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Smoothing effects for Schrödinger equations with electro-magnetic potentials and applications to the Maxwell–Schrödinger equations

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Abstract

We consider Schrödinger equations in $\boldsymbol{R}^{1+2}$ with electro-magnetic potentials. The potentials belong to H^1, and typically they are time-independent or determined as solutions to inhomogeneous wave equations. We prove Kato type smoothing estimates for solutions. We also apply this result to the Maxwell–Schrödinger equations in the Lorentz gauge and prove unique solvability of this system in the energy space.

Keywords: Smoothing effect; Maxwell–Schrödinger equations; Well-posedness

1. Introduction

This paper is devoted to the study of Kato type smoothing effects for Schrödinger equations with electro-magnetic potentials and an application to the solvability of the Maxwell–Schrödinger equations (MS) in $\boldsymbol{R}^{1+2}$ spacetime.

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The system (MS) stated below describes the time evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electro-magnetic field it generates. In the following, greek indices run from 0 to 2, latin indices run from 1 to 2, indices are raised and lowered with the metric tensor $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1)$ so that for any vector field $A = (A_\mu)$, $A_0 = A^0$ and $A_k = -A^k$, and we use the standard summation convention on repeated indices. With this notation, (MS) is written as

$$iD_0 u = D_k D^k u, \tag{1.1}$$

$$\partial_\mu F^{\mu\nu} = J^\nu, \tag{1.2}$$

where u and A are respectively a complex scalar-valued and a real vector-valued function defined in $\boldsymbol{R}^{1+2}$, $D_\mu = \partial_\mu + iA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $J_0 = |u|^2$, and $J_k = -2\,\mathrm{Im}(\bar{u} D_k u)$. The conservation of charge, namely $\partial_\mu J^\mu = 0$ follows from (1.1). The system (MS) conserves the total charge $\mathscr{Q} \equiv \|u(t)\|_2{}^2$ and the total energy

$$\mathscr{E} \equiv \int \left(-D_k u \overline{D^k u} - F_{0k} F^{0k} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) dx.$$

Since (MS) is gauge invariant, we shall consider this system in a fixed gauge. If space dimension $n \geqslant 3$, the Coulomb gauge $\partial_k A^k = 0$ seems the most convenient for the purpose of analysis mainly because the equation for A_0 reduces to the Poisson equation $-\Delta A_0 = J_0$ which is immediately solved by the Newtonian potential. Indeed, in the preceding papers [12,23,24] concerning the 3-dimensional case, (MS) is mainly treated in the Coulomb gauge; the system in the other gauges is solved by gauge transforms. However in 2-dimensional case the Coulomb gauge is difficult to treat because in this gauge we cannot expect A_0, which is expressed by the logarithmic potential, to belong usual Lebesgue spaces. Therefore in 2-dimensional case we usually adopt the Lorentz gauge $\partial_\mu A^\mu = 0$. Nakamitsu and Tsutsumi [22] treated this case and showed the global well-posedness in sufficiently regular spaces. However in both physical and mathematical point of view, it is desirable to show the well-posedness in the energy space which is in this case H^1. The present paper aims at solving this problem.

Generally speaking, smoothing property is one of main tools for solving nonlinear dispersive equations in satisfactory low regularity spaces, especially in the case where nonlinear terms contain derivatives of unknown functions, which is the case for (MS). The first application of smoothing property to the nonlinear problem was the seminal work by Kato [16], in which he showed that the solutions to the K–dV equation $\partial_t u + \partial_x^3 u + u \partial_x u = 0$ with L^2 initial data belong to the class $L_T^2 H_{\mathrm{loc}}^1 \equiv L^2(0, T; H_{\mathrm{loc}}^1(\boldsymbol{R}_x))$ and the corresponding norm is controlled by the L^2-norm of the initial data. He used this fact to construct L^2-solutions of the K–dV equation. Afterwards the study of smoothing effects has developed for various kind of dispersive equations. Roughly speaking, if we consider the equation $i\partial_t u = P(-i\nabla)u$ where $P(\xi)$ is real and $P(\xi) \sim |\xi|^m$ with $m > 1$, the solutions satisfy the estimate $\|u; L_T^2 H^{(m-1)/2}(Q)\| \leqslant C\|u(0)\|_2$, where Q is a unit cube and C is independent of Q; for example see [2,9,10,17,19,20,25]. Recently smoothing estimates for Eq. (1.1) with given potential A has been studied by several authors [3–6]. Unfortunately these deep results are not sufficient to be directly applied to our problem since these results concern the case $n \geqslant 3$, and assume exact decay or smallness of the potentials. In this paper we introduce a smoothing estimate for (1.1) which is satisfactory to our analysis of (MS). For the statement, we need some notation. Let $\Omega_A = (1 + D_k D^k)^{1/2}$, and let

Q_α be the unit square centered at $\alpha \in \boldsymbol{Z}^2$. We put $|||F|||_T = (\sum_\alpha \|F; L^\infty_T L^2(Q_\alpha)\|^2)^{1/2}$. With this notation, we can state one of the main results in the present paper:

Theorem 1.1. *Let $\theta > 1/2$. Let $A \in C_T H^1 \cap C^1_T L^2$ with $|||F_{12}|||_T < \infty$. Then for any initial data $u(0) \in L^2$, the solution to* (1.1) *belongs to $L^2_T H^{1-\theta}_{\mathrm{loc}}$ and satisfies the estimate*

$$\sup_\alpha \sum_k \left\| D_k {\Omega_A}^{-\theta} u; L^2_T L^2(Q_\alpha)\right\|^2 \leqslant C_A \langle T\rangle^{2+\epsilon} \langle |||F_{12}|||_T\rangle^2 \left\|u(0)\right\|_2^2. \tag{1.3}$$

Here $\epsilon > 0$ is some positive number, and C_A is some polynomial of $\|A; L^\infty_T H^1\|$.

More complete statements shall appear below as Proposition 3.2 and Corollary 3.1. At first sight, the norm $|||\cdot|||_T$ may look complicated, but at least in the following two important cases, this norm is easy to handle. First, if A is time-independent, $|||F_{12}|||_T$ is nothing but $\|F_{12}\|_2$. Second, if A is the solution to (1.2), $|||F_{12}|||_T$ is controlled by the initial energy of A and $L^1_T L^2$-norm of the forcing term J by virtue of finite propagation property (see Lemma 2.2).

Applying Theorem 1.1 to (MS) in the Lorentz gauge, we can show that this system is uniquely solvable in the energy space; we can obtain the following:

Theorem 1.2. *For any initial data $(u(0), A(0), \partial_t A(0)) \in H^1 \times H^1 \times L^2$ satisfying*

$$\partial_\mu A^\mu(0) = \partial_k \partial_t A^k(0) - \partial_k \partial^k A_0(0) + \left|u(0)\right|^2 = 0, \tag{1.4}$$

the system (1.1)–(1.2) *with $\partial_\mu A^\mu = 0$ has a unique solution (u, A) satisfying $u \in C(\boldsymbol{R}; H^1) \cap C^1(\boldsymbol{R}; H^{-1})$ and $A \in C(\boldsymbol{R}; H^1) \cap C^1(\boldsymbol{R}; L^2)$.*

The assumption (1.4) is the compatibility condition so that the Lorentz gauge condition $\partial_\mu A^\mu = 0$ is conserved (see Section 4).

This paper is organized as follows. In Section 2, we summarize basic facts and estimates repeatedly used in the proof of main theorems such as well-known Strichartz estimates, and several estimates related to covariant derivatives. In Section 3, we prove Theorem 1.1. The proof is based on energy methods together with calculations of commutators as in [9,10,16]. We systematically use calculations by covariant derivatives by which we can avoid producing harmless terms. We also use a modification of Strichartz estimates by Koch–Tzvetkov (Lemma 2.3). In Section 4, we prove Theorem 1.2. We show a priori estimates of solutions (Lemma 4.1) to prove the existence of local solutions by compactness argument. We also derive estimates of the difference of solutions (Lemma 4.2), from which we can prove uniqueness of solutions. Once we establish the unique existence of time-local solution, we can prolong it time-globally for the sake of the conservation of the charge and the energy.

We conclude this section by giving the notation used in this paper. For linear operators P_1, P_2, we define $[P_1, P_2] = P_1 P_2 - P_2 P_1$ and $[P_1, P_2]_+ = P_1 P_2 + P_2 P_1$. For any $\alpha = (\alpha^1, \alpha^2) \in \boldsymbol{Z}^2$, Q_α means the unit square centered at α, and $\chi_\alpha(x) = \chi(x - \alpha)$, where $\chi(x)$ is a nonnegative smooth function of $x = (x^1, x^2) \in \boldsymbol{R}^2$ such that $\chi(x) = 1$ if $\max_k |x^k| \leqslant 1/2$ and $\chi(x) = 0$ if $\max_k |x^k| \geqslant 1$. $\Omega = (1 - \Delta)^{1/2}$. $L^p = L^p(\boldsymbol{R}^2)$ is the usual Lebesgue space and its norm is denoted by $\|\cdot\|_p$. $p' = p/(p-1)$ is the dual exponent of p; this symbol is used only for Lebesgue exponents. $(f, g) = \int_{\boldsymbol{R}^2} f(x)\overline{g(x)}\,dx$ is the inner-product in L^2. $H^s_p = \Omega^{-s} L^p$ is the usual Sobolev space. If $p = 2$, we simply write $H^s = H^s_2$. For any interval $I \subset \boldsymbol{R}$ and Banach space X, $L^p(I; X)$ denotes the space of X-valued strongly measurable functions on I whose X-norm belong to $L^p(I)$. This space is often abbreviated to $L^p_T X$ for $I = (0, T)$. Similarly we use the abbreviation $C^m_T X = C^m([0, T]; X)$. C_M means various polynomials of M. Especially we simply write $C_A = C_{\|A; L^\infty_T H^1\|}$.

2. Preliminaries

Lemma 2.1. *Let $1 < r < \infty$, $m \in \boldsymbol{R}$, and $p(\xi) \in C^\infty(\boldsymbol{R}^2)$ satisfy the estimate $|\partial_\xi^\gamma p(\xi)| \leqslant C\langle \xi\rangle^{m-|\gamma|}$ for any multi-index γ. Then the following estimate holds*:

$$\left(\sum_\alpha \big\|\big[p(-i\nabla), \chi_\alpha\big] f\big\|_r{}^r\right)^{1/r} \lesssim \big\|f; H^{m-1}_r\big\|. \tag{2.1}$$

Especially we have $(\sum_\alpha \|\chi_\alpha f; H^m_r\|_r{}^r)^{1/r} \lesssim \|f; H^m_r\|$.

Proof. As is well known in the theory of pseudodifferential operators, we have the formula

$$\begin{aligned}\big[p(-i\nabla), \chi_\alpha\big] f(x) = {} & \sum_{1 \leqslant |\gamma| \leqslant N-1} \frac{(-i)^{|\gamma|}}{\gamma!} \chi_\alpha^{(\gamma)}(x) p^{(\gamma)}(-i\nabla) f(x) \\ & + \sum_{|\gamma| = N} \int L_{\alpha,\gamma}(x, y)(1 - \Delta_y)^{-l} f(y)\, dy,\end{aligned}$$

where $2l \geqslant \max\{1 - m, 0\}$, $N > 2l + m + 2$, $\chi_\alpha^{(\gamma)} = \partial^\gamma \chi_\alpha$, etc. and

$$L_{\alpha,\gamma}(x, y) = \frac{N(-i)^N}{\gamma!} (1 - \Delta_y)^l \left(\tilde{p}^{(\gamma)}(x - y) \int_0^1 (1 - \theta)^{N-1} \chi_\alpha^{(\gamma)}\big(\theta y + (1 - \theta)x\big)\, d\theta \right)$$

with $\tilde{p}^{(\gamma)}(x) = (2\pi)^{-2} \int e^{ix\xi} p^{(\gamma)}(\xi)\, d\xi$. If $|\gamma| = N$ and $|\gamma'| \leqslant l$, we can show $|\partial_x^{\gamma'} \tilde{p}^{(\gamma)}(x)| \lesssim \langle x\rangle^{-3}$. Since $\{\operatorname{supp} \chi_\alpha\}$ overlap finitely, we can show $\sum_\alpha |\chi_\alpha^{(\gamma)}(x)|^r \leqslant C$ for any γ. These estimates imply $(\sum_\alpha |L_{\alpha,\gamma}(x, y)|^r)^{1/r} \lesssim \langle x - y\rangle^{-3}$. Therefore, using the Young inequality, we obtain

$$\begin{aligned}&\left(\sum_\alpha \left\| \int L_{\alpha,\gamma}(\cdot, y)(1 - \Delta_y)^{-l} f(y)\, dy \right\|_r^r\right)^{1/r} \\ &\quad \leqslant \left\| \int \left(\sum_\alpha \big|L_{\alpha,\gamma}(\cdot, y)\big|^r\right)^{1/r} \big|(1 - \Delta_y)^{-l} f(y)\big|\, dy \right\|_r \\ &\quad \lesssim \big\| \langle\cdot\rangle^{-3} * \big|(1 - \Delta)^{-l} f\big| \big\|_r \lesssim \big\|f; H^{m-1}_r\big\|.\end{aligned}$$

On the other hand, we obtain

$$\Big(\sum_{\alpha}\big\|\chi_\alpha^{(\gamma)}p^{(\gamma)}(-i\nabla)f\big\|_r^r\Big)^{1/r}\lesssim\big\|p^{(\gamma)}(-i\nabla)f\big\|_r\lesssim\big\|f;H_r^{m-|\gamma|}\big\|.$$

Here we have used the Mihlin–Hörmander theorem to prove the second inequality. Thus the lemma has been proved. □

Lemma 2.2. *Let $T>0$, let (q,r) satisfy $0\leqslant 2/q=1/2-1/r$ with $2\leqslant r<\infty$, and let $\beta=3/q$. Then a solution A to the equation*

$$\big(\partial_\mu\partial^\mu+1\big)A=F \tag{2.2}$$

with $(A(0),\partial_t A(0))\in H^1\times L^2$ and $F\in L_T^1L^2$ belongs to $C_TH^1\cap C_T^1L^2$ and satisfies the estimates:

$$\big\|A;L_T^qH_r^{1-\beta}\big\|+\big\|\partial_tA;L_T^qH_r^{-\beta}\big\|\lesssim\big\|\big(A(0),\partial_tA(0)\big);H^1\times L^2\big\|+\big\|F;L_T^1L^2\big\|; \tag{2.3}$$

$$\begin{aligned}&\Big(\sum_\alpha\big\|\chi_\alpha A;L_T^qH_r^{1-\beta}\big\|^2+\sum_\alpha\big\|\chi_\alpha\partial_\mu A;L_T^qH_r^{-\beta}\big\|^2\Big)^{1/2}\\&\quad\lesssim\langle T\rangle^2\bigg\{\big\|\big(A(0),\partial_tA(0)\big);H^1\times L^2\big\|+\Big(\sum_\alpha\big\|F;L_T^1L^2(Q_\alpha)\big\|^2\Big)^{1/2}\bigg\}.\end{aligned} \tag{2.4}$$

Proof. The inequality (2.3) is the well-known Strichartz estimate for the Klein–Gordon equation (see for example [1,8,11,26]). To prove (2.4), we use the finite propagation property for hyperbolic equations. If $0<t<T$, the value of A in $\operatorname{supp}\chi_\alpha$ is determined by the values of data and force term at x satisfying $\operatorname{dist}(x,\operatorname{supp}\chi_\alpha)\leqslant T$. Let $\tilde\chi_{\alpha,T}$ be a smooth function which is equal to 1 if $\operatorname{dist}(x,\operatorname{supp}\chi_\alpha)\leqslant T$ and is equal to 0 if $\operatorname{dist}(x,\operatorname{supp}\chi_\alpha)\geqslant T+1$. For $0<t<T$ and $x\in\operatorname{supp}\chi_\alpha$, A satisfies (2.2) with data and force term multiplied by $\tilde\chi_{\alpha,T}$. Therefore, usual Strichartz estimate yields

$$\begin{aligned}&\big\|\chi_\alpha A;L_T^qH_r^{1-\beta}\big\|^2+\big\|\chi_\alpha\partial_\mu A;L_T^qH_r^{-\beta}\big\|^2\\&\quad\lesssim\big\|\tilde\chi_{\alpha,T}\big(A(0),\partial_tA(0)\big);H^1\times L^2\big\|^2+\big\|\tilde\chi_{\alpha,T}F;L_T^1L^2\big\|^2.\end{aligned}$$

We notice that for any $\alpha_0\in\boldsymbol{Z}$, the number of $\alpha\in\boldsymbol{Z}$ such that $\operatorname{supp}\tilde\chi_{\alpha,T}$ intersects Q_{α_0} is less than $C\langle T\rangle^2$. On account of this, we take the summation with respect to α and obtain (2.4). □

Lemma 2.3. *Let $T>0$, $s\in\boldsymbol{R}$, $\alpha>0$ and $0\leqslant 2/q_1=1-2/r_1<1$. Let $f\in L_T^2H^{s-2\alpha}$. Then a solution $u\in C_TH^s$ to the equation*

$$i\partial_tu=-\Delta u+f$$

belongs to $L_T^{q_1}H_{r_1}^{s-\alpha}$ and satisfies the estimate

$$\big\|u;L_T^{q_1}H_{r_1}^{s-\alpha}\big\|^2\lesssim T^{-1}\big\|u;L_T^2H^s\big\|^2+T\big\|f;L_T^2H^{s-2\alpha}\big\|^2. \tag{2.5}$$

Proof. See [13,18,21,24]. □

Lemma 2.4. *Let $1 < r < \infty$, $-1 \leqslant s \leqslant 2$ and $A_k \in H^1$. Then the following estimate holds*:

$$\|D_k u; H_r^{s-1}\| \lesssim \langle \|A_k; H^1\| \rangle \|u; H_r^s\|.$$

Proof. It suffices to show $\|A_k u; H_r^{s-1}\| \lesssim \|A_k; H^1\| \|u; H_r^s\|$. If $s > 0$, this estimate follows by the use of the Sobolev inequality, together with the Leibniz rule if $s > 1$. If $s \leqslant 0$, the estimate is proved by duality. □

In the following we summarize some properties of ${\Omega_A}^2 \equiv 1 + D_k D^k$.

Lemma 2.5. *Let $A_k \in H^1$. Then ${\Omega_A}^2$ is a positive self-adjoint operator with domain $\mathscr{D}({\Omega_A}^2) = H^2$. Moreover, for $-2 \leqslant s \leqslant 2$, the fractional power ${\Omega_A}^s \equiv ({\Omega_A}^2)^{s/2}$ satisfies the following*:

(i) *${\Omega_A}^s$ satisfies the estimate*

$$C_A{}^{-1} \|u; H^s\| \leqslant \|{\Omega_A}^s u\|_2 \leqslant C_A \|u; H^s\|; \tag{2.6}$$

(ii) *for $\lambda \in \boldsymbol{C}$ satisfying $|\arg \lambda| \leqslant \pi - \delta$, and for $-2 \leqslant \sigma \leqslant 2$ satisfying $0 \leqslant s - \sigma \leqslant 2$, the estimate*

$$\|{\Omega_A}^s ({\Omega_A}^2 + \lambda)^{-1} f\| \leqslant C \langle \lambda \rangle^{-1+(s-\sigma)/2} \|{\Omega_A}^\sigma f\| \tag{2.7}$$

holds for any $f \in H^\sigma$.

Proof. Let $V = 2i A^k \partial_k + i(\partial_k A^k) - A_k A^k$ so that ${\Omega_A}^2 = 1 - \Delta + V$. By virtue of the Sobolev inequality, V satisfies

$$\begin{aligned}
\|Vu\|_2 &\leqslant \|A^k\|_6 \|\partial_k u\|_3 + \|\partial_k A^k\|_2 \|u\|_\infty + \Big(\sum_k \|A_k\|_6\Big)^2 \|u\|_6 \\
&\leqslant C\Big(\sum_k \|A_k; H^1\|\Big) \|u\|_2{}^{1/3} \|(1-\Delta)u\|_2{}^{2/3} \\
&\quad + C\Big(\sum_k \|A_k; H^1\|\Big)^2 \|u\|_2{}^{2/3} \|(1-\Delta)u\|_2{}^{1/3} \\
&\leqslant \epsilon \|(1-\Delta)u\|_2 + C\epsilon^{-2} \Big(\sum_k \|A_k; H^1\|\Big)^3 \|u\|_2.
\end{aligned} \tag{2.8}$$

Here $\epsilon > 0$ can be taken arbitrary small. Therefore the Kato–Rellich theorem shows that ${\Omega_A}^2$ is a self-adjoint operator with $\mathscr{D}({\Omega_A}^2) = H^2$. The positivity is clear since $({\Omega_A}^2 u, u) = \|u\|_2{}^2 + \sum_k \|D_k u\|_2{}^2 \geqslant \|u\|_2{}^2$. The estimate (2.8) also shows (2.6) with $s = 2$ and $C_A = C\langle \sum_k \|A_k; H^1\| \rangle^3$. Therefore we can show (i) by the Heinz–Kato theorem (see [27]). The assertion (ii) is easily proved by the spectral representation of ${\Omega_A}^2$. □

Lemma 2.6. *Let θ, s, σ satisfy $-2 < \sigma \leqslant s < 2$ and $s - \sigma < \theta < 2$. Let $A_\mu \in H^1$ and $\partial^k F_{\mu k} \in L^2$. Then the estimate*

$$\left\| \left[\Omega_A{}^{-\theta}, D_\mu\right] f; H^s \right\| \leqslant C_A \left\{ \left(\sum_k \|F_{\mu k}\|_2 \right) + \left\| \partial^k F_{\mu k} \right\|_2 \right\} \left\| f; H^\sigma \right\| \tag{2.9}$$

holds for any $f \in H^\sigma$. Furthermore, if $-1 < \sigma \leqslant s < 1$, then we do not need to assume $\partial^k F_{\mu k} \in L^2$, and we can omit $\|\partial^k F_{\mu k}\|_2$ in the right-hand side.

Proof. We put $R(\lambda) = (\Omega_A{}^2 + \lambda)^{-1}$. Then we can write $\Omega_A{}^{-\theta} = \pi^{-1} \sin(\pi\theta/2) \times \int_0^\infty \lambda^{-\theta/2} R(\lambda)\, d\lambda$. Using the relation $[D_\mu, D_k D^k] = i[F_{\mu k}, D^k]_+$, we see

$$\left[\Omega_A{}^{-\theta}, D_\mu\right] = \frac{i \sin(\pi\theta/2)}{\pi} \int_0^\infty \lambda^{-\theta/2} R(\lambda) \left[F_{\mu k}, D^k\right]_+ R(\lambda)\, d\lambda. \tag{2.10}$$

To prove the lemma, we shall estimate $\|R(\lambda)[F_{\mu k}, D^k]_+ R(\lambda) f; H^s\|$ as follows by the use of Lemmas 2.4, 2.5 and the Sobolev inequality. If $-1 < \sigma \leqslant s < 2$ and $s - \sigma < \theta < 2$, we can choose $0 < \epsilon < 1$ satisfying $0 \leqslant s + \epsilon \leqslant 2$ and $0 \leqslant 2 - \sigma - \epsilon \leqslant 2$. With this choice we obtain

$$\begin{aligned} \left\| R(\lambda) F_{\mu k} D^k R(\lambda) f; H^s \right\| &\leqslant C_A \langle\lambda\rangle^{-1+(s+\epsilon)/2} \left\| F_{\mu k} D^k R(\lambda) f; H^{-\epsilon} \right\| \\ &\leqslant C_A \langle\lambda\rangle^{-1+(s+\epsilon)/2} \left(\sum_k \|F_{\mu k}\|_2 \right) \left\| R(\lambda) f; H^{2-\epsilon} \right\| \\ &\leqslant C_A \langle\lambda\rangle^{-1+(s-\sigma)/2} \left(\sum_k \|F_{\mu k}\|_2 \right) \left\| f; H^\sigma \right\|. \end{aligned} \tag{2.11}$$

If $-2 < \sigma \leqslant s < 1$ and $s - \sigma < \theta < 2$, we can choose $0 < \epsilon' < 1$ satisfying $0 \leqslant 1 + s + \epsilon' \leqslant 2$ and $0 \leqslant 1 - \sigma - \epsilon' \leqslant 2$. With this choice we similarly obtain

$$\begin{aligned} \left\| R(\lambda) D^k F_{\mu k} R(\lambda) f; H^s \right\| &\leqslant C_A \langle\lambda\rangle^{-1+(1+s+\epsilon')/2} \sum_k \left\| F_{\mu k} R(\lambda) f; H^{-\epsilon'} \right\| \\ &\leqslant C_A \langle\lambda\rangle^{-1+(1+s+\epsilon')/2} \left(\sum_k \|F_{\mu k}\|_2 \right) \left\| R(\lambda) f; H^{1-\epsilon'} \right\| \\ &\leqslant C_A \langle\lambda\rangle^{-1+(s-\sigma)/2} \left(\sum_k \|F_{\mu k}\|_2 \right) \left\| f; H^\sigma \right\|. \end{aligned} \tag{2.12}$$

Therefore, if $-1 < \sigma \leqslant s < 1$, we have

$$\left\| R(\lambda) \left[F_{\mu k}, D^k\right]_+ R(\lambda) f; H^s \right\| \leqslant C_A \langle\lambda\rangle^{-1+(s-\sigma)/2} \left(\sum_k \|F_{\mu k}\|_2 \right) \left\| f; H^\sigma \right\|. \tag{2.13}$$

This estimate shows (2.9) without the term $\|\partial^k F_{\mu k}\|_2$ since the integral (2.10) converges uniformly. If $\sigma \leqslant -1$ or $s \geqslant 1$, we can avoid either of (2.11) and (2.12) by the use of the relation $[D^k, F_{\mu k}] = (\partial^k F_{\mu k})$, and the following estimate which is proved similarly as (2.11) or (2.12):

$$\| R(\lambda)(\partial_k F_{\mu k}) R(\lambda) f; H^s \| \leqslant C_A \langle\lambda\rangle^{-1+(s-\sigma)/2} \|\partial_k F_{\mu k}\|_2 \| f; H^\sigma \|. \tag{2.14}$$

Thus the lemma has been proved. □

Lemma 2.7. *Let r, s, β satisfy $0 < s < 2$, $0 < \beta = 3/4 - 3/2r < 3/4$ and $0 < s + \beta \leqslant 2$. Let $A_\mu \in H^1$, $F_{\mu k} \in H_r^{-\beta}$ and $\partial^k F_{\mu k} \in L^2$. Then the estimate*

$$\| \Omega_A{}^s [\Omega_A{}^{-s}, D_\mu] f \|_2 \leqslant C_A \left\{ \left(\sum_k \| F_{\mu k}; H_r^{-\beta} \| \right) + \| \partial^k F_{\mu k} \|_2 \right\} \| f \|_2 \tag{2.15}$$

holds for any $f \in L^2$. Furthermore if $s + \beta \leqslant 1$, then we do not need to assume $\partial^k F_{\mu k} \in L^2$, and we can omit $\|\partial^k F_{\mu k}\|_2$ in the right-hand side.

Proof. Similarly as in the proof of Lemma 2.6, we begin with (2.10) with θ replaced by s. We shall estimate $\|\Omega_A{}^s R(\lambda)[F_{\mu k}, D^k]_+ R(\lambda) f\|_2$ by duality. Let g be an arbitrary L^2-function. By the Leibniz rule,

$$\begin{aligned}
&\left| \left(\Omega_A{}^s R(\lambda) F_{\mu k} D^k R(\lambda) f, g \right) \right| \\
&\quad \leqslant \| F_{\mu k}; H_r^{-\beta} \| \, \| \overline{\left(D^k R(\lambda) f \right)} R(\lambda) \Omega_A{}^s g; H_{r'}^{\beta} \| \\
&\quad \lesssim \| F_{\mu k}; H_r^{-\beta} \| \left\{ \| D^k R(\lambda) f; H_{r_*}^{\beta} \| \, \| R(\lambda) \Omega_A{}^s g \|_2 + \| D^k R(\lambda) f \|_{r_*} \| R(\lambda) \Omega_A{}^s g; H^\beta \| \right\}.
\end{aligned}$$

Here $1/r_* = 1/2 - 1/r$. By the Sobolev inequality together with Lemma 2.5, the right-hand side is bounded by $C_A \langle\lambda\rangle^{-1+s/2-\beta/6} (\sum_k \|F_{\mu k}; H_r^{-\beta}\|) \|f\|_2 \|g\|_2$, which yields

$$\| \Omega_A{}^s R(\lambda) F_{\mu k} D^k R(\lambda) f \|_2 \leqslant C_A \langle\lambda\rangle^{-1+s/2-\beta/6} \left(\sum_k \| F_{\mu k}; H_r^{-\beta} \| \right) \| f \|_2. \tag{2.16}$$

If $s + \beta \leqslant 1$, we can similarly show that $\|\Omega_A{}^s R(\lambda) D^k F_{\mu k} R(\lambda) f\|_2$ does not exceed the right-hand side of (2.16). If $s + \beta > 1$, we use the relation $[D^k, F_{\mu k}] = (\partial^k F_{\mu k})$. Hence it suffices to show

$$\| \Omega_A{}^s R(\lambda) \left(\partial^k F_{\mu k} \right) R(\lambda) f \|_2 \leqslant C_A \langle\lambda\rangle^{-3/2+s/2} \| \partial^k F_{\mu k} \|_2 \| f \|_2.$$

This inequality can easily be shown by Lemma 2.5. These estimates prove the lemma. □

3. Smoothing effects

In this section we shall show the smoothing effect of the following Schrödinger equation:

$$iD_0u = D_kD^ku + f. \tag{3.1}$$

Here f is defined on $[0,T] \times \boldsymbol{R}^2$. In this section we assume the following:

Assumption 3.1. $A = (A_\mu)$ satisfies *either* of the following:

(i) A does not depend on t and belongs to H^1;
(ii) $A \in C_TH^1$ with $F_{0k} \in L_T^1H_r^{-\beta}$ for some $2 < r < \infty$, where $\beta = 3/4 - 3/2r$, and $\partial^kF_{0k} \in L_T^1L^2$.

We call u an H^s-solution to (3.1) if $u \in C_TH^s \cap C_T^1H^{s-2}$ and satisfies (3.1). We first consider the homogeneous case, namely $f = 0$. Under the assumption (i), the operator $\mathscr{H} = D_kD^k + A_0$ with $\mathscr{D}(\mathscr{H}) = H^2$ is self-adjoint. Accordingly by Stone's theorem, the solution to (3.1) with $f = 0$ and $u(t_0,\cdot) = \phi \in L^2$ is expressed as

$$u(t) = \exp\bigl\{-i(t-t_0)\mathscr{H}\bigr\}\phi.$$

On the other hand, under the assumption (ii) we have the following:

Proposition 3.1. *Let* $1 \leqslant s < 2$ *and let* $0 \leqslant t_0 \leqslant T$. *We assume* (ii) *in Assumption* 3.1 *with* $s + \beta \leqslant 2$. *Then there exists a unique* H^s*-solution to* (3.1) *with* $f = 0$ *and* $u(t_0,\cdot) = \phi \in H^s$. *The solution* u *satisfies* $\|u(t)\|_2{}^2 = \|\phi\|_2{}^2$ *and*

$$\bigl\|\bigl(\varOmega_A{}^su\bigr)(t)\bigr\|_2 \leqslant \bigl\|\bigl(\varOmega_A{}^su\bigr)(t_0)\bigr\|_2 \exp\left\{C_A\left|\int_{t_0}^{t}\left(\left(\sum_k\bigl\|F_{0k}(\tau);H_r^{-\beta}\bigr\|\right) + \bigl\|\partial^kF_{0k}(\tau)\bigr\|_2\right)d\tau\right|\right\}. \tag{3.2}$$

If $1 \leqslant s < 2$, Proposition 3.1 defines the propagator $\{U(t,t_0)\}$ for (3.1) by the relation $u(t) = U(t,t_0)\phi$, where $\phi \in H^s$. By virtue of the L^2-norm conservation law, $\{U(t,t_0)\}$ is extended as a two-parameter family of unitary operators in L^2. Therefore, we can construct H^s-solutions for $0 \leqslant s < 1$. By Lemma 2.7, such solutions clearly satisfy (3.2) without the term $\|\partial^kF_{0k}\|_2$. If $f \not\equiv 0$, we can solve (3.1) by the Duhamel principle. Indeed, for $f \in C_TH^s$, the formula

$$u(t) = U(t,t_0)\phi - i\int_{t_0}^{t}U(t,\tau)f(\tau)\,d\tau$$

gives the solution to (3.1) with $u(t_0) = \phi$.

Proof of Proposition 3.1. If A and ϕ are sufficiently smooth, Kato's abstract theory of "hyperbolic" evolution equations (see [14,15,27]) proves the unique existence of smooth solutions, and once the a priori estimate (3.2) is obtained, we can obtain solutions for rough coefficients and data by compactness method. Indeed, approximating A and ϕ by some sequences of smooth functions, we can obtain a sequence $\{u_n\}$ of smooth solutions which converges star-weakly in $L_T^\infty H^s$. Let $u = \text{w}^*\text{-}\lim_n u_n$. Since each u_n satisfies (3.2), so as u. We can easily show that u is weakly continuous in H^s, absolutely continuous in H^{s-2} and satisfies (3.1) almost all $t \in [0, T]$. Such a function is unique, for

$$\partial_t \|u(t)\|_2^{\,2} = 2\,\mathrm{Im}\big(\big(D_k D^k + A_0\big)u, u\big)_{H^{-1}\times H^1} = 0, \tag{3.3}$$

from which the conservation of the L^2-norm yields. Once we have proved uniqueness, we immediately find the group property $U(t,t_1)U(t_1,t_0) = U(t,t_0)$. We obtain $\Omega_A{}^s u \in C_T L^2$ since it is weakly continuous and satisfies $\liminf_{t\to t_0} \|(\Omega_A{}^s u)(t)\|_2 \leqslant \|(\Omega_A{}^s u)(t_0)\|_2$ by virtue of the a priori estimate (3.2). This implies that $u \in C_T H^s \cap C_T^1 H^{s-2}$ and that u satisfies (3.1) for any t. Therefore u is the unique solution to (3.1) with $f = 0$, $u(t_0) = \phi$. Now we shall prove (3.2). $\Omega_A{}^s u$ satisfies the equation

$$i D_0 \Omega_A{}^s u = D_k D^k \Omega_A{}^s u + i\Omega_A{}^s\big[\Omega_A{}^{-s}, D_0\big]\Omega_A{}^s u.$$

Accordingly, standard energy method shows

$$\begin{aligned}
\big\|\big(\Omega_A{}^s u\big)(t)\big\|\Big|_{t=t_0}^{t=t} &\leqslant \left|\int_{t_0}^{t} \big\|\Omega_A{}^s\big[\Omega_A{}^{-s}, D_0\big]\Omega_A{}^s u\big\|_2\, d\tau\right| \\
&\leqslant \left| C_A \int_{t_0}^{t} \left(\left(\sum_k \big\|F_{0k}(\tau); H_r^{-\beta}\big\|\right) + \big\|\partial^k F_{0k}(\tau)\big\|_2\right) \big\|\Omega_A{}^s u(\tau)\big\|_2\, d\tau\right|.
\end{aligned}$$

Here we have used Lemma 2.7. Therefore we obtain (3.2) by the Gronwall inequality. □

Remark. (i) A function u is called a weak H^s-solution to (3.1) if $u \in L_T^\infty H^s \cap W_T^{1,\infty} H^{s-2}$ and satisfies (3.1) almost every t. By the proof above, weak H^1-solutions to (3.1) are unique under the assumption of Proposition 3.1.

(ii) If $0 < s < 1$, uniqueness of H^s-solutions can be proved in the same way as in the proof of Lemma 3.2 in [23]. However, if $s = 0$, we do not know whether the solutions are unique or not. This is because we use the unique existence of H^{2-s}-solutions to prove uniqueness of H^s-solutions.

Lemma 3.1. *We assume* (i) *or* (ii) *in Assumption* 3.1. *Let* $0 \leqslant s < 2$, $0 < \theta < 1$ *and* $(\theta - 1)/2 < m < \theta - 1/2$. *Let* $0 < 2/q_1 = 1 - 2/r_1 < 1$. *Then the solution* u *to* (3.1) *satisfies the following estimate*:

$$\sum_\alpha \big\|\chi_\alpha \Omega_A{}^{s-\theta} u; L_T^{q_1} H_{r_1}^m\big\|^2 \leqslant C_A \langle T\rangle^2 \big\|u; L_T^\infty H^s\big\|^2 + C_A T\big\|f; L_T^2 H^{s-2\theta+2m}\big\|^2. \tag{3.4}$$

Proof. By computation $\chi_\alpha {\Omega_A}^{s-\theta} u$ satisfies the equation

$$
\begin{aligned}
(i\partial_t+\Delta)\chi_\alpha {\Omega_A}^{s-\theta}u &= i\big[\partial_k, A^k\big]_+ \chi_\alpha {\Omega_A}^{s-\theta}u - \chi_\alpha A_k A^k {\Omega_A}^{s-\theta}u + \chi_\alpha A_0 {\Omega_A}^{s-\theta}u \\
&\quad + \big[\chi_\alpha, D_k D^k\big]{\Omega_A}^{s-\theta}u + i\chi_\alpha\big[D_0, {\Omega_A}^{s-\theta}\big]u + \chi_\alpha {\Omega_A}^{s-\theta} f \\
&\equiv g_\alpha + \chi_\alpha {\Omega_A}^{s-\theta} f.
\end{aligned}
$$

Applying Lemmas 2.1 and 2.3, we immediately show that the left-hand side of (3.4) is bounded by

$$
T^{-1}\big\|{\Omega_A}^{s-\theta}u; L_T^2 H^\theta\big\|^2 + T\sum_\alpha \big\|g_\alpha; L_T^2 H^{-\theta+2m}\big\|^2 + T\big\|{\Omega_A}^{s-\theta}f; L_T^2 H^{-\theta+2m}\big\|^2.
$$

By virtue of Lemma 2.5, we only have to estimate the middle term to prove (3.4). Under the assumption, $0<1-\theta<\theta-2m$, and hence $L^{2/(2-\theta)}\subset H^{-\theta+2m}$. Therefore, by the use of Lemmas 2.1 and 2.5 together with the Sobolev inequality, we obtain

$$
\begin{aligned}
\sum_\alpha \big\|\chi_\alpha\big(-A_0+A_kA^k\big){\Omega_A}^{s-\theta}u; H^{-\theta+2m}\big\|^2 &\lesssim \big\|\big(-A_0+A_kA^k\big){\Omega_A}^{s-\theta}u\big\|_{2/(2-\theta)}{}^2 \\
&\lesssim \big\|\big(-A_0+A_kA^k\big)\big\|_2{}^2 \big\|{\Omega_A}^{s-\theta}u\big\|_{2/(1-\theta)}{}^2 \\
&\lesssim C_A\big\|u; H^s\big\|^2.
\end{aligned}
$$

Next we estimate $\sum_\alpha \|[\partial_k, A^k]_+\chi_\alpha {\Omega_A}^{s-\theta}u; H^{-\theta+2m}\|^2$. We split this norm into two parts according to the relation $[\partial_k, A^k]_+ = -(\partial_k A^k)+2\partial_k A^k$ and estimate each part. Similarly as above, we can show $\sum_\alpha \|(\partial_k A^k)\chi_\alpha {\Omega_A}^{s-\theta}u; H^{-\theta+2m}\|^2 \leqslant C_A\|u; H^s\|^2$. Choosing $2/p_1 = 1-2/p_2 = 2\theta-2m-1$ and applying the Leibniz rule, we can estimate the second part as

$$
\begin{aligned}
&\sum_\alpha \big\|\partial_k\big(A^k\chi_\alpha {\Omega_A}^{s-\theta}u\big); H^{-\theta+2m}\big\|^2 \\
&\quad\lesssim \sum_k \big\|A_k {\Omega_A}^{s-\theta}u; H^{1-\theta+2m}\big\|^2 \\
&\quad\lesssim \sum_k \big(\big\|A_k; H^{1-\theta+2m}_{2/\theta}\big\|\big\|{\Omega_A}^{s-\theta}u\big\|_{2/(1-\theta)} + \|A_k\|_{p_1}\big\|{\Omega_A}^{s-\theta}u; H^{1-\theta+2m}_{p_2}\big\|\big)^2 \\
&\quad\lesssim C_A\big\|u; H^s\big\|^2.
\end{aligned}
$$

By the relation $[\chi_\alpha, D_kD^k] = -2(\partial_k\chi_\alpha)D^k + (\Delta\chi_\alpha)$, and the fact that $\{\operatorname{supp}\chi_\alpha\}$ overlap finitely,

$$
\sum_\alpha \big\|\big[\chi_\alpha, D_kD^k\big]{\Omega_A}^{s-\theta}u; H^{-\theta+2m}\big\|^2 \lesssim \big\|{\Omega_A}^{s-\theta}u; H^{1-\theta+2m}\big\|^2 \lesssim C_A\big\|u; H^s\big\|^2.
$$

Finally, we estimate $\sum_\alpha \|\chi_\alpha[D_0, {\Omega_A}^{s-\theta}]u; H^{-\theta+2m}\|^2$. If $s-\theta<0$, Lemma 2.6 shows that this is estimated by $\|[D_0, {\Omega_A}^{s-\theta}]u\|_2^2 \leqslant C_A\|u\|_2$. On the other hand if $s-\theta>0$, it is estimated by

$$
\big\|{\Omega_A}^{s-\theta}\big[{\Omega_A}^{-s+\theta}, D_0\big]{\Omega_A}^{s-\theta}u; H^{-\theta+2m}\big\|^2 \leqslant C_A\big\|\big[{\Omega_A}^{-s+\theta}, D_0\big]{\Omega_A}^{s-\theta}u; H^{s-2\theta+2m}\big\|^2,
$$

which is again estimated by $C_A\|\Omega_A{}^{s-\theta}u\|_2^2$ by virtue of Lemma 2.6 provided $-1 < s - 2\theta + 2m < \min\{s-\theta, 1\}$, which follows from the assumption of the lemma. Collecting these estimates, we obtain the desired result. □

Proposition 3.2. *We assume* (i) *or* (ii) *in Assumption* 3.1. *Let* $1/2 < \theta < 1$. *Then the solution* u *to* (3.1) *satisfies the following estimate*:

$$\sup_\alpha \sum_k \bigl\| D_k \Omega_A{}^{-\theta} u; L_T^2 L^2(Q_\alpha)\bigr\|^2$$
$$\leqslant C_A \langle T\rangle^{2+\epsilon} \bigl\langle |\!|\!| F_{12} |\!|\!|_T\bigr\rangle^2 \bigl\|u; L_T^\infty L^2\bigr\|^2 + C_A\langle T\rangle^{1+\epsilon}\bigl\|f; L_T^2 H^{1-2\theta}\bigr\|^2. \tag{3.5}$$

Here $0 < \epsilon < 1$ *is some positive number.*

Proof. In the proof we keep the index j to indicate a specific direction and assume that we do not take summation for j even if it appears repeatedly. Let $L_j = \Omega_A{}^{-\theta} h D_j \Omega_A{}^{-\theta}$, where $h = h(x^j)$ is a real-valued function. Then $L_j u$ satisfies

$$i D_0 L_j u = D_k D^k L_j u + L_j f + i[D_0, L_j]u + \bigl[L_j, D_k D^k\bigr]u.$$

Since $\Omega_A{}^{-\theta}$ clearly commutes with $D_k D^k$, a direct computation shows

$$\bigl[L_j, D_k D^k\bigr] = \Omega_A{}^{-\theta}\bigl\{ih\bigl[F_{jk}, D^k\bigr]_+ + 2D_j h' D_j - h'' D_j\bigr\}\Omega_A{}^{-\theta}. \tag{3.6}$$

Using (3.6) and the relation $\partial_t(L_j u, u) = (D_0 L_j u, u) + (L_j u, D_0 u)$, we can derive

$$2\bigl(h' D_j \Omega_A{}^{-\theta} u, D_j \Omega_A{}^{-\theta} u\bigr) = -i\partial_t(L_j u, u) + (L_j f, u) - (L_j u, f) + i\bigl([D_0, L_j]u, u\bigr)$$
$$+ i\bigl(h\bigl[F_{jk}, D^k\bigr]_+ \Omega_A{}^{-\theta} u, \Omega_A{}^{-\theta} u\bigr) - \bigl(\Omega_A{}^{-\theta} h'' D_j \Omega_A{}^{-\theta} u, u\bigr).$$

Now let h_0 be an increasing function such that $h_0'(t) = 1$ if $|t| \leqslant 1/2$ and $h_0'(t) = 0$ if $|t| \geqslant 1$. Then for any $\alpha \in \boldsymbol{Z}^2$, we choose $h(x^j) = h_0(x^j - \alpha^j)$ so that we can obtain

$$2\bigl\|D_j \Omega_A{}^{-\theta} u; L_T^2 L^2(Q_\alpha)\bigr\|^2$$
$$\leqslant -i(L_j u, u)\big|_{t=0}^T + \int_0^T \bigl\{\bigl|(L_j f, u)\bigr| + \bigl|(L_j u, f)\bigr|\bigr\}\,dt + \int_0^T \bigl|\bigl([D_0, L_j]u, u\bigr)\bigr|\,dt$$
$$+ 2\int_0^T \bigl|\bigl(hF_{jk} D^k \Omega_A{}^{-\theta} u, \Omega_A{}^{-\theta} u\bigr)\bigr|\,dt + \int_0^T \bigl|\bigl(\Omega_A{}^{-\theta} h'' D_j \Omega_A{}^{-\theta} u, u\bigr)\bigr|\,dt$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \tag{3.7}$$

Since L_j and $\Omega_A{}^{-\theta} h'' D_j \Omega_A{}^{-\theta}$ map $H^{1-2\theta}$ to L^2, we obtain

$$\mathrm{I} + \mathrm{II} + \mathrm{V} \lesssim \langle T\rangle \bigl\|u; L_T^\infty L^2\bigr\|^2 + \bigl\|f; L_T^2 H^{1-2\theta}\bigr\|^2.$$

In order to estimate III, we write

$$[D_0, L_j] = \left[D_0, \Omega_A{}^{-\theta}\right] h D_j \Omega_A{}^{-\theta} + \Omega_A{}^{-\theta} h [D_0, D_j] \Omega_A{}^{-\theta} + \Omega_A{}^{-\theta} h D_j \left[D_0, \Omega_A{}^{-\theta}\right].$$

The middle term is equal to $i\Omega_A{}^{-\theta} h F_{0j} \Omega_A{}^{-\theta}$. Then, by the use of Lemmas 2.4 and 2.5, we see that III in (3.7) is bounded by $C_A T \|u; L_T^\infty L^2\|^2$. In order to estimate IV, we decompose $\boldsymbol{R}^2$ into squares $\{Q_\alpha\}$ and apply Hölder's inequality for x, t and α. Then we obtain

$$\mathrm{IV} \leqslant \frac{1}{2} \sup_\alpha \sum_k \left\| D_k \Omega_A{}^{-\theta} u; L_T^2 L^2(Q_\alpha)\right\|^2 + 2 |||F_{12}|||_T{}^2 \sum_\alpha \left\| \Omega_A{}^{-\theta} u; L_T^2 L^\infty(Q_\alpha)\right\|^2.$$

Collecting these estimates, we obtain

$$\begin{aligned} &\sup_\alpha \sum_k \left\| D_k \Omega_A{}^{-\theta} u; L_T^2 L^2(Q_\alpha)\right\|^2 \\ &\quad \lesssim \langle T\rangle \left\|u; L_T^\infty L^2\right\|^2 + \left\| f; L_T^1 H^{1-2\theta}\right\|^2 + |||F_{12}|||_T{}^2 \sum_\alpha \left\| \Omega_A{}^{-\theta} u; L_T^2 L^\infty(Q_\alpha)\right\|^2. \quad (3.8) \end{aligned}$$

Let q_1, r_1, ϵ satisfy $0 < 2/r_1 = 1 - 2/q_1 < \epsilon < \theta - 1/2$. By the Sobolev inequality and Lemma 3.1,

$$\begin{aligned} \sum_\alpha \left\| \Omega_A{}^{-\theta} u; L_T^2 L^\infty(Q_\alpha)\right\|^2 &\lesssim T^{1-2/q_1} \sum_\alpha \left\| \chi_\alpha \Omega_A{}^{-\theta} u; L_T^{q_1} H_{r_1}^\epsilon\right\|^2 \\ &\leqslant \langle T\rangle^\epsilon \left\{C_A \langle T\rangle^2 \left\|u; L_T^\infty L^2\right\|^2 + C_A T \left\| f; L_T^2 H^{-2\theta+2\epsilon}\right\|^2\right\}. \end{aligned}$$

These estimates complete the proof. □

Corollary 3.1. *We assume* (i) *or* (ii) *in Assumption* 3.1. *Let* $0 < s_1 + s_2 = s < 2$ *with* $0 \leqslant s_1 < 1$ *and let* $1/2 < \theta < 1$. *Then the solution* u *to* (3.1) *satisfies the following estimate*:

$$\begin{aligned} &\sup_\alpha \left\{ \sum_k \left\| \chi_\alpha D_k \Omega_A{}^{s-\theta} u; L_T^2 L^2\right\|^2 + \sum_k \left\| \chi_\alpha D_k \Omega_A{}^{s_2-\theta} u; L_T^2 H^{s_1}\right\|^2 \right. \\ &\qquad \left. + \left\| \chi_\alpha \Omega_A{}^{s_2-\theta} u; L_T^2 H^{s_1+1}\right\|^2 \right\} \\ &\quad \leqslant C_A \langle T\rangle^{2+\epsilon} \left\langle \sum_{\mu,\nu} |||F_{\mu\nu}|||_T \vee \left\| \partial^k F_{0k}; L_T^\infty L^2\right\| \right\rangle^2 \left\|u; L_T^\infty H^s\right\|^2 \\ &\qquad + C_A \langle T\rangle^{1+\epsilon} \left\| f; L_T^2 H^{s+1-2\theta}\right\|^2. \quad (3.9) \end{aligned}$$

Here $0 < \epsilon < 1$ *is some positive number. If* $s < 2\theta$, *we do not need* $\|\partial^k F_{0k}; L_T^\infty L^2\|$ *in the right-hand side.*

Proof. One of the three norms in the left-hand side is, if it is added to $\|u; L_T^2 H^s\|^2$, greater than the other two norms. This is easily shown by the use of the identity $\|\Omega_A u\|_2{}^2 = \sum_k \|D_k u\|_2{}^2 + \|u\|_2{}^2$ and the estimates

$$\big\|\big[\chi_\alpha, {\Omega_A}^{s_1}\big]u\big\|_2 \leqslant C_A \big\|u; H^{s_1-1+0}\big\| \quad \text{and} \quad \big\|\big[D_k, {\Omega_A}^{s_1}\big]u\big\|_2 \leqslant C_A \big\|u; H^{s_1+0}\big\|,$$

which can be shown similarly as in the proof of Lemma 2.6 provided $s_1 < 1$. Therefore, it suffices to show that $\sup_\alpha \sum_k \|\chi_\alpha D_k {\Omega_A}^{s-\theta} u; L_T^2 L^2\|^2$ is bounded by the right-hand side of (3.9). Since ${\Omega_A}^s u$ satisfies the equation

$$i D_0 {\Omega_A}^s u = D_k D^k {\Omega_A}^s u + i\big[D_0, {\Omega_A}^s\big]u + {\Omega_A}^s f,$$

we can apply Proposition 3.2 with u and f replaced by ${\Omega_A}^s u$ and $i[D_0, {\Omega_A}^s]u + {\Omega_A}^s f$ respectively. Therefore we have only to estimate $\|[D_0, {\Omega_A}^s]u; H^{1-2\theta}\| = \|{\Omega_A}^s[{\Omega_A}^{-s}, D_0]{\Omega_A}^s u; H^{1-2\theta}\|$. An application of Lemma 2.6 shows that this is estimated by $C_A\{(\sum_k \|F_{0k}\|_2) + \|\partial^k F_{0k}\|_2\}\|u; H^s\|$. We find that the term $\|\partial^k F_{0k}\|_2$ can be omitted if $s < 2\theta$ by the assertion of the lemma. □

4. Maxwell–Schrödinger equations

In this section we consider the Maxwell–Schrödinger equations (MS), namely (1.1)–(1.2). The system (MS) is invariant under the gauge transform $u \to ue^{-i\phi}$, $A_\mu \to A_\mu + \partial_\mu\phi$, where ϕ is an arbitrary real-valued function. Therefore, uniqueness of solutions to (MS) without gauge condition clearly fails. This is why we consider (MS) in some fixed gauge. We consider the Lorentz gauge in which we assume

$$\partial_\mu A^\mu = 0. \tag{4.1}$$

In this gauge, the Maxwell equations (1.2) reduce to a system of wave equations. Therefore we consider the following system:

$$i D_0 u = D_k D^k u, \tag{4.2}$$

$$\partial_\nu \partial^\nu A^\mu = J^\mu. \tag{4.3}$$

Here $J_0 = |u|^2$ and $J_k = -2\,\mathrm{Im}(\bar{u} D_k u)$. We aim to prove the unique solvability of (4.2)–(4.3) in the energy space $X = H^1 \times H^1 \times L^2$ that the initial data $(u(0), A(0), \partial_t A(0))$ should belong to. We also assume that the data satisfy the compatibility condition

$$\partial_\mu A^\mu(0) = \partial_k \partial_t A^k(0) - \partial_k \partial^k A_0(0) + \big|u(0)\big|^2 = 0 \tag{4.4}$$

to ensure that the Lorentz gauge condition holds for any t. To see this, let $u \in C_T H^1 \cap C_T^1 H^{-1}$ and $A \in C_T H^1 \cap C_T^1 L^2$ satisfy (4.2)–(4.3). Then, $J_0 \in C_T^1 H_p^{-1}$, $J_k \in C_T L^p$ for any $1 < p < 2$, and $\partial_\mu J^\mu = 0$. Therefore, by (4.3), we find that $\partial_\nu \partial^\nu \partial_\mu A^\mu = \partial_\mu J^\mu = 0$. Furthermore, $\partial_\mu A^\mu(0) = \partial_t(\partial_\mu A^\mu)(0) = 0$ by virtue of (4.4). By the uniqueness of solutions to the wave equation, (4.1) holds for any t.

In the following we treat (4.3) as a system of Klein–Gordon equations by adding A_μ to the both-sides, namely $(\partial_\nu\partial^\nu + 1)A_\mu = A_\mu + J_\mu$. This is because we want to avoid using homogeneous Sobolev spaces which are suitable for wave equations, and as long as we only consider time local problem, this lower order term does not harm at all. Once we establish the unique existence of time-local solutions, we can prolong the solution globally in time for the sake of the conservation laws of total charge and total energy.

To prove the solvability of (4.2)–(4.3), we shall estimate A_μ by the norm

$$|||A_\mu|||_{1,T} = \Big(\sum_\alpha \big\|A_\mu; L_T^\infty H^1(Q_\alpha)\big\|^2\Big)^{1/2} \vee \big\|\partial_t A_\mu; L_T^\infty L^2\big\|.$$

We also introduce the operator $p_k^l = \delta_k^l - (1-\Delta)^{-1}\partial_k\partial^l$, where δ_k^l is the Kronecker delta. This operator is clearly bounded in L^2. (p_k^l) can be regarded as a modification of the Helmholtz projection because $\partial_l p_k^l = (1-\Delta)^{-1}\partial_k$ and $\partial^k p_k^l = (1-\Delta)^{-1}\partial^l$, which are exactly 0 in the usual Helmholtz projection case, are smoothing operators which map H_r^{-1} to L^r.

Lemma 4.1. *Let $2/q = 1/2 - 1/r = 2\beta/3$ with $2 < r < \infty$. Let (u, A) be a solution to the system* (4.2)–(4.3) *with initial data $(u(0), A(0), \partial_t A(0)) \in X$ satisfying* (4.4). *Then there exists a positive number T such that*

$$\big\|\Omega_A u; L_T^\infty L^2\big\| \leqslant 2\big\|(\Omega_A u)(0)\big\|_2,$$
$$|||A_\mu|||_{1,T} \vee \big\|\partial_\nu A_\mu; L_T^q H_r^{-\beta}\big\| \leqslant 2C \max_\mu \big\|\big(A_\mu(0), \partial_t A_\mu(0)\big); H^1 \times L^2\big\|.$$

Here T depends only on $\|(u(0), A(0), \partial_t A(0)); X\|$.

Proof. Without loss of generality we can assume $0 < T \leqslant 1$. We put $S(T) = \|\Omega_A u; L_T^\infty L^2\|$ and $M(T) = \max_{\mu,\nu}(|||A_\mu|||_{1,T} \vee \|\partial_\nu A_\mu; L_T^q H_r^{-\beta}\|)$. We estimate $S(T)$ by Proposition 3.1. By the relation $\partial^k F_{0k} = -J_0$ together with the Sobolev inequality for covariant derivatives (see [7]),

$$\int_0^T \Big(\Big(\sum_k \big\|F_{0k}; H_r^{-\beta}\big\|\Big) + \big\|\partial^k F_{0k}\big\|_2\Big)\, dt \leqslant T^{1-1/q} \sum_k \big\|F_{0k}; L_T^q H_r^{-\beta}\big\| + CT\big\|\Omega_A u; L_T^\infty L^2\big\|^2.$$

Therefore the estimate (3.2) yields

$$S(T) \leqslant S_0 \exp\big(C\langle M(T)\rangle^m T^{1-1/q}\langle S(T)\rangle^2\big), \tag{4.5}$$

where $S_0 = \|(\Omega_A u)(0)\|_2$ and m is a positive integer. We next estimate $M(T)$. By virtue of Lemma 2.1, we obtain

$$\begin{aligned}\big\|\partial_\nu A_\mu; H_r^{-\beta}\big\| &\leqslant \Big(\sum_\alpha \big\|\chi_\alpha \Omega^{-\beta}\partial_\nu A_\mu\big\|_r^{\,r}\Big)^{1/r} \\ &\lesssim \Big(\sum_\alpha \big\|\chi_\alpha \partial_\nu A_\mu; H^{-\beta}\big\|^r\Big)^{1/r} + \big\|\partial_\nu A_\mu; H_r^{-1-\beta}\big\|,\end{aligned}$$

which yields $\|\partial_\nu A_\mu; L_T^q H_r^{-\beta}\| \lesssim (\sum_\alpha \|\chi_\alpha \partial_\nu A_\mu; L_T^q H_r^{-\beta}\|^2)^{1/2} + \|\partial_\nu A_\mu; L_T^q H_r^{-1-\beta}\|$ since $q, r > 2$. We shall show the following inequalities:

$$\begin{aligned}
&\left\|\partial_\nu A_\mu; L_T^q H_r^{-1-\beta}\right\| \\
&\quad \leqslant C\left\|\left(A_\mu(0), \partial_t A_\mu(0)\right); L^2 \times H^{-1}\right\| + CT\left\|A_\mu; L_T^\infty H^{-1}\right\| + CT\left\|\Omega_A u; L_T^\infty L^2\right\|^2, \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
&|||A_\mu|||_{1,T} + \left(\sum_\alpha \left\|\chi_\alpha \partial_\nu A_\mu; L_T^q H_r^{-\beta}\right\|^2\right)^{1/2} \\
&\quad \leqslant C\left\|\left(A_\mu(0), \partial_t A_\mu(0)\right); H^1 \times L^2\right\| + CT\left\|A_\mu; L_T^\infty L^2\right\| \\
&\qquad + CT^{1/2}\begin{cases} \|\Omega_A u; L_T^\infty L^2\|^2, & \mu = 0, \\ \|\Omega_A u; L_T^\infty L^2\| \sup_\alpha \|\chi_\alpha D_k u; L_T^2 H^{1-\theta}\|, & \mu = k = 1, 2. \end{cases} \qquad (4.7)
\end{aligned}$$

The inequalities (4.6) and (4.7) are obtained by Lemma 2.2, combined with the estimates of $\|J_\mu; L_T^1 H^{-1}\|$ and $(\sum_\alpha \|J_\mu; L_T^1 L^2(Q_\alpha)\|^2)^{1/2}$ respectively. Therefore we can easily show (4.6) for $0 \leqslant \mu \leqslant 2$ and (4.7) for $\mu = 0$ by the Sobolev inequality for covariant derivatives. We next estimate $(\sum_\alpha \|J_k; L_T^1 L^2(Q_\alpha)\|^2)^{1/2}, k = 1, 2,$ to prove (4.7). By the Sobolev inequalities both for usual and covariant derivatives,

$$\begin{aligned}
\left\|J_k; L^2(Q_\alpha)\right\| &\lesssim \left\|u; L^{2/(1-\theta)}(Q_\alpha)\right\| \left\|D_k u; L^{2/\theta}(Q_\alpha)\right\| \\
&\lesssim \left(\sum_l \left\|D_l u; L^2(Q_\alpha)\right\|\right) \left\|\chi_\alpha D_k u; H^{1-\theta}\right\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_\alpha \left\|J_k; L_T^1 L^2(Q_\alpha)\right\|^2 &\lesssim \sum_\alpha \left(\sum_l \left\|D_l u; L_T^2 L^2(Q_\alpha)\right\|^2\right) \left\|\chi_\alpha D_k u; L_T^2 H^{1-\theta}\right\|^2 \\
&\lesssim \left\|\Omega_A u; L_T^2 L^2\right\|^2 \sup_\alpha \left\|\chi_\alpha D_k u; L_T^2 H^{1-\theta}\right\|^2,
\end{aligned}$$

which completes the proof of (4.7). On the other hand, from Corollary 3.1 we have

$$\sup_\alpha \left\|\chi_\alpha D_k u; L_T^2 H^{1-\theta}\right\| \leqslant C\langle M(T)\rangle^m \left\|\Omega_A u; L_T^\infty L^2\right\|,$$

where m is a positive integer. Combining this inequality with (4.6)–(4.7), we obtain

$$M(T) \leqslant M_0 + CTM(T) + C\langle M(T)\rangle^m T^{1/2} S(T)^2, \qquad (4.8)$$

where $M_0 = C \max_\mu \|(A_\mu(0), \partial_t A_\mu(0)); H^1 \times L^2\|$. After replacing m in (4.5) by greater number if necessary, we choose T so that $\exp(C\langle 2M_0\rangle^m T^{1-1/q}\langle 2S_0\rangle^2) < 2$ and that $2CTM_0 + C\langle 2M_0\rangle^m T^{1/2}(2S_0)^2 < M_0$. Then we can obtain the desired estimates $S(T) \leqslant 2S_0$ and $M(T) \leqslant 2M_0$. $\square$

Lemma 4.2. *Let $2/q = 1/2 - 1/r = 2\beta/3$ with $2 < r < \infty$, and $2/q_1 = 1 - 2/r_1$ with $4 < r_1 < \infty$. Let $0 < \sigma < 1/2$ and $1/2 - \sigma < \epsilon < \min\{(4/q_1 - 1)\sigma; (1-\sigma)/2\}$. Let $0 < T \leqslant 1$. Let (u, A) be a solution to the system* (4.2)–(4.3) *with initial data $(u(0), A(0), \partial_t A(0)) \in X$ satisfying* (4.4). *Let (u, A) satisfy the estimates $\|u; L_T^\infty H^1\| \leqslant S$ and $|||A_\mu|||_{1,T} \vee \|\partial_\nu A_\mu; L_T^q H_r^{-\beta}\| \leqslant M$. Here S, M are positive numbers. Let (u', A') be another solution to* (4.2)–(4.3) *having the same initial data and satisfying the same estimates as above. Then the following estimates hold*:

$$\big\|\big(\Omega_A{}^\sigma u\big)_-; L_T^\infty L^2\big\| \leqslant C_M S T^{1/2} \exp\big(C_M T^{1-1/q}\big) \max_\mu |||A_{\mu-}|||_{1,T}, \tag{4.9}$$

$$\big\|u_-; L_T^\infty L^2\big\| \leqslant C_M S T^{1/2} \max_\mu |||A_{\mu-}|||_{1,T}, \tag{4.10}$$

$$\Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^{q_1} H_{r_1}^{-\epsilon}\big\|^2\Big)^{1/2} \leqslant C_M\Big(\big\|u_-; L_T^\infty H^\sigma\big\| + S \max_\mu \big\|A_{\mu-}; L_T^\infty H^1\big\|\Big), \tag{4.11}$$

$$\big\|u_-; L_T^\infty H^\sigma\big\| \leqslant C_M\Big(\big\|\big(\Omega_A{}^\sigma u\big)_-; L_T^\infty L^2\big\| + \max_k \big\|A_{k-}; L_T^\infty H^1\big\|\Big), \tag{4.12}$$

$$\max_\mu |||A_{\mu-}|||_{1,T} \leqslant C_M T^{1/2-1/q_1} \langle S\rangle^2 \Big(\max_\mu |||A_{\mu-}|||_{1,T}\Big) \vee \big\|u_-; L_T^\infty H^\sigma\big\|. \tag{4.13}$$

Here $u_- = u - u'$, $A_{\mu-} = A_\mu - A'_\mu$.

Proof. We begin with the proof of (4.9). $\Omega_A{}^\sigma u$ satisfies the equation

$$i D_0 \Omega_A{}^\sigma u = D_k D^k \Omega_A{}^\sigma u + i\big[D_0, \Omega_A{}^\sigma\big] u. \tag{4.14}$$

We find that $(\Omega_A{}^\sigma u)_-$ satisfies the following equation, by taking the difference of (4.14) and the corresponding equation for $(\Omega_A{}^\sigma u)'$:

$$\begin{aligned} i D_0\big(\Omega_A{}^\sigma u\big)_- &= D_k D^k\big(\Omega_A{}^\sigma u\big)_- + i\Omega_A{}^\sigma\big[\Omega_A{}^{-\sigma}, D_0\big]\big(\Omega_A{}^\sigma u\big)_- \\ &\quad + A_{0-}\big(\Omega_A{}^\sigma u\big)' + \big(D_k D^k\big)_-\big(\Omega_A{}^\sigma u\big)' + i\big\{\Omega_A{}^\sigma\big[\Omega_A{}^{-\sigma}, D_0\big]\big\}_-\big(\Omega_A{}^\sigma u\big)'. \end{aligned}$$

A standard energy estimate shows

$$\begin{aligned} \big\|\big(\Omega_A{}^\sigma u\big)_-; L_T^\infty L^2\big\| \leqslant \int_0^T &\big\{\big\|\Omega_A{}^\sigma\big[\Omega_A{}^{-\sigma}, D_0\big]\big(\Omega_A{}^\sigma u\big)_-\big\|_2 + \big\|A_{0-}\big(\Omega_A{}^\sigma u\big)'\big\|_2 \\ &+ \big\|\big(D_k D^k\big)_-\big(\Omega_A{}^\sigma u\big)'\big\|_2 + \big\|\big\{\Omega_A{}^\sigma\big[\Omega_A{}^{-\sigma}, D_0\big]\big\}_-\big(\Omega_A{}^\sigma u\big)'\big\|_2\big\}\, dt. \end{aligned}$$

By Lemma 2.7, the first term in the integrand is bounded by $C_M \max_k \|F_{0k}; H_r^{-\beta}\| \|(\Omega_A{}^\sigma u)_-\|_2$. The second term is easily estimated by $C_M S\|A_{0-}; H^1\|$. We skip the estimate of the third term and proceed to the last term. On account of the formula (2.10), we write as follows, and the right-hand side is to be estimated term by term:

$$\begin{aligned}
&\{\Omega_A{}^\sigma[\Omega_A{}^{-\sigma}, D_0]\}_-(\Omega_A{}^\sigma u)' \\
&= c\int_0^\infty d\lambda\,\lambda^{-\sigma/2}\{\Omega_A{}^\sigma R(\lambda)[F_{0k}, D^k]_+ R(\lambda)\}_- \Omega_{A'}{}^\sigma u' \\
&= c\int_0^\infty d\lambda\,\lambda^{-\sigma/2}\{\Omega_A{}^\sigma R(\lambda)[F_{0k}, D^k]_+ R(\lambda)_- \Omega_{A'}{}^\sigma + \Omega_A{}^\sigma R(\lambda)[F_{0k}, D^k_-]_+ R(\lambda)'\Omega_{A'}{}^\sigma \\
&\quad + \Omega_A{}^\sigma R(\lambda)[F_{0k-}, D^{k\prime}]_+ R(\lambda)'\Omega_{A'}{}^\sigma + \Omega_A{}^\sigma R(\lambda)_-[F'_{0k}, D^{k\prime}]_+ R(\lambda)'\Omega_{A'}{}^\sigma \\
&\quad + (\Omega_A{}^\sigma)_- R(\lambda)'[F'_{0k}, D^{k\prime}]_+ R(\lambda)'\Omega_{A'}{}^\sigma\}u' = c\int_0^\infty d\lambda\,\lambda^{-\sigma/2}\{B_1(\lambda)+\cdots+B_5(\lambda)\}u',
\end{aligned}$$

where $R(\lambda) = (\Omega_A{}^2 + \lambda)^{-1}$, and the difference thereof is written as $R(\lambda)_- = -R(\lambda) \times (D_k D^k)_- R(\lambda)'$. Here we remark that $(D_k D^k)_- = i D_k A^k_- + i A_{k-} D^{k\prime}$ maps $H^{1+\delta}$ to L^2 for any $\delta > 0$ and

$$\|(D_k D^k)_-; H^{1+\delta} \to L^2\| \leqslant C_M \max_k \|A_{k-}; H^1\|. \tag{4.15}$$

This estimate, combined with the integral representation of $(\Omega_A{}^\sigma)_- = -\Omega_A{}^\sigma(\Omega_A{}^{-\sigma})_-\Omega_{A'}{}^\sigma$, shows that $\|(\Omega_A{}^\sigma)_-; H^{\sigma-1+2\delta} \to L^2\| \leqslant C_M \max_k \|A_{k-}; H^1\|$. Indeed, applying Lemma 2.5 together with (4.15), we can obtain

$$\begin{aligned}
\|(\Omega_A{}^\sigma)_-\phi\|_2 &\lesssim \int_0^\infty d\lambda\,\lambda^{-\sigma/2}\|\Omega_A{}^\sigma R(\lambda)(D_k A^k_- + A_{k-}D'_k)R(\lambda)'\Omega_{A'}{}^\sigma\phi\|_2 \\
&\leqslant \int_0^\infty d\lambda\,\lambda^{-\sigma/2} C_M\langle\lambda\rangle^{-1+\sigma/2}\max_k\|A_{k-}; H^1\|\langle\lambda\rangle^{-\delta/2}\|\phi; H^{\sigma-1+2\delta}\| \\
&\leqslant C_M \max_k\|A_{k-}; H^1\|\|\phi; H^{\sigma-1+2\delta}\|
\end{aligned} \tag{4.16}$$

for any $\phi \in H^{\sigma-1+2\delta}$. We proceed to the estimates of $B_j(\lambda)u'$. By the estimate (2.13) in the proof of Lemma 2.6 together with (4.15),

$$\begin{aligned}
\|B_1(\lambda)u'\|_2 &\leqslant C_M\langle\lambda\rangle^{-1+\sigma/2}\|(D_k D^k)_- R(\lambda)'(\Omega_A{}^\sigma u)'\|_2 \\
&\leqslant C_M\langle\lambda\rangle^{-1+\sigma/2}\max_k\|A_{k-}; H^1\|\|R(\lambda)'(\Omega_A{}^\sigma u)'; H^{1+\delta}\| \\
&\leqslant C_M\langle\lambda\rangle^{-1+(2\sigma+\delta-1)/2}\max_k\|A_{k-}; H^1\|\|u'\|_2.
\end{aligned}$$

$\|B_4(\lambda)u'\|_2$ is also bounded by the same quantity because B_4 can be regarded as the adjoint operator of B_1 with A and A' interchanged. By Lemma 2.5,

$$\begin{aligned}\|B_2(\lambda)u'\|_2 &= 2\|\Omega_A{}^\sigma R(\lambda) F_{0k} A^k_- R(\lambda)'(\Omega_A{}^\sigma u)'\|_2\\ &\leqslant C_M\langle\lambda\rangle^{-1+(\sigma+\delta)/2}\|F_{0k}A^k_- R(\lambda)'(\Omega_A{}^\sigma u)'; H^{-\delta}\|\\ &\leqslant C_M\langle\lambda\rangle^{-1+(\sigma+\delta)/2}\|F_{0k}\|_2\|A^k_-; H^1\|\|R(\lambda)'(\Omega_A{}^\sigma u)'; H^1\|\\ &\leqslant C_M\langle\lambda\rangle^{-1+(2\sigma+\delta-1)/2}\max_k\|A_{k-}; H^1\|\|u'\|_2.\end{aligned}$$

In the same way as in the proof of Lemma 2.6, we can show

$$\|B_3(\lambda)u'\|_2 \leqslant C_M\langle\lambda\rangle^{-1+(\sigma-\delta)/2}\max_\mu\|A_{\mu-}; H^1\|\|(\Omega_A{}^{\sigma+\delta}u)'\|_2.$$

By the estimate (2.13) together with (4.16),

$$\begin{aligned}\|B_5(\lambda)u'\|_2 &\leqslant C_M\max_k\|A_{k-}; H^1\|\|R(\lambda)'[F'_{0k}, D^{k\prime}]_+ R(\lambda)'(\Omega_A{}^\sigma u)'; H^{\sigma-1+2\delta}\|\\ &\leqslant C_M\langle\lambda\rangle^{-1+(2\sigma+2\delta-1)/2}\max_k\|A_{k-}; H^1\|\|u'\|_2.\end{aligned}$$

Collecting these estimates, we obtain

$$\|\{\Omega_A{}^\sigma[\Omega_A{}^{-\sigma}, D_0]\}_-(\Omega_A{}^\sigma u)'\|_2 \leqslant C_M S\max_\mu\|A_{\mu-}; H^1\|$$

provided $\sigma < 1$. As a result, we obtain the inequality

$$\begin{aligned}\|(\Omega_A{}^\sigma u)_-; L^\infty_T L^2\| \leqslant C_M\int_0^T \Big\{&\max_k\|F_{0k}; H_r^{-\beta}\|\|(\Omega_A{}^\sigma u)_-\|_2\\ &+\|(D_kD^k)_-(\Omega_A{}^\sigma u)'\|_2 + S\max_\mu\|A_{\mu-}; H^1\|\Big\}\,dt.\end{aligned}$$

Therefore, the Gronwall inequality yields

$$\begin{aligned}\|(\Omega_A{}^\sigma u)_-; L^\infty_T L^2\| \leqslant C_M&\exp\left(\int_0^T C_M\max_k\|F_{0k}; H_r^{-\beta}\|\,dt\right)\\ &\times\int_0^T\Big\{\|(D_kD^k)_-(\Omega_A{}^\sigma u)'\|_2 + S\max_\mu\|A_{\mu-}; H^1\|\Big\}\,dt.\end{aligned}$$

As in the proof of Lemma 4.1, $\int_0^T\|F_{0k}; H_r^{-\beta}\|\,dt \leqslant T^{1-1/q}\|F_{0k}; L^q_T H_r^{-\beta}\| \leqslant CMT^{1-1/q}$. Therefore, to obtain (4.9), it suffices to estimate $\|(D_kD^k)_-(\Omega_A{}^\sigma u)'; L^1_T L^2\|$. Decomposing $\boldsymbol{R}^2$ into squares $\{Q_\alpha\}_\alpha$, using (4.15), the Leibniz rule, the Hölder and the Sobolev inequality for x, and the Hölder inequality for t, α, we obtain

$$\begin{aligned}\big\|\big(D_kD^k\big)_-\big(\Omega_A{}^\sigma u\big)';L^1_TL^2\big\|^2 &\leqslant T\int_0^T\sum_\alpha\big\|\big(D_kA^k_-+A_{k-}D^{k\prime}\big)\chi_\alpha\big(\Omega_A{}^\sigma u\big)'\big\|_2{}^2\,dt\\ &\leqslant C_MT\sum_\alpha\int_0^T\max_k\big\|A_{k-};H^1(Q_\alpha)\big\|^2\big\|\chi_\alpha\big(\Omega_A{}^\sigma u\big)';H^{1+\delta}\big\|^2\,dt\\ &\leqslant C_MT\max_k|||A_{k-}|||_{1,T}{}^2\sup_\alpha\big\|\chi_\alpha\big(\Omega_A{}^\sigma u\big)';L^2_TH^{1+\delta}\big\|^2,\end{aligned}$$

which is further estimated by $C_MTS^2\max_k|||A_{k-}|||_{1,T}{}^2$ on account of Corollary 3.1. This completes the proof of (4.9).

We next prove (4.10) and (4.11). Taking the difference of (4.2) and the corresponding equation for u', we see

$$iD_0u_-=D_kD^ku_-+A_{0-}u'+\big(D_kD^k\big)_-u'.$$

A standard energy estimate shows $\|u_-;L^\infty_TL^2\|\leqslant\|A_{0-}u'+(D_kD^k)_-u';L^1_TL^2\|$. Therefore (4.10) follows from a similar and actually much easier estimate as in the proof of (4.9). An application of Lemma 3.1 with $(s,\theta,m)=(\sigma,\sigma,-\epsilon)$ shows

$$\Big(\sum_\alpha\big\|\chi_\alpha u_-;L^{q_1}_TH^{-\epsilon}_{r_1}\big\|^2\Big)^{1/2}\leqslant C_M\big\|u_-;L^\infty_TH^\sigma\big\|+C_M\big\|A_{0-}u'+\big(D_kD^k\big)_-u';L^\infty_TH^{-\sigma}\big\|.$$

Using the relation $(D_kD^k)_-=iD_kA^k_-+iA_{k-}D'_k$ and Lemma 2.4 together with the Sobolev inequality and the Leibniz rule, we can obtain $\|A_{0-}u'+(D_kD^k)_-u';H^{-\sigma}\|\leqslant C_MS\times\max_\mu\|A_{\mu-};H^1\|$. Combining this estimate with the inequality above, we obtain (4.11).

To prove (4.12), we write $(\Omega_A{}^\sigma u)_-=(\Omega_A{}^\sigma)u_-+(\Omega_A{}^\sigma)_-u'$. Applying Lemma 2.5 and (4.16), we can immediately show (4.12).

For the proof of (4.13), we split A_k into $p^l_kA_l$ and $(\delta^l_k-p^l_k)A_l$. On account of (4.1), we find $(\delta^l_k-p^l_k)A_l=-(1-\Delta)^{-1}\partial_k\partial_tA_0$ and $(\delta^l_k-p^l_k)\partial_tA_l=-(1-\Delta)^{-1}\partial_k(\Delta A_0+J_0)$, thereby obtaining

$$\big|\big|\big|\big(\delta^l_k-p^l_k\big)A_l\big|\big|\big|_{1,T}\lesssim T|||A_{0-}|||_{1,T}+\big\|J_{0-};L^1_TL^2\big\|+\big\|J_{0-};L^\infty_TH^{-1}\big\|$$

by virtue of Lemma 2.2. By the Sobolev inequality we can show $\|J_{0-}\|_2\lesssim S\|u_-;H^\sigma\|$ and $\|J_{0-};H^{-1}\|\lesssim S\|u_-\|_2$. Combined with (4.10), these estimates prove

$$\big|\big|\big|\big(\delta^l_k-p^l_k\big)A_l\big|\big|\big|_{1,T}\leqslant C_M\langle S\rangle T^{1/2}\Big(\max_\mu|||A_{\mu-}|||_{1,T}\Big)\vee\big\|u_-;L^\infty_TH^\sigma\big\|.$$

On the other hand, we have $|||p^l_kA_{l-}|||_{1,T}\lesssim T|||A_{k-}|||_{1,T}+(\sum_\alpha\|\chi_\alpha p^l_kJ_{l-};L^1_TL^2\|^2)^{1/2}$ by Lemma 2.2. To estimate the right-hand side, we split $(\bar uD_lu)_-=\partial_l(\bar uu_-)-(\overline{D_lu})u_-+i\bar uA_{l-}u'+\bar u_-(D_lu)'$ and correspondingly

$$\begin{aligned}
&\Big(\sum_\alpha \big\|\chi_\alpha p_k^l J_{l-}; L_T^1 L^2\big\|^2\Big)^{1/2} \\
&\quad\lesssim \big\|p_k^l \partial_l(\bar u u_-); L_T^1 L^2\big\| + \Big(\sum_\alpha \big\|\chi_\alpha p_k^l\big\{(\overline{D_l u})u_-\big\}; L_T^1 L^2\big\|^2\Big)^{1/2} \\
&\qquad + \sum_l \big\|\bar u A_{l-} u'; L_T^1 L^2\big\| + \Big(\sum_\alpha \big\|\chi_\alpha p_k^l\big\{\bar u_-(D_l u)'\big\}; L_T^1 L^2\big\|^2\Big)^{1/2}. \qquad (4.17)
\end{aligned}$$

The first term, as well as the third, is easily treated by the Sobolev inequality since $p_k^l \partial_l$ is bounded in L^2; these terms are estimated by $CT\langle S\rangle^2(\max_\mu |||A_{\mu-}|||_{1,T}) \vee \|u_-; L_T^\infty H^\sigma\|$. The second term is treated by using local smoothing property of u. Since p_k^l is a nonlocal operator, we take the commutator $[\chi_\alpha, p_k^l]$ and directly multiply χ_α by $\overline{D_l u}$. Estimating $[\chi_\alpha, p_k^l]\{(\overline{D_l u})u_-\}$ by the use of Lemma 2.1, we obtain

$$\begin{aligned}
&\Big(\sum_\alpha \big\|\chi_\alpha p_k^l\big\{(\overline{D_l u})u_-\big\}; L_T^1 L^2\big\|^2\Big)^{1/2} \\
&\quad\lesssim \Big(\sum_\alpha \big\|p_k^l\big\{\chi_\alpha(\overline{D_l u})u_-\big\}; L_T^1 L^2\big\|^2\Big)^{1/2} + \big\|(\overline{D_l u})u_-; L_T^1 H^{-1}\big\|.
\end{aligned}$$

We can easily estimate the second term in the right-hand side and obtain $\|(\overline{D_l u})u_-; L_T^1 H^{-1}\| \lesssim TS\|u_-; L_T^\infty H^\sigma\|$. For the estimate of the first term, we put $b = \epsilon/(\sigma+\epsilon)$ and $\theta = b\sigma + (1-b)(2/q_1 - \epsilon)$. Then by the Hölder and the Sobolev inequality, and Corollary 3.1

$$\begin{aligned}
\Big(\sum_\alpha \big\|\chi_\alpha(\overline{D_l u})u_-; L_T^1 L^2\big\|^2\Big)^{1/2} &\leqslant C\sup_\alpha \big\|\chi_\alpha D_l u; L_T^2 H^{1-\theta}\big\| \Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^2 L^{2/(1-\theta)}\big\|^2\Big)^{1/2} \\
&\leqslant C_M S\Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^2 L^{2/(1-\theta)}\big\|^2\Big)^{1/2}.
\end{aligned}$$

By the Sobolev inequality for x, and the Hölder inequality for t, α, together with (4.11), the right-hand side is estimated by

$$\begin{aligned}
&C_M S\Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^2 H^\sigma\big\|^2\Big)^{b/2}\Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^2 H_{r_1}^{-\epsilon}\big\|^2\Big)^{(1-b)/2} \\
&\quad\leqslant C_M S T^{1/2-1/q_1}\big\|u_-; L_T^\infty H^\sigma\big\|^b\Big(\sum_\alpha \big\|\chi_\alpha u_-; L_T^{q_1} H_{r_1}^{-\epsilon}\big\|^2\Big)^{(1-b)/2} \\
&\quad\leqslant C_M\langle S\rangle^2 T^{1/2-1/q_1}\big\|u_-; L_T^\infty H^\sigma\big\| \vee \max_\mu |||A_\mu|||_{1,T}.
\end{aligned}$$

The last term in the right-hand side of (4.17) is estimated analogously. These estimates prove (4.13). □

Proof of Theorem 1.2. Let $\{(u^{(n)}(0), A^{(n)}(0), \partial_t A^{(n)}(0))\}_n$ be a sequence of smooth functions converging to $(u(0), A(0), \partial_t A(0))$ in X. Then we can obtain the sequence $\{(u^{(n)}, A_\mu^{(n)})\}_n$ of corresponding solutions since the unique existence of solutions has established for smooth initial data (see [22]). For sufficiently small $T > 0$, the sequence $\{(u^{(n)}, A^{(n)}, \partial_t A^{(n)})\}_n$ is bounded in $L_T^\infty X$ by virtue of Lemma 4.1, and hence we can extract a subsequence which converges in the weak* sense. The weak* limit $(u, A, \partial_t A)$ satisfies (4.2)–(4.3) in the sense of distribution. Continuity of A for time variable is recovered by the representation formula

$$A_\mu(t) = \cos(\Omega t)A_\mu(0) + \frac{\sin(\Omega t)}{\Omega}(\partial_t A_\mu)(0) + \int_0^t \frac{\sin(\Omega(t-\tau))}{\Omega}\bigl\{A_\mu(\tau) + J_\mu(\tau)\bigr\}\,d\tau. \tag{4.18}$$

Then Proposition 3.1 recovers the continuity of $u(t)$ because this is the unique solution of (4.2) (see remark for Proposition 3.1). The uniqueness of solutions to (4.2)–(4.3) follows from Lemma 4.2. Indeed, from (4.9), (4.12) and (4.13), we can show

$$\bigl\|u_-; L_T^\infty H^\sigma\bigr\| \vee \max_\mu |||A_{\mu-}|||_{1,T} \leqslant C_{S,M} T^{1/2-1/q} \bigl\|u_-; L_T^\infty H^\sigma\bigr\| \vee \max_\mu |||A_{\mu-}|||_{1,T}$$

for sufficiently small T, thereby obtaining $u_- = A_{\mu-} = 0$. Finally we show the global existence. In the following C_1 denotes various constants depending only on $\|(u(0), A(0), \partial_t A(0)); X\|$. The conservation laws of the charge and the energy imply $\|\Omega_A u\|_2{}^2 + (1/2)\sum_{\mu<\nu}\|F_{\mu\nu}\|_2{}^2 \leqslant \mathscr{Q} + \mathscr{E} \leqslant C_1$. Moreover, multiplying $\partial_t A_0$ to (4.3) with $\mu = 0$, we obtain

$$\sum_\mu \bigl\|\partial_\mu A_0(t)\bigr\|_2{}^2\Big|_{t=0}^t \leqslant \int_0^t \bigl\|u(\tau)\bigr\|_4{}^2\,d\tau \leqslant C\int_0^t \bigl\|\Omega_A u(\tau)\bigr\|_2{}^2\,d\tau \leqslant C_1 t. \tag{4.19}$$

Clearly (4.19) ensures $\|\partial_\mu A_0(t)\| \leqslant C_1\langle t\rangle^{1/2}$. Since $\|F_{\mu\nu}(t)\|_2 = \|\partial_\mu A_\nu(t) - \partial_\nu A_\mu(t)\|_2 \leqslant C_1$, we also have $\|\partial_0 A_\mu(t)\|_2 \leqslant C_1\langle t\rangle^{1/2}$. Integrating this estimate with respect to t, we obtain $\|A_\mu(t)\|_2 \leqslant C_1\langle t\rangle^{3/2}$. On account of the relation $\partial^k F_{kl} = \partial^k\partial_k A_l - \partial_l\partial^k A_k$ together with (4.1), we obtain $\|(-\Delta)^{1/2}A_l(t)\|_2 \leqslant \sum_k \|F_{kl}(t)\|_2 + \|\partial_t A_l(t)\|_2 \leqslant C_1\langle t\rangle^{1/2}$. Therefore by Lemma 2.4, $\|u(t); H^1\| \leqslant C_1\langle t\rangle^{3/2}$. These estimates ensure that (u, A) does not blow up in finite time. □

Remark. In the proof above, we proved the existence of solutions by compactness method. This is a matter of taste and we can alternatively prove Theorem 1.2 by iteration. For given (u, A), we define $(u', A') = \Phi(u, A)$ by the equation

$$iD_0 u' = D_k D^k u', \qquad \bigl(\partial_\nu\partial^\nu + 1\bigr)A'_\mu = A_\mu + J'_\mu.$$

Here $J'_0 = |u'|^2$, $J'_k = -2\,\mathrm{Im}(\bar{u}' D_k u')$, and the initial data for (u', A') is the same as that for (MS). We put $(u^{(0)}, A^{(0)}) = (0, 0)$ and successively define $(u^{(n)}, A^{(n)}) = \Phi(u^{(n-1)}, A^{(n-1)})$. If we choose T sufficiently small, then $(u^{(n)}, A^{(n)}, \partial_t A^{(n)}) \in C_T X$, $n = 1, 2, \ldots$, and $(u^{(n)}, A^{(n)})$ satisfies the estimates in Lemma 4.1. Furthermore, in exactly the same way as in the proof of Lemma 4.2, we can show

$$\begin{aligned}&\left\|u^{(n+1)}-u^{(n)}; L_T^\infty H^\sigma\right\| \vee \left\|\!\left|A^{(n+1)}-A^{(n)}\right|\!\right\|_{1,T}\\&\quad\leqslant CT^{1/2-1/q}\left\|u^{(n)}-u^{(n-1)}; L_T^\infty H^\sigma\right\| \vee \left\|\!\left|A^{(n)}-A^{(n-1)}\right|\!\right\|_{1,T}.\end{aligned}$$

From this inequality we find that $(u^{(n)}, A^{(n)}, \partial_t A^{(n)})$ converges in $L_T^\infty(H^\sigma \times H^1 \times L^2)$, and the limit satisfies (MS). The uniqueness is proved as above, and the regularity of the solution is recovered by (4.18) and Proposition 3.1.

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