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Dedicated to Academician A.A. Dorodnicyn on the Occasion of the Centenary of His Birth

Application of a 14-Point Averaging Operator in the Grid Method

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Abstract—The Dirichlet problem for Laplace's equation in a rectangular parallelepiped is solved by applying the grid method. A 14-point averaging operator is used to specify the grid equations on the entire grid introduced in the parallelepiped. Given boundary values that are continuous on the parallelepiped edges and have first derivatives satisfying the Lipschitz condition on each parallelepiped face, the resulting discrete solution of the Dirichlet problem converges uniformly and quadratically with respect to the mesh size. Assuming that the boundary values on the faces have fourth derivatives satisfying the Hölder condition and the second derivatives on the edges obey an additional compatibility condition implied by Laplace's equation, the discrete solution has uniform and quartic convergence with respect to the mesh size. The convergence of the method is also analyzed in certain cases when the boundary values are of intermediate smoothness.

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1. STATEMENT OF THE PROBLEM

Suppose that $R = \{(x_1, x_2, x_3): 0 < x_i < a_i, i = 1, 2, 3\}$ is a rectangular parallelepiped; Γ_j $(j = 1, \ldots, 6)$ are its faces, including the edges; the face Γ_j with $j = 2i - 1$ $(j = 2i)$ lies in the plane $x_i = 0$ $(x_i = a_i)$; $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_6$ is the boundary of the parallelepiped; γ is the union of the edges of R; $\gamma_{lm} = \Gamma_l \cap \Gamma_m$; $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$; and $C_{k,\lambda}(E)$ is the class of functions that have continuous kth derivatives on E satisfying the Hölder condition with an exponent $\lambda \in (0, 1]$, which becomes the Lipschitz condition at $\lambda = 1$.

Consider the Dirichlet problem

$$\Delta u = 0 \text{ on } R, \quad u = \varphi_j \text{ on } \Gamma_j, \quad j = 1, \ldots, 6, \tag{1.1}$$

where φ_j are given functions. Assume that

$$\varphi_j \in C_{1,1}(\Gamma_j), \quad j = 1, \ldots, 6, \tag{1.2}$$

$$\varphi_l = \varphi_m \quad \text{on } \gamma_{lm}, \tag{1.3}$$

where $1 \le l \le 4$, $2[(l + 1)/2] + 1 \le m \le 6$ and $[a]$ is the integer part of a.

Problem (1.1) has a unique classical solution (see [1]) that is continuous on the closed parallelepiped $\bar{R}$.

Lemma 1. *The solution u of Dirichlet problem* (1.1) *belongs to the class* $C_{1,\mu}(\bar{R})$ *for any* $\mu \in (0, 1)$.

The proof follows from Theorem 2.2 in [2].

Let M_j be the constant (coefficient) in the Lipschitz condition satisfied, according to (1.2), by the first derivatives of φ_j $(j = 1, \ldots, 6)$ on Γ_j, and let

$$M = \max\{M_1, \ldots, M_6\}. \tag{1.4}$$

Lemma 2. *Under conditions* (1.2) *and* (1.3),

$$\max_{0 \le i \le 3} \sup_{(x_1, x_2, x_3) \in R} \left|\frac{\partial^2 u}{\partial x_i^2}\right| \le 2M,$$

where u solves Dirichlet problem (1.1) *and M is given by* (1.4).

The proof follows from Theorem 4.1 in [2].

We introduce a cubic grid with a step $h > 0$ defined by the planes $x_i = 0, h, 2h, \ldots, i = 1, 2, 3$. It is assumed that the edge lengths of R and h are such that $a_i/h \geq 4$ ($i = 1, 2, 3$) are integers. In what follows, h always satisfies this condition.

Let D_h be the set of nodes of the grid constructed, $\bar{R}_h = \bar{R} \cap D_h$, $R_h = R \cap D_h$, $R_h^k \subset R_h$ be the set of nodes of R_h lying at a distance of kh away from the boundary Γ of R, and $\Gamma_h = \Gamma \cap D_h$.

The 14-point averaging operator S on the grid is defined as

$$Su(x_1, x_2, x_3) \equiv \left(8\sum_{p=1}^{6}{}_1 u_p + \sum_{q=7}^{14}{}_3 u_q\right)(56)^{-1}, \quad (x_1, x_2, x_3) \in R_h, \tag{1.5}$$

where Σ_m is the sum extending over the nodes lying at a distance of $m^{1/2}h$ away from the point (x_1, x_2, x_3) and u_p and u_q are the values of u at the corresponding nodes.

On the boundary Γ of R, we define the function φ

$$\varphi = \begin{cases} \varphi_1 \text{ on } \Gamma_1 \\ \varphi_j \text{ on } \Gamma_1 \backslash \left(\bigcup_{i=1}^{j-1} \Gamma_i\right), \quad j = 2, \ldots, 6. \end{cases} \tag{1.6}$$

By virtue of condition (1.3), φ is continuous on the entire boundary of R, i.e., including the edges. Obviously,

$$\varphi = \varphi_j \text{ on } \Gamma_j, \quad j = 1, \ldots, 6.$$

Consider the following system of grid equations approximating Dirichlet problem (1.1):

$$u_h = Su_h \text{ on } R_h, \quad u_h = \varphi \text{ on } \Gamma_h, \tag{1.7}$$

where S is the averaging operator given by (1.5) and φ is the function defined by (1.6). By the maximum principle, which obviously holds for this system, system (1.7) has a unique solution.

Let $c, c_0, c_1, \ldots$ denote positive constants independent of the nearby multiplier, of which some possibly have identical values.

Theorem 1. *Under conditions* (1.2) *and* (1.3),

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \leq c_0 M h^2,$$

where u solves Dirichlet problem (1.1) *and u_h solves the system of grid equations* (1.7).

Before proving Theorem 1, we formulate several lemmas.

2. AUXILIARY LEMMAS

Consider two systems of grid equations

$$v_h = Sv_h + g_h \text{ on } R_h, \quad v_h = 0 \text{ on } \Gamma_h, \tag{2.1}$$

$$\bar{v}_h = S\bar{v}_h + \bar{g}_h \text{ on } R_h, \quad \bar{v}_h = 0 \text{ on } \Gamma_h, \tag{2.2}$$

where g_h and $\bar{g}_h$ are given functions and $|\bar{g}_h| \leq g_h$ on R_h.

Lemma 3. *The solutions v_h and $\bar{v}_h$ of systems* (2.1) *and* (2.2) *satisfy the inequality*

$$|\bar{v}_h| \leq v_h \text{ on } R_h.$$

The proof of Lemma 3 is similar to that of the comparison theorem in [3, Chapter III, Section 3].

Define

$$N(h) = [\min\{a_1, a_2, a_3\}/(2h)],$$

$$g_h^k = \begin{cases} 1, & (x_1, x_2, x_3) \in R_h^k \\ 0, & (x_1, x_2, x_3) \in R_h \backslash R_h^k, \end{cases} \tag{2.3}$$

and, for a fixed k such that $1 \le k \le N(h)$, consider the system of grid equations

$$v_h^k = Sv_h^k + g_h^k \text{ on } R_h, \quad v_h^k = 0 \text{ on } \Gamma_h. \tag{2.4}$$

Lemma 4. *The solution* v_h^k *of system* (2.4) *satisfies the inequality*

$$\max_{(x_1, x_2, x_3) \in R_h} v_h^k \le 5k, \quad 1 \le k \le N(h).$$

The proof of Lemma 4 is similar to that of Lemma 2 in [4].

Let $f = f(x)$ be a one-variable function on the interval $[0, a]$ of the x axis. Assume that this function has a derivative $f'(x)$ satisfying on $[0, a]$ the Lipschitz condition with a constant $\mathcal{F}$.

Lemma 5. *Let* $0 < x_0 < a$ *and* $0 < h \le \min\{x_0, a - x_0\}$. *Then*

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + \delta h^2,$$

where $\delta = \delta(x_0, h)$ *satisfies the inequality*

$$|\delta(x_0, h)| \le 2\mathcal{F}.$$

Lemma 5 is easily proved by applying the mean value theorem.

Lemma 6. *Let* $\bar{\sigma}_i$, $i \in \{1, 2, 3\}$, *be the segment formed by the nonempty intersection of a straight line parallel to the* x_i *axis and the closed parallelepiped* $\bar{R}$.

Then the trace on $\bar{\sigma}_i$ *of the solution u to Dirichlet problem* (1.1), *which is a function of the single variable* x_i, *has a derivative along* $\bar{\sigma}_i$ *that satisfies the Lipschitz condition with the constant* $2M$.

Proof. If $\bar{\sigma}_i$, $i \in \{1, 2, 3\}$, lies entirely on the boundary Γ of R, i.e., $\bar{\sigma}_i \cap R = \varnothing$, then the assertion of the lemma follows directly from (1.2)–(1.4). If $\bar{\sigma}_i \cap R \ne \varnothing$, then Lemma 6 follows from Lemma 1, according to which the solution of the Dirichlet problem has continuous first derivatives on $\bar{R}$, and from Lemma 2.

Lemma 6 is proved.

Lemma 7. *It is true that*

$$\max_{(x_1, x_2, x_3) \in R_h} |Su - u| \le 3Mh^2,$$

where u is a solution of Dirichlet problem (1.1), *S is the averaging operator* (1.5), *M is given by* (1.4), *and h is the mesh size of* R_h.

Proof. For notational brevity, we introduce $x_i^{+h} = x_i + h$, $x_i^{-h} = x_i - h$, $i = 1, 2, 3$.

Let $(x_1, x_2, x_3) \in R_h$. By Lemmas 5 and 6, we have

$$\begin{aligned}
&[u(x_1^{+h}, x_2^{+h}, x_3^{+h}) + u(x_1^{-h}, x_2^{+h}, x_3^{+h})] + [u(x_1^{+h}, x_2^{-h}, x_3^{+h}) + u(x_1^{-h}, x_2^{-h}, x_3^{+h})] \\
&+ [u(x_1^{+h}, x_2^{+h}, x_3^{-h}) + u(x_1^{-h}, x_2^{+h}, x_3^{-h})] + [u(x_1^{+h}, x_2^{-h}, x_3^{-h}) + u(x_1^{-h}, x_2^{-h}, x_3^{-h})] \\
&= 2[u(x_1, x_2^{+h}, x_3^{+h}) + u(x_1, x_2^{-h}, x_3^{+h})] + 2[u(x_1, x_2^{+h}, x_3^{-h}) + u(x_1, x_2^{-h}, x_3^{-h})] + \delta_1 h^2 \\
&= 4[u(x_1, x_2, x_3^{+h}) + u(x_1, x_2, x_3^{-h})] + \delta_2 h^2 = 8u(x_1, x_2, x_3) + \delta_3 h^2,
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&[u(x_1^{+h}, x_2, x_3) + u(x_1^{-h}, x_2, x_3)] + [u(x_1, x_2^{+h}, x_3) + u(x_1, x_2^{-h}, x_3)] \\
&+ [u(x_1, x_2, x_3^{+h}) + u(x_1, x_2, x_3^{-h})] = 6u(x_1, x_2, x_3) + \delta_4 h^2,
\end{aligned} \tag{2.6}$$

where u solves Dirichlet problem (1.1) and $\delta_k = \delta_k(x_1, x_2, x_3)$ satisfy the inequalities

$$|\delta_1| \le 16M, \quad |\delta_2| \le 32M, \quad |\delta_3| \le 48M, \quad |\delta_4| \le 12M \tag{2.7}$$

for any point $(x_1, x_2, x_3) \in R_h$.

Adding the initial sum in (2.5) times 1/56 to the left-hand side of (2.6) times 1/7 and equating the result to the sum of the final part of (2.5) times 1/56 and the right-hand side of (2.6) times 1/7, we obtain

$$Su = u + \delta h^2, \quad (x_1, x_2, x_3) \in R_h, \tag{2.8}$$

where S is the averaging operator defined by (1.5), u is a solution of Dirichlet problem (1.1), and $\delta = \delta(x_1, x_2, x_3)$ is such that

$$\delta = \frac{\delta_3}{56} + \frac{\delta_4}{7}$$

and, by virtue of (2.7),

$$|\delta| \le \frac{|\delta_3|}{56} + \frac{|\delta_4|}{7} \le 3M, \quad (x_1, x_2, x_3) \in R_h.$$

Combining this with (2.8), we deduce the assertion of Lemma 7.

Let $\rho = \rho(x_1, x_2, x_3)$ be the distance from the current point (x_1, x_2, x_3) in the open parallelepiped R to its boundary.

Lemma 8. *It is true that*

$$\max_{0 \le \mu \le 6} \max_{0 \le \nu \le 6-\mu} \left| \frac{\partial^6 u(x_1, x_2, x_3)}{\partial x_1^\mu \partial x_2^\nu \partial x_3^{6-\mu-\nu}} \right| \le cM\rho^{-4}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R, \tag{2.9}$$

where u solves Dirichlet problem (1.1) *and M is given by* (1.4).

Proof. Obviously, any sixth derivative of u can be obtained by differentiating a certain of its unmixed second derivatives four times. Applying Lemma 3 from [5, Chapter IV, Section 3] and Lemma 2, we derive inequality (2.9).

Lemma 8 is proved.

Introducing for brevity the notation $x_0 = (x_{01}, x_{02}, x_{03})$, we use the Taylor formula to represent the solution of the Dirichlet problem around some point $x_0 \in R_h$:

$$u(x_1, x_2, x_3) = p_5(x_1, x_2, x_3; x_0) + r_6(x_1, x_2, x_3; x_0), \tag{2.10}$$

where p_5 is the fifth-degree Taylor polynomial and r_6 is the remainder. Here,

$$p_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30}), \quad r_6(x_{10}, x_{20}, x_{30}; x_0) = 0.$$

Lemma 9. *It is true that*

$$Su(x_{10}, x_{20}, x_{30}) = u(x_{10}, x_{20}, x_{30}) + Sr_6(x_{10}, x_{20}, x_{30}; x_0), \quad (x_{10}, x_{20}, x_{30}) \in R_h, \tag{2.11}$$

where u solves the Dirichlet problem, r_6 *is the remainder in the Taylor formula*, *and S is the averaging operator defined by* (1.5).

Proof. Let $p_5(x_1, x_2, x_3; x_0)$ be a Taylor polynomial. Direct calculations yield

$$Sp_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30}) + \frac{3}{14}h^2 \sum_{i=1}^{3} \frac{\partial^2 u(x_{10}, x_{20}, x_{30})}{\partial x_i^2} + \frac{h^4}{56} \sum_{i=1}^{3} \frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_i^4}$$
$$+ \frac{h^4}{28}\left(\frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4 u(x_{10}, x_{20}, x_{30})}{\partial x_3^2 \partial x_1^2} \right).$$

Since u is a harmonic function, the second term on the right-hand side of this equality vanishes, while the third and fourth terms cancel each other out. Thus,

$$Sp_5(x_{10}, x_{20}, x_{30}; x_0) = u(x_{10}, x_{20}, x_{30}).$$

Combining this with (2.10) and recalling the linearity of S, we derive (2.11).

Lemma 9 is proved.

Lemma 10. *It is true that*

$$\max_{(x_1, x_2, x_3) \in R_h^k} |Su - u| \le c_1 M \frac{h^2}{k^4}, \quad k = 1, 2, \ldots, N(h), \tag{2.12}$$

where u solves the Dirichlet problem, M is given by (1.4), R_h^k *is the subset of nodes of* R_h *lying at a distance of kh away from the boundary of the parallelepiped R, and N(h) is given by* (2.3).

Proof. For $k = 1$, inequality (2.12) holds according to Lemma 7. Let us prove this inequality for the other values of k. Let $x_0 \in R_h^{k_0}$ be an arbitrary point for arbitrary k_0 such that $2 < k_0 < N(h)$. Let $r_6(x_1, x_2, x_3; x_0)$ be the Lagrange remainder corresponding to this point in Taylor formula (2.10). Then $Sr_6(x_{10}, x_{20}, x_{30}; x_0)$ can be expressed linearly in terms of a fixed number of sixth derivatives of u at some points of the open cube

$$\kappa_0 = \{(x_1, x_2, x_3): |x_i - x_{i0}| < h, i = 1, 2, 3\},$$

which is a distance of at least $k_0 h/2$ away from the boundary of R. The sum of the absolute values of the coefficients multiplying the sixth derivatives does not exceed ch^6, which is independent of k_0 ($2 \le k_0 \le N(h)$) or the point $x_0 \le R_h^{k_0}$. By Lemma 8, we have

$$Sr_6(x_{10}, x_{20}, x_{30}; x_0) \le c_1 M \frac{h^6}{(k_0 h)^4} = c_1 M \frac{h^2}{k_0^4},$$

where c_1 is a constant independent of k_0 ($2 \le k_0 \le N(h)$) or the point $x_0 \in R_h^{k_0}$. Combining (2.11) with this estimate for the averaged remainder yields (2.12) for $k = 2, 3, \ldots, N(h)$.

Lemma 10 is proved.

3. PROOF OF THEOREM 1

Let

$$\varepsilon_h = u_h - u \text{ on } R_h \cup \Gamma_h, \tag{3.1}$$

where u is the solution of Dirichlet problem (1.1) and u_h is the solution of system (1.7). Obviously, the error ε_h satisfies the system of grid equations

$$\varepsilon_h = S\varepsilon_h + (Su - u) \text{ on } R_h, \quad \varepsilon_h = 0 \text{ on } \Gamma_h, \tag{3.2}$$

where S is the averaging operator defined by (1.5).

Let $1 \le k \le N(h)$,

$$\varepsilon_h^k = S\varepsilon_h^k + \sigma_h^k \text{ on } R_h, \quad \varepsilon_h^k = 0 \text{ on } \Gamma_h, \tag{3.3}$$

where $N(h)$ is given by (2.4) and

$$\sigma_h^k = \begin{cases} Su - u & \text{on} \quad R_h^k \\ 0 \quad \text{on} & R_h \backslash R_h^k. \end{cases} \tag{3.4}$$

By Lemmas 3, 4, and 10, the solution ε_h^k of system (3.3) satisfies the inequality

$$\max_{(x_1, x_2, x_3) \in R_h} |\varepsilon_h^k| \le cM \frac{h^2}{k^3}, \quad 1 \le k \le N(h). \tag{3.5}$$

According to (3.1)–(3.4),

$$u_h - u = \varepsilon_h = \varepsilon_h^1 + \ldots + \varepsilon_h^{N(h)}.$$

Combining this with (3.5) produces

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le cMh^2 \sum_{k=1}^{N(h)} \frac{1}{k^3} \le c_0 Mh^2.$$

Theorem 1 is proved.

4. DIRICHLET PROBLEM WITH SECOND DERIVATIVES OF THE BOUNDARY VALUES COMPARTIBLE ON THE EDGES

Consider Dirichlet problem (1.1), assuming that the boundary values specified on the faces of R satisfy the conditions

$$\varphi_j \in C_{p,\lambda}(\Gamma_j), \quad 0 < \lambda < 1, \quad j = 1, \ldots, 6, \quad p \in \{2, 3\}, \tag{4.1}$$

$$\varphi_l = \varphi_m, \quad \frac{\partial^2 \varphi_l}{\partial t_l^2} + \frac{\partial^2 \varphi_m}{\partial t_m^2} + \frac{\partial^2 \varphi_l}{\partial t_{lm}^2} = 0 \quad \text{on} \quad \gamma_{lm}, \tag{4.2}$$

where $1 \le l \le 4$; $2[(l+1)/2] + 1 \le m \le 6$; $[a]$ is the integer part of a; t_{lm} is the arc length along the edge γ_{lm}; and t_l and t_m are normals to γ_{lm} on the faces Γ_l and Γ_m, respectively.

Lemma 11. *Under conditions* (4.1) *and* (4.2), *the solution u of Dirichlet problem* (1.1) *belong to the Hölder class* $C_{p,\lambda}(\bar{R})$, $0 < \lambda < 1$.

Lemma 11 follows from Theorem 2.1 in [2].

As before, we assume that $\rho = \rho(x_1, x_2, x_3)$ is the distance from the point (x_1, x_2, x_3) in R to its boundary.

Lemma 12. *Let u be the solution of Dirichlet problem* (1.1) *with conditions* (4.1) *and* (4.2), *and let* $\partial/\partial l \equiv \alpha_1\partial/\partial x_1 + \alpha_2\partial/\partial x_2 + \alpha_3\partial/\partial x_3$ *and* $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$.

Then

$$\left|\frac{\partial^6 u(x_1, x_2, x_3)}{\partial l^6}\right| \le c\rho^{p+\lambda-6}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R, \tag{4.3}$$

where c is a constant independent of the direction of the differentiation operator $\partial/\partial l$.

Proof. We choose an arbitrary point $(x_{10}, x_{20}, x_{30}) \in R$. Let $\rho_0 = \rho(x_{10}, x_{20}, x_{30})$ and $\bar{\sigma}_0 \subset \bar{R}$ be the closed ball of radius ρ_0 centered at (x_{10}, x_{20}, x_{30}).

Consider the harmonic function on R

$$v(x_1, x_2, x_3) = \partial^p u(x_1, x_2, x_3)/\partial l^p - \partial^p u(x_{10}, x_{20}, x_{30})/\partial l^p.$$

By Lemma 11, $u \in C_{p,\lambda}(\bar{R})$ for $0 < \lambda < 1$. Therefore,

$$\max_{(x_1, x_2, x_3) \in \bar{\sigma}_0} |v(x_1, x_2, x_3)| \le c_1\rho_0^{\lambda}, \tag{4.4}$$

where c_1 is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$. Using estimate (4.4) and applying Lemma 3 from [5, Chapter IV, Section 3], we obtain

$$\left|\frac{\partial^6 u(x_{10}, x_{20}, x_{30})}{\partial l^6}\right| \le c\rho_0^{p+\lambda-6},$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$.

Since the point $(x_{10}, x_{20}, x_{30}) \in R$ is arbitrary, inequality (4.3) holds true.

Lemma 12 is proved.

Lemma 13. *It is true that*

$$\max_{(x_1, x_2, x_3) \in R_h^1} |Su - u| \le ch^{p+\lambda}, \tag{4.5}$$

where u is the solution of Dirichlet problem (1.1) *with conditions* (4.1) *and* (4.2) *and S is the averaging operator defined by* (1.5).

Proof. Let (x_{10}, x_{20}, x_{30}) be a node of the grid $R_h^1 \subset R_h$, and let

$$\Theta_0 = \{(x_1, x_2, x_3): |x_i - x_{i0}| < h, i = 1, 2, 3\} \tag{4.6}$$

be an elementary cube some of whose faces lie on the boundary of R. The nodes of the operator S calculating the averaged value $Su(x_{10}, x_{20}, x_{30})$ of u lie at the vertices of the cube and at the centers of its faces.

Let us estimate the remainder r_6 in (2.10) at the point $(x_{10} + h, x_{20} + h, x_{30} + h)$, which is one of the nodes of S. Consider the function

$$\tilde{u}(s) = u(x_{10} + s/\sqrt{3}, x_{20} + s/\sqrt{3}, x_{30} + s/\sqrt{3}), \quad -\sqrt{3}h \le s \le \sqrt{3}h, \tag{4.7}$$

of single variable s, which is the arc length along the straight line through the points $(x_{10} - h, x_{20} - h, x_{30} - h)$ and $(x_{10} + h, x_{20} + h, x_{30} + h)$. Regardless of whether or not $(x_{10} + h, x_{20} + h, x_{30} + h)$ lies on the boundary of R, by Lemma 12, we have

$$|\tilde{u}^{(6)}(s)| \le c(\sqrt{3}h - s)^{p+\lambda-6}, \quad 0 < \lambda < 1, \quad 0 \le s < \sqrt{3}h, \tag{4.8}$$

where c is a constant independent of the chosen point $(x_{10}, x_{20}, x_{30}) \in R_h^1$.

By using the Taylor formula, function (4.7) around the point $s = 0$ can be represented as

$$\tilde{u}(s) = \tilde{p}_5(s) + \tilde{r}_6(s),$$

where $\tilde{p}_5(s)$ is the fifth-degree Taylor polynomial (in single variable s) and $\tilde{r}_6(s)$ is the remainder. Since

$$\tilde{p}_5(s) \equiv p_5(x_{10} + s/\sqrt{3}, x_{20} + s/\sqrt{3}, x_{30} + s/\sqrt{3}; x_0),$$

where $p_5(x_1, x_2, x_3; x_0)$ is the Taylor polynomial in (2.10), we have

$$r_6(x_{10} + s/\sqrt{3}, x_{20} + s/\sqrt{3}, x_{30} + s/\sqrt{3}; x_0) = \tilde{r}_6(s), \quad 0 \le |s| < \sqrt{3}h. \tag{4.9}$$

Since the remainder r_6 in (2.10) is continuous on the closure of cube (4.6) and $\tilde{r}_6(s)$ is continuous on the interval $[-\sqrt{3}h, \sqrt{3}h]$, it follows from (4.9) that

$$r_6(x_{10} + h, x_{20} + h, x_{30} + h; x_0) = \lim_{\varepsilon \to +0} \tilde{r}_6(\sqrt{3}h - \varepsilon). \tag{4.10}$$

Applying an integral representation for $\tilde{r}_6$ gives

$$\tilde{r}_6(\sqrt{3}h - \varepsilon) = \frac{1}{5!}\int_0^{\sqrt{3}h-\varepsilon} (\sqrt{3}h - \varepsilon - t)^5 \tilde{u}^{(6)}(t)dt, \quad 0 < \varepsilon \le \frac{\sqrt{3}h}{2}.$$

Using estimate (4.8), we obtain

$$|\tilde{r}_6(\sqrt{3}h - \varepsilon)| \le c_1 \int_0^{\sqrt{3}h-\varepsilon} (\sqrt{3}h - \varepsilon - t)^5 (\sqrt{3}h - t)^{p+\lambda-6} dt \le c_2 \int_0^{\sqrt{3}h-\varepsilon} (\sqrt{3}h - t)^{p+\lambda-1} dt \le ch^{p+\lambda},$$
$$0 < \varepsilon \le \frac{\sqrt{3}h}{2}. \tag{4.11}$$

Thus, combining (4.8), (4.10), and (4.11) yields

$$|r_6(x_{10} + h, x_{20} + h, x_{30} + h; x_0)| \le ch^{p+\lambda},$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R_h^1$.

Proceeding in a similar manner, we can find the same estimates of r_6 at the other vertices of cube (4.6) and at the centers of its faces. Since the norm of S in the uniform metric is equal to unity, we have

$$|Sr_6(x_{10}, x_{20}, x_{30}; x_0)| \le ch^{p+\lambda},$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R_h^1$. Combining this with equality (2.11) in Lemma 9 yields inequality (4.5).

Lemma 13 is proved.

Lemma 14. *Under conditions* (4.1) *and* (4.2),

$$\max_{(x_1, x_2, x_3) \in R_h^k} |Su - u| \le c_1 \frac{h^{p+\lambda}}{k^{6-p-\lambda}}, \quad k = 1, 2, \ldots, N(h), \tag{4.12}$$

where u solves Dirichlet problem (1.1) *and $N(h)$ is given by* (2.3).

Proof. For $k = 1$, inequality (4.12) holds by Lemma 13. For $k = 2, 3, \ldots, N(h)$, inequality (4.12) is proved in a similar fashion to inequality (2.12) with the only difference being that Lemma 8 is replaced by Lemma 12.

Lemma 14 is proved.

Theorem 2. *Let u be the solution of Dirichlet problem* (1.1) *with conditions* (4.1) *and* (4.2). *Then*

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le c_0 h^{p+\lambda}, \tag{4.13}$$

where u_h is the solution of the system of grid equations (1.7).

The proof of Theorem 2 is similar to that of Theorem 1 (see Section 3). The only difference is that Lemma 10 is replaced by Lemma 14 when the solution ε_h^k of system (3.3) is estimated. As a result, we obtain

$$\max_{(x_1, x_2, x_3) \in R_h} |\varepsilon_h^k| \le c \frac{h^{p+\lambda}}{k^{5-p-\lambda}}, \quad 0 < \lambda < 1, \quad k = 1, 2, \ldots, N(h),$$

and the final estimates become

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le \sum_{k=1}^{N(h)} \max_{(x_1, x_2, x_3) \in R_h} |\varepsilon_h^k| \le c h^{p+\lambda} \sum_{k=1}^{N(h)} \frac{1}{k^{5-p-\lambda}} \le c_0 h^{p+\lambda}.$$

Thus, inequality (4.13) holds.

Theorem 3. *Assume that* $\varphi_j \in C_{3,1}(\Gamma_j)$, $j = 1, \ldots, 6$; *i.e., on the faces of the parallelepiped R, the boundary data given in* (1.1) *have third derivatives satisfying the Lipschitz condition. Additionally, let conditions* (4.2) *hold.*

Then

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le c_0 h^4 (1 + |\ln h|), \tag{4.14}$$

where u is the solution of Dirichlet problem (1.1) *and u_h is the solution of the system of grid equations* (1.7).

The proof of Theorem 3 is based on the following three lemmas.

Lemma 15. *Under the conditions of Theorem* 3, *let u be the solution of Dirichlet problem* (1.1), $\rho(x_1, x_2, x_3)$ *be the distance from the current point of the open parallelepiped R to its boundary*, $\partial/\partial l \equiv \alpha_1 \partial/\partial x_1 + \alpha_2 \partial/\partial x_2 + \alpha_3 \partial/\partial x_3$, *and* $\alpha_1^2 + \alpha_2 + \alpha_3^2 = 1$.

Then

$$\left| \frac{\partial^6 u(x_1, x_2, x_3)}{\partial l^6} \right| \le c_1 \rho^{-2}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R, \tag{4.15}$$

where c_1 is a constant independent of the direction of the differentiation operator $\partial/\partial l$.

Proof. According to Theorem 1 in [4], we have

$$\max_{0 \le p \le 2} \max_{0 \le q \le 2-p} \sup_{(x_1, x_2, x_3) \in R} \left| \frac{\partial^4 u(x_1, x_2, x_3)}{\partial x_1^{2p} \partial x_2^{2q} \partial x_3^{4-2p-2q}} \right| \le c < \infty, \tag{4.16}$$

where u solves the Dirichlet problem in question. Any of the sixth derivatives of u can be obtained by the double differentiation of one of the derivatives included in the absolute value bars on the left-hand side of (4.16). Therefore, by Lemma 3 in [5, Chapter IV, Section 3], we conclude that

$$\max_{0 \le \mu \le 6} \max_{0 \le \nu \le 6-\mu} \left| \frac{\partial^6 u(x_1, x_2, x_3)}{\partial x_1^{\mu} \partial x_2^{\nu} \partial x_3^{6-\mu-\nu}} \right| \le c_2 \rho^{-2}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R. \tag{4.17}$$

Inequality (4.17) obviously implies (4.15).

Lemma 15 is proved.

Lemma 16. *It is true that*

$$\max_{(x_1, x_2, x_3) \in R_h^1} |Su - u| \le c_3 h^4,$$

where u is the solution of Dirichlet problem (1.1) *under the conditions of Theorem* 3 *and S is the averaging operator defined by* (1.5).

The proof of Lemma 16 nearly word-for-word follows the proof of Lemma 13 if we formally set $p + \lambda = 4$ in the latter and replace Lemma 12 by Lemma 15.

Lemma 17. *Under the conditions of Theorem* 3,

$$\max_{(x_1, x_2, x_3) \in R_h^k} |Su - u| \le c_4 \frac{h^4}{k^2}, \quad k = 1, 2, \ldots, N(h), \tag{4.18}$$

where u is the solution of the Dirichlet problem and N(h) *is given by* (2.3).

Proof. For $k = 1$, inequality (4.18) holds according to Lemma 16. For $k = 2, 3, \ldots, N(h)$, inequality (4.18) is proved in a similar manner to (2.12) with the only difference being that Lemma 8 is replaced by Lemma 15.

Lemma 17 is proved.

The proof of Theorem 3 is similar to that of Theorem 1 (see Section 3). The only difference is that Lemma 10 is replaced by Lemma 17 when the solution ε_h^k of system (3.3) is estimated. As a result, we have

$$\max_{(x_1, x_2, x_3) \in R_h} |\varepsilon_h^k| \le c\frac{h^4}{k}, \quad 1 \le k \le N(h),$$

and the final estimates become

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le \sum_{k=1}^{N(h)} \max_{(x_1, x_2, x_3) \in R_h} |\varepsilon_h^k| \le ch^4 \sum_{k=1}^{N(h)} \frac{1}{k} \le c_0 h^4 (1 + |\ln h|),$$

where u is the solution of Dirichlet problem (1.1) under the conditions of Theorem 3 and u_h is the solution of the system of grid equations (1.7).

Thus, inequality (4.14) holds.

Theorem 4. *Assume that* $\varphi_j \in C_{4,\lambda}(\Gamma_j)$, $\lambda > 0$, $j = 1, \ldots, 6$; *i.e.*, *on the faces of the parallelepiped R*, *the boundary values specified in* (1.1) *have fourth derivatives satisfying the Hölder condition. Additionally*, *let conditions* (4.2) *hold*.

Then

$$\max_{(x_1, x_2, x_3) \in R_h} |u_h - u| \le c_0 h^4,$$

where u is the solution of Dirichlet problem (1.1) *and* u_h *is the solution of the system of grid equations* (1.7).

The conditions of Theorem 4 and the assertion that the discrete solution of the Dirichlet problem on the rectangular parallelepiped has uniform and quartic convergence with respect to the mesh size coincide with the conditions and assertion of Theorem 1 in [6], respectively. The difference is in the method of defining the grid equations. In this paper, the grid equations on the entire grid (i.e., on R_h) are constructed using the 14-point averaging operator (1.5), while a combined grid method is applied in [6]. At boundary nodes, i.e., on R_h^1, the equations are defined using the classical 6-point averaging operator, while an 18-point averaging operator is used on $R_h \backslash R_h^1$. Thus, the grid equations in this paper are simpler.

The proof of Theorem 1 in [6] occupies nearly the entire paper and relies on novel subtle methods. The same methods can be used to prove Theorem 4. However, this proof is omitted for reasons of space.

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