Journal of Mathematical Analysis and Applications **263,** 501–509 (2001)
doi:10.1006/jmaa.2001.7626, available online at http://www.idealibrary.com on IDEAL®

Precise Coefficient Estimates for Close-to-Convex Harmonic Univalent Mappings

Xiao-Tian Wang and Xiang-Qian Liang

Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

and

Yu-Lin Zhang

Department of Mathematics, Northwest University, Shaanxi Xi'an 710069, People's Republic of China

Submitted by Alexander Isaev

Received May 8, 2001

The class S_H consists of harmonic, univalent, and sense-preserving functions f in the open unit disk $U = \{z : |z| < 1\}$, such that $f = h + \bar{g}$, where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$. Let S_H^0, C_H, and C_H^0 denote the subclass of S_H with $a_{-1} = 0$, the subclass of S_H with f being a close-to-convex mapping, and the intersection of S_H^0 and C_H, respectively. In this paper, for $f \in C_H^0$ and $f \in C_H$, we prove that the harmonic analogue of the Bieberbach conjecture and the generalization of the Bieberbach conjecture are true. © 2001 Academic Press

1. INTRODUCTION

Let S_H denote the class of all harmonic, complex valued, orientation-preserving, and univalent mappings f defined in the open unit disk $U = \{z : \ |z| < 1\}$ normalized at the origin by $f(0) = 0$ and $f_z(0) = 1$. Such functions can be written in the form

$$f = h + \bar{g},$$

where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$ are analytic and $|g'(z)| < |h'(z)|$ in U. It follows that $|a_{-1}| < 1$, and hence $(f - a_{-1}\bar{f})/(1 -$

0022-247X/01 $35.00

$|a_{-1}|$) belongs to S_H. Thus we may restrict our attention to the subclass

$$S_H^0 = \{f \in S_H : f_{\bar{z}}(0) = 0\}.$$

Let C_H denote the subset of S_H, such that for any $f \in C_H$, $f(U)$ is a close-to-convex domain. Let C_H^0 denote $S_H^0 \cap C_H$, and let $\widetilde{C}_H$ denote the closure of C_H. Clunie and Sheil-Small [1] posed the following conjecture:

I. Harmonic analogue of the Bieberbach conjecture. If $f \in S_H^0$, then

$$||a_n| - |a_{-n}|| \le n \qquad (n = 2, 3, \dots)$$
$$|a_{-n}| \le (n-1)(2n-1)/6 \qquad (n = 2, 3, \dots).$$

Later Sheil-Small [2] developed the above conjecture and posed a generalization of the Bieberbach conjecture:

II. If $f \in S_H$, then

$$|a_n| < (2n^2 + 1)/3 \qquad (|n| = 2, 3, \dots).$$

It was proved in [1, 2] that conjecture I is true for typically real functions and all functions $f \in S_H^0$ for which $f(U)$ is starlike with respect to the origin or $f(U)$ is convex in one direction. However, it remains open for the close-to-convex class C_H^0. For conjecture II, Clunie and Sheil-Small proved the following facts:

If $f \in \widetilde{C}_H$, then $|a_n| \le (2n^2 + 1)/3 \quad (|n| = 1, 2, \dots)$.

In this paper, we prove that the conjectures I and II are also true for $f \in C_H^0$ and $f \in C_H$, respectively.

For convenience, we introduce some notation. Let P denote the set of analytic functions in U satisfying $\operatorname{Re} G > 0$, for $G \in P$. Let H^p denote the class of analytic functions G in U which have bounded integral mean $M_p(r, G)$ as $r \to 1^-$, where

$$M_p(r, G) = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} |G(re^{i\theta})|^p \, d\theta \right)^{1/p}, & 0 < p < \infty \\ \displaystyle\max_{0 \le \theta \le 2\pi} |G(re^{i\theta})|, & p = \infty. \end{cases}$$

A real-valued function $u(z)$ in U is said to be harmonic if $M_p(r, u)$ is bounded as $r \to 1^-$. The class of harmonic functions is denoted by h^p, $0 < p \le \infty$.

2. LEMMAS

LEMMA 1 [3]. *Let $t(\theta)$ be a real-valued continuous function on* $(-\infty, +\infty)$. *If $t(\theta)$ satisfies $t(\theta+2\pi) = t(\theta) + 2\pi$ and $t(\theta_2) - t(\theta_1) > -\pi$, where $\theta_1 < \theta_2 < \theta_1 + 2\pi$, then there is a continuous nondecreasing function $s(\theta)$, such that*

$$s(\theta+2\pi) = s(\theta) + 2\pi \quad \textit{and} \quad |s(\theta) - t(\theta)| \le \pi/2.$$

LEMMA 2 [4]. *If $u \in h^p$ for some p, $1 < p < \infty$, then its harmonic conjugate v with $v(0) = 0$ also belongs to h^p. Furthermore, for all $u \in h^p$, there is a constant $A_p = (p/(p-1))^{1/p}$ depending only on p, such that*

$$M_p(r, v) \le A_p M_p(r, u), \qquad 0 \le r < 1.$$

LEMMA 3 [5]. *Let $u(z)$ be real-valued harmonic in U. If $u(z)$ satisfies*

$$\int_0^{2\pi} |u(re^{i\theta})|^p d\theta = O(1), \qquad 0 \le r < 1, \ p > 1,$$

then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} f_1(\varphi)\, d\varphi, \qquad z = re^{i\theta},$$

where $f_1 \in L^p$, and for almost all $e^{i\theta}$, $\lim_{z\to e^{i\theta}} u(z) = f_1(\theta)$ *uniformly when $z \to e^{i\theta}$ from the inside of any fixed Stolz domain in U.*

LEMMA 4. *If $f = h + \bar{g} \in C_H$, then there exist real numbers μ, θ_0 and a function $H(z) \in P$ such that*

$$H(z)\{ie^{i\theta_0}(1-z^2)[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)]\} \in P \qquad (|z| < 1).$$

Proof of Lemma 4. Using the approximation method of [1, Theorem 3.7], we may assume that f extends continuously and smoothly to $\overline{U}$, with $f(|z| = 1)$ being a smooth curve whose interior is a close-to-convex domain and

$$|h'(e^{i\theta})| > |g'(e^{i\theta})|.$$

Let $l_1(\theta) = \partial f(e^{i\theta})/\partial\theta = e^{i\theta}ih'(e^{i\theta}) + \overline{ie^{i\theta}g'(e^{i\theta})}$. Then

$$\arg l_1(2\pi+\theta) = \arg l_1(\theta) + 2\pi,$$

$$\arg l_1(\theta_2) - \arg l_1(\theta_1) > -\pi \qquad (\theta_1 < \theta_2 < \theta_1 + 2\pi).$$

By Lemma 1, there is a continuous nondecreasing function $s(\theta)$ such that

$$s(\theta+2\pi) = s(\theta) + 2\pi,$$

$$|s(\theta) - \arg l_1(\theta)| \le \pi/2.$$

Now setting $l_2(\theta) = e^{is(\theta)}$ and $R(\theta) = l_1(\theta)/l_2(\theta)$, we obtain

$$\arg R(\theta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]. \tag{2.1}$$

From the proof of [1, Lemma 5.11], we know that there exists a real number θ_0, such that

$$\operatorname{Re}\left[(1 - e^{2i\theta}) e^{-i(\theta + s(\theta_0))} l_2(\theta + \theta_0)\right] \geq 0 \qquad \text{for } \theta \in [0, 2\pi]. \tag{2.2}$$

For simplicity, we define

$$e^{-is(\theta_0)} = e^{-i\mu},$$

$$l_3(\theta) = (1 - e^{2i\theta}) e^{-i(\theta+\mu)} l_1(\theta + \theta_0) \qquad \text{for } \theta \in [0, 2\pi], \tag{2.3}$$

$$l_4(\theta) = (1 - e^{2i\theta}) e^{-i(\theta+\mu)} l_2(\theta + \theta_0) \qquad \text{for } \theta \in [0, 2\pi] \tag{2.4}$$

and

$$Q(\theta) = \begin{cases} -\dfrac{\pi}{2}, & \text{for } \theta \in E[\theta : 0 \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) \leq \pi] \\ \dfrac{\pi}{2}, & \text{for } \theta \in E[\theta : -\pi \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) < 0] \\ 0, & \text{for } \theta \in \{0, \pi, 2\pi\}, \end{cases} \tag{2.5}$$

where $E = (0, \pi) \cup (\pi, 2\pi)$. Here, noting (2.1)–(2.4), we have $-\pi \leq \arg l_4(\theta) + \arg R(\theta + \theta_0) \leq \pi$, for $\theta \in E$.

Next we put

$$v(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) Q(t)}{1 + r^2 - 2r\cos(\theta - t)}\, dt, \qquad z = re^{i\theta} \in U.$$

then

$$v(e^{i\theta}) = Q(\theta) \qquad \text{a.e. for } \theta \in [0, 2\pi] \tag{2.6}$$

and

$$|v(re^{i\theta})| \leq \pi/2.$$

Since $v(re^{i\theta})$ is real harmonic in U, we have

$$|v(re^{i\theta})| < \pi/2. \tag{2.7}$$

Define

$$H(z) = e^{\varphi(z)}, \tag{2.8}$$

where $\varphi(z)$ is the analytic function in U and satisfies

$$\operatorname{Im} \varphi(z) = v(z) \tag{2.9}$$

and $\varphi(0) = iv(0)$. Let $u(z) = \operatorname{Re}\varphi(z)$. For $p \geq 2$, we get from Lemma 2 the following inequality:

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |u(z)|^p \, d\theta\right)^{1/p} \leq 2\left(\frac{p}{p-1}\right)^{1/p}\left(\frac{1}{2\pi}\int_0^{2\pi} |v(z)|^p \, d\theta\right)^{1/p}$$
$$\leq 2^{1+1/p}\left(\frac{1}{2\pi}\int_0^{2\pi} |v(z)|^p \, d\theta\right)^{1/p}.$$

By (2.7), we have

$$\int_0^{2\pi} |u(z)|^p d\theta \leq 4\pi^{p+1}.$$

For $p > 1$, using Hölder's inequality, we deduce

$$\begin{aligned}\int_0^{2\pi} |H(z)|^p \, d\theta &= \int_0^{2\pi} e^{pu(z)} \, d\theta \\ &\leq \int_0^{2\pi} p|u(z)| \, d\theta + \sum_{n=2}^{\infty} \frac{1}{n!}\int_0^{2\pi} |pu(z)|^n \, d\theta + 2\pi \\ &\leq \int_0^{2\pi} p|u(z)| \, d\theta + \sum_{n=1}^{\infty} \frac{4p^n\pi^{n+1}}{n!} \\ &< 4p\pi^3 + 4\pi e^{p\pi}.\end{aligned}$$

From Fatou's lemma, we get

$$H(z) \in H^p \qquad (p > 1)$$

and

$$H(z)\{ie^{i\theta_0}(1-z^2)[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)]\} \in H^p \qquad (p > 1).$$

Finally set

$$v_1(z) = \operatorname{Re}\{H(z)ie^{i\theta_0}(1-z^2)[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)]\},$$

then

$$v_1(z) \in h^p \qquad (p > 1).$$

Hence applying Lemma 3, we can write $v_1(z)$ as

$$v_1(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)f_2(t)}{1+r^2-2r\cos(\theta-t)} \, dt \qquad (z = re^{i\theta}) \tag{2.10}$$

and

$$f_2(\theta) = v_1(e^{i\theta}) \qquad \text{for a.e. } \theta \in [0, 2\pi]. \tag{2.11}$$

By virtue of (2.1)–(2.9), we find that for a.e. $\theta \in [0, 2\pi]$,

$$\arg H(e^{i\theta}) = Q(\theta)$$

and

$$\begin{aligned} H(e^{i\theta}) l_3(\theta) &= H(e^{i\theta}) \frac{l_3(\theta)}{l_4(\theta)} l_4(\theta) \\ &= H(e^{i\theta}) \frac{l_1(\theta+\theta_0)}{l_2(\theta+\theta_0)} l_4(\theta) \\ &= H(e^{i\theta}) R(\theta+\theta_0) l_4(\theta). \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{\pi}{2} \le \arg\,[H(e^{i\theta}) l_3(\theta)] &= Q(\theta) + \arg\, l_4(\theta) \\ &\quad + \arg\, R(\theta+\theta_0) \le \frac{\pi}{2} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} v_1(e^{i\theta}) &= \operatorname{Re}\big[H(e^{i\theta}) i(1-e^{2i\theta}) e^{i(\theta_0-\mu)} h'(e^{i(\theta_0+\theta)}) \\ &\qquad + H(e^{i\theta}) i(1-e^{2i\theta}) e^{i(\theta_0+\mu)} g'(e^{i(\theta_0+\theta)})\big] \\ &= \operatorname{Re}\big[H(e^{i\theta}) i(1-e^{2i\theta}) e^{i(\theta_0-\mu)} h'(e^{i(\theta_0+\theta)}) \\ &\qquad + \overline{H(e^{i\theta}) i(1-e^{2i\theta}) e^{i(\theta_0+\mu)} g'(e^{i(\theta_0+\theta)})}\big] \\ &= \operatorname{Re}\big\{H(e^{i\theta}) e^{-i\mu}\big[i(1-e^{2i\theta}) e^{i\theta_0} h'(e^{i(\theta_0+\theta)}) \\ &\qquad - \overline{i(1-e^{2i\theta}) e^{i\theta_0} g'(e^{i(\theta_0+\theta)})}\big]\big\} \\ &= \operatorname{Re}\big\{H(e^{i\theta}) e^{-i\mu}\big[i(1-e^{2i\theta}) e^{i\theta_0} h'(e^{i(\theta_0+\theta)}) \\ &\qquad - (e^{i\theta}-e^{-i\theta}) \overline{i e^{i(\theta_0+\theta)} g'(e^{i(\theta_0+\theta)})}\big]\big\} \\ &= \operatorname{Re}\big\{H(e^{i\theta}) e^{-i(\mu+\theta)}\big[i(1-e^{2i\theta}) e^{i(\theta_0+\theta)} h'(e^{i(\theta_0+\theta)}) \\ &\qquad - (e^{2i\theta}-1) \overline{i e^{i(\theta_0+\theta)} g'(e^{i(\theta_0+\theta)})}\big]\big\} \\ &= \operatorname{Re}\big\{H(e^{i\theta}) e^{-i(\mu+\theta)} (1-e^{2i\theta})\big[i e^{i(\theta+\theta_0)} h'(e^{i(\theta_0+\theta)}) \\ &\qquad + \overline{i e^{i(\theta+\theta_0)} g'(e^{i(\theta_0+\theta)})}\big]\big\} \\ &= \operatorname{Re}\,[H(e^{i\theta}) l_3(\theta)] \\ &= \operatorname{Re}\big[H(e^{i\theta}) l_4(\theta) R(\theta+\theta_0)\big] > 0 \quad \text{for a.e. } \theta \in [0, 2\pi]. \end{aligned} \tag{2.13}$$

By using (2.10)–(2.13), we have $v_1(z) \ge 0$.

Applying an argument similar to that in the proof of (2.7), we can prove

$$v_1(z) > 0 \qquad \text{for } z \in U.$$

Consequently,

$$H(z)\{ie^{i\theta_0}(1-z^2)[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)]\} \in P. \tag{2.14}$$

From (2.5)–(2.9), we get $H(z) \in P$. Hence the assertion of the lemma holds.

3. PROOFS ON CONJECTURES I AND II FOR $f \in C_H^0$ AND $f \in \widetilde{C}_H$

For conjecture I, we prove the following theorem:

THEOREM 1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} a_{-n} z^n} \in C_H^0$, *then*

$$||a_n| - |a_{-n}|| \le n \qquad (n = 2, 3, \ldots) \tag{3.1}$$

$$|a_{-n}| \le \frac{(n-1)(2n-1)}{6} \qquad (n = 2, 3, \ldots) \tag{3.2}$$

and

$$|a_n| \le \frac{(n+1)(2n+1)}{6} \qquad (n = 2, 3, \ldots). \tag{3.3}$$

Proof. From (2.14), we have

$$p_1(z) = H(z)\{ie^{i\theta_0}(1-z^2)[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)]\} \in P.$$

Let $p_2(z) = 1/H(z)$, then $p_2(z) \in P$. We therefore have

$$ie^{i\theta_0}[e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)] = \frac{p_1(z)p_2(z)}{1-z^2}. \tag{3.4}$$

Where $|p_1(0)p_2(0)| = 1$, $p_1(z)p_2(z) = p_1(z)/|p_1(0)| \cdot p_2(z)/|p_2(0)|$.

Since $(1+z)/(1-z)$ and $1/(1-z^2)$ have positive coefficients, by (3.4) we obtain

$$e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z) \ll \left(\frac{1+z}{1-z}\right)^2 \frac{1}{1-z^2} = \frac{1+z}{(1-z)^3},$$

where the symbol $\ll$ means the moduli of the coefficients of the function on the left are bounded by the corresponding coefficients of the function on the right. Hence,

$$|e^{-i\mu}a_n + e^{i\mu}a_{-n}| \le n \qquad \text{for } n = 1, 2, \ldots.$$

The result of (3.1) holds.

On the other hand, we have

$$g'(z) = w(z)h'(z) \qquad (z \in U),$$

where $|w(z)| \le |z|$. Again by (3.4), we get

$$h'(e^{i\theta_0}z) = -\frac{ie^{(i\mu-\theta_0)}p_1(z)p_2(z)}{(1-z^2)(1+e^{2i\mu}w(e^{i\theta_0}z))} \ll \frac{1+z}{(1-z)^3}\frac{1}{1-z} = \frac{1+z}{(1-z)^4}$$

and

$$g'(e^{i\theta_0}z) = -\frac{ie^{i(\mu-\theta_0)}p_1(z)p_2(z)w(e^{i\theta_0}z)}{(1-z^2)(1+e^{2i\mu}w(e^{i\theta_0}z))} \ll \frac{1+z}{(1-z)^3}\frac{z}{1-z} = \frac{z(1+z)}{(1-z)^4}.$$

These inequalities give (3.2) and (3.3).

Remark. The estimates (3.1)–(3.3) are sharp, for we may choose a function

$$k_0(z) = \operatorname{Re}\left(\frac{z+(1/3)z^3}{(1-z)^3}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^3}\right) \in C_H^0,$$

which is extremal (see [1]).

For conjecture II, we improve Clunie and Sheil-Small's result and prove the following theorem:

THEOREM 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} a_{-n} z^n} \in C_H$, *then*

$$||a_n| - |a_{-n}|| \le (1+|a_{-1}|)n \qquad (n = 2, 3, \ldots) \tag{3.5}$$

$$|a_n| \le \frac{(n+1)(2n+1)}{6} + \frac{|a_{-1}|}{6}(n-1)(2n-1) < \frac{2n^2+1}{3} \qquad (n = 1, 2, 3, \ldots) \tag{3.6}$$

$$|a_{-n}| \le \frac{(n-1)(2n-1)}{6} + \frac{|a_{-1}|}{6}(n+1)(2n+1) < \frac{2n^2+1}{3} \qquad (n = 1, 2, 3, \ldots). \tag{3.7}$$

Proof. As in the proof of Theorem 1, by (2.14) we have a similar result,

$$\frac{e^{-i\mu}h'(e^{i\theta_0}z) + e^{i\mu}g'(e^{i\theta_0}z)}{|e^{-i\mu} + a_{-1}e^{i\mu}|} \ll \left(\frac{1+z}{1-z}\right)^2 \frac{1}{1-z^2} = \frac{1+z}{(1-z)^3}.$$

This yields

$$||a_n| - |a_{-n}|| < (1+|a_{-1}|)n \qquad (n = 2, 3, \ldots).$$

Since the class C_H is affine and linear invariant, for any $f \in C_H$, we conclude

$$f_0 = \frac{f - a_{-1}\bar{f}}{1 - |a_{-1}|^2} \in C_H^0.$$

Therefore we can write $f = f_0 + a_{-1}\bar{f_0}$. Now (3.6) and (3.7) follow from (3.2) and (3.3), respectively. In particular, we have

$$||a_n| - |a_{-n}|| < 2n \qquad (n = 1, 2, \ldots)$$

and

$$|a_n| < \frac{2n^2 + 1}{3} \qquad (|n| \geq 2).$$

This completes the proof.

Remark. Obviously, Theorem 2 implies Theorem 1, so that we may rewrite the generalization of the Bieberbach conjecture as follows:

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} a_{-n} z^n} \in S_H$, then

$$||a_n| - |a_{-n}|| \leq (1 + |a_{-1}|)n \qquad (n = 1, 2, \ldots)$$

$$|a_n| \leq \frac{(n+1)(2n+1)}{6} + \frac{|a_{-1}|}{6}(n-1)(2n-1) \qquad (n = 2, 3, \ldots)$$

$$|a_{-n}| \leq \frac{(n-1)(2n-1)}{6} + \frac{|a_{-1}|}{6}(n+1)(2n+1) \qquad (n = 2, 3, \ldots).$$

In particular,

$$||a_n| - |a_{-n}|| < 2n \qquad (n = 2, 3, \ldots)$$

and

$$|a_n| < \frac{2n^2 + 1}{3} \qquad (|n| \geq 2).$$

ACKNOWLEDGMENT

The authors express their heartfelt thanks to Prof. Liu Liquan for his kind suggestion.

REFERENCES

1. J. Clunie and T. Sheil–Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.* **9** (1984), 3–25.
2. T. Sheil–Small, Constants for planar harmonic mappings, *J. London Math. Soc.* **42** (1990), 237–248.
3. P. L. Duren, "Univalent Functions," Springer-Verlag, New York, 1983.
4. P. L. Duren, "Theory of H^p Space," Academic Press, New York, 1970.
5. M. Tsuji, "Potential Theory in Modern Function Theory," Manruzen, Tokyo, 1959.