambda = f' \upharpoonright \Lambda$ , then  $(f', g) \in W$  since (3) follows from  $\text{dom}(\varphi(\lambda, \alpha)) \subset \Lambda \cup (\kappa \times \omega_1)$  whenever  $g(\lambda, \alpha) \neq \kappa$ . But  $\{(f', g) : f' \upharpoonright \Lambda = f \upharpoonright \Lambda\} \subset W$  is homeomorphic to  $2^\kappa$ , and the proof is complete.

**Corollary 1.** Every point in  $W$  is the limit of a nontrivial convergent sequence in  $W$  and thus  $W$  has no (weak)  $P$ -points.

Note that, by replacing  $2^\kappa$  by  $[0, 1]^\kappa$  in the construction, we get an almost  $P$ -space which can be partitioned into closed unit intervals.

Note that we can choose any  $\kappa$  which satisfies the equations  $\text{cf}(\kappa) > \omega_1$  and  $\kappa^\omega = \kappa$ . So  $\kappa = 2^{\aleph_0}$  suffices unless  $\text{cf}(2^{\aleph_0}) = \aleph_1$ . Without any hypothesis,  $\kappa = (2^{\aleph_0})^+$  works.

Our original construction used a nonlinear inverse limit in which the bonding maps are the projections from a kind of Alexandroff duplicate (see example 3.1.77 in [3]). The present exposition is a translation of this method into the product of which the inverse limit is a subspace.

We became aware of this question by reading a recent preprint of Williams and Zhou [4]. We thank the referee who suggested a substantial improvement.

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