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A Subclass of Harmonic Functions with Varying Arguments Defined by Hypergeometric Functions

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Abstract—We define the generalized Dziok-Srivastava operator for harmonic functions and introduce a new subclass of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. We investigate the coefficient bounds, distortion inequalities and extreme points for this generalized class of functions.

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1. INTRODUCTION

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| z^m + \overline{\sum_{m=1}^{\infty} |b_m| z^m}, \qquad |b_1| < 1 \tag{2}$$

where the analytic functions h and g are in the forms

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = b_1 z + \sum_{m=2}^{\infty} b_m z^m \qquad (0 \le b_1 < 1).$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, that is $g \equiv 0$. Let the Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{m=2}^{\infty} \phi_m z^m \tag{3}$$

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and

$$\psi(z) = z + \sum_{m=2}^{\infty} \psi_m z^m$$

in $\mathcal{S}$ be defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{m=2}^{\infty} \phi_m \psi_m z^m.$$

For complex parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ $(\beta_j \neq 0, -1, -2 \ldots; \; j = 1, \ldots, q)$ the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) := \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{z^m}{m!}, \tag{4}$$

$$(p \leq q+1; p, q \in N_0 := \; N \cup \{0\}; z \in \mathcal{U})$$

where N denotes the set of all positive integers and $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \begin{cases} 1, & m = 0 \\ a(a+1)(a+2)\ldots(a+m-1), & m \in \; N. \end{cases} \tag{5}$$

For positive real values of $\alpha_i > 0$, $(i = 1, \ldots, p)$, $\beta_j > 0$ $(j = 1, \ldots, q)$, $p \leq q+1; p, q \in N_0 = N \cup \{0\}$, let

$$H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q))(\phi)](z) &:= z\,{}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) * \phi(z) \\ &= z + \sum_{m=2}^{\infty} \omega_m(\alpha_1; p; q) \phi_m z^m, \end{aligned} \tag{6}$$

where

$$\omega_m(\alpha_1; p; q) = \frac{(\alpha_1)_{m-1} \ldots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \ldots (\beta_q)_{m-1}} \frac{1}{(m-1)!}. \tag{7}$$

For notational simplicity, we use a shorter notation $H_q^p[\alpha_1, \beta_1]$ for $H(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q)$ in the sequel.By using the generalized hypergeometric function, Dziok and Srivastava [3] introduced a linear operator which was subsequently extended for harmonic functions by the authors [10]. Now we recall generalized linear operator involving hypergeometric function $\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1}$ as follows:

$$\mathcal{L}_{\lambda,\alpha_1}^{0} \phi(z) = \phi(z), \tag{8}$$

$$\mathcal{L}_{\lambda,p,q}^{1,\alpha_1} \phi(z) = (1-\lambda) H_q^p[\alpha_1] \phi(z) + \lambda (H_q^p[\alpha_1] \phi(z))' = \mathcal{L}_{\lambda,p,q}^{\alpha_1} \phi(z), (\lambda \geq 0).$$

$$\mathcal{L}_{\lambda,p,q}^{2,\alpha_1} \phi(z) = \mathcal{L}_{\lambda,p,q}^{\alpha_1} (\mathcal{L}_{\lambda,p,q}^{1,\alpha_1} \phi(z)) \tag{9}$$

and in general,

$$\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} \phi(z) = \mathcal{L}_{\lambda,p,q}^{\alpha_1} (\mathcal{L}_{\lambda,p,q}^{\tau-1,\alpha_1} \phi(z)), \}; z \in U). \tag{10}$$

If the function $\phi(z)$ is given by$(p \leq q+1; p, q \in N_0 = N \cup \{0\})$ (3), then we see from (6), (7), (8), and (10) that

$$\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} \phi(z) := z + \sum_{m=2}^{\infty} \omega_m^{\tau}(\alpha_1; \lambda; p; q) \phi_m z^m \tag{11}$$

where

$$\omega_m^\tau(\alpha_1;\lambda;p;q) = \left(\frac{(\alpha_1)_{m-1}\dots(\alpha_l)_{m-1}}{(\beta_p)_{m-1}\dots(\beta_q)_{m-1}}\frac{[1+\lambda(m-1)]}{(m-1)!}\right)^\tau, \quad (m \in N\setminus\{1\}, \tau \in N_0) \tag{12}$$

unless otherwise stated. We note that when $\tau = 1$ and $\lambda = 0$ the linear operator$\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1}$ would reduce to the familiar Dziok-Srivastava linear operator given by [3] (also see, [4]).

In view of the relationship (12) and the linear operator (11) for the harmonic function $f = h + \overline{g}$ given by (1), we define the operator

$$\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(z) = \mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} g(z)}. \tag{13}$$

Goodman [5] introduced two interesting subclasses of S, namely, uniformly convex functions (UCV) and uniformly starlike functions (UST), and Ronning [11] introduced a subclass of starlike functions S_p corresponding to the class (UCV). Motivated by the earlier works of [6–9] on the subject of harmonic functions, we introduce here a new subclass $\mathcal{GL}_H(\tau;\lambda;k,\gamma)$ of $\mathcal{H}$ in terms of the operator defined by (13).

Let $\mathcal{GL}_H(\tau;\lambda;k,\gamma)$ denote a subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \overline{g}$ given by (2) satisfying the condition that

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(z))'}{z'(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(z))} - e^{i\psi}\right\} \geq \gamma \tag{14}$$

$$= \text{Re}\left\{(1+e^{i\psi})\frac{z(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} h(z))' - \overline{z(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} g(z))'}}{\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} g(z)}} - e^{i\psi}\right\} \geq \gamma,$$

$(z = re^{i\theta};\ 0 \leq \theta < 2\pi;\ 0 \leq r < 1;\ 0 \leq \gamma < 1;\ z \in \mathbb{U})$ where $(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(z))' = \frac{\partial}{\partial\theta}(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(re^{i\theta}))$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$ and $\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} f(z)$ is defined by (13).

We also let $\mathcal{VL}_H(\tau;\lambda;k,\gamma) = \mathcal{GL}_H(\tau;\lambda;k,\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [6], consisting of functions f of the form (1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi(\text{mod}2\pi), \qquad \delta_m + (m-1)\phi \equiv 0, m \geq 2, \tag{15}$$

where $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

In this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $\mathcal{GL}_H(\tau;\lambda;k,\gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$. Further, distortion results and extreme points for functions in $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ are also obtained.

2. THE CLASS $\mathcal{GL}_H(\tau;\lambda;k,\gamma)$

We begin deriving a sufficient coefficient condition for the functions belonging to the class $\mathcal{GL}_H(\tau;\lambda;k,\gamma)$. This result is contained in the following.

Theorem 1. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{m=2}^{\infty}\left(\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\right)\omega_m^\tau(\alpha_1;\lambda;p;q) \leq 1 - \frac{3+\gamma}{3-\gamma}b_1 \tag{16}$$

$0 \leq \gamma < 1$, then $f \in \mathcal{GL}_H(\tau;\lambda;k,\gamma)$.

Proof. We first show that if the inequality (16) holds for the coefficients of $f = h + \overline{g}$, then the required condition (14) is satisfied. Using (13) and (14), we can write

$$\text{Re}\left\{(1+e^{i\psi})\left[\frac{z(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} h(z))' - \overline{z(\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} g(z))'}}{\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} h(z) + \overline{\mathcal{L}_{\lambda,p,q}^{\tau,\alpha_1} g(z)}} - e^{i\psi}\right]\right\} = \text{Re}\frac{A(z)}{B(z)},$$

where

$$A(z) = (1+e^{i\psi})[z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}h(z))' - \overline{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}g(z))'}] - e^{i\psi}\left[\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}g(z)}\right]$$

and

$$B(z) = \mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}g(z)}.$$

In view of the simple assertion that $\mathrm{Re}(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{17}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (17), we get

$$\begin{aligned} &|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\ &\geq (2-\gamma)|z| - \sum_{m=2}^{\infty}(2m-\gamma)\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m||z|^m - \sum_{m=1}^{\infty}(2m+\gamma)\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m|||z|^m \\ &- \gamma|z| - \sum_{m=2}^{\infty}(2m-2-\gamma)\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m||z|^m - \sum_{m=1}^{\infty}(2m+2+\gamma)\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m||z|^m. \\ &\geq 2(1-\gamma)|z|\left\{1 - \frac{3+\gamma}{1-\gamma}b_1 - \left(\sum_{m=2}^{\infty}\left[\frac{2m-1-\gamma}{1-\gamma}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m|\right.\right.\right. \\ &\left.\left.\left. + \frac{2m+1+\gamma}{1-\gamma}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m|\right]\right)\right\} \geq 0 \end{aligned}$$

by virtue of the inequality (16). This implies that $f \in \mathcal{GL}_H(\tau;\lambda;k,\gamma),\gamma)$. □

Now we obtain the necessary and sufficient condition for function $f = h + \overline{g}$ be given with condition (15).

Theorem 2. *Let $f = h + \overline{g}$ be given by (2). Then $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$ if and only if*

$$\sum_{m=2}^{\infty}\left\{\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\right\}\omega^{\tau}_m(\alpha_1;\lambda;p;q) \leq 1 - \frac{3+\gamma}{1-\gamma}b_1 \tag{18}$$

$0 \leq \gamma < 1$ *and $\omega^{\tau}_m(\alpha_1;\lambda;p;q)$ is given by (12).*

Proof. Since $\mathcal{VL}_H(\tau;\lambda;k,\gamma) \subset \mathcal{GL}_H(\tau;\lambda;k,\gamma)$, we only need to prove the necessary part of the theorem. Assume that $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$, then by virtue of (13) to (14), we obtain

$$\mathrm{Re}\left\{(1+e^{i\psi})\left[\frac{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}h(z))' - \overline{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}g(z))'}}{\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,p,q}g(z)}} - (e^{i\psi}+\gamma)\right]\right\} \geq 0.$$

The above inequality is equivalent to

$$\mathrm{Re}\left\{\frac{z + \sum_{m=2}^{\infty}[m(1+e^{i\psi}) - \gamma - e^{i\psi}]\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m|z^m}{z + \sum_{m=2}^{\infty}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m|z^m + \sum_{m=1}^{\infty}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m|\overline{z}^m}\right\}$$

$$-\,\mathrm{Re}\left\{\frac{\sum_{m=1}^{\infty}[m(1+e^{i\psi}) + \gamma + e^{i\psi}]\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m|\overline{z}^m}{z + \sum_{m=2}^{\infty}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|a_m|z^m + \sum_{m=1}^{\infty}\omega^{\tau}_m(\alpha_1;\lambda;p;q)|b_m|\overline{z}^m}\right\}$$

$$= \operatorname{Re}\left\{\frac{(1-\gamma)+\sum_{m=2}^{\infty}[m(1+e^{i\psi})-e^{i\psi}-\gamma]\omega_m^\tau(\alpha_1;\lambda;p;q)|a_m|z^{m-1}}{1+\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum_{m=1}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|b_m|\overline{z}^{m-1}}\right\}$$

$$-\operatorname{Re}\left\{\frac{\frac{\overline{z}}{z}\sum_{m=1}^{\infty}[m(1+e^{i\psi})+e^{i\psi}+\gamma]\omega_m^\tau(\alpha_1;\lambda;p;q)|b_m|\overline{z}^{m-1}}{1+\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum_{m=1}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|b_m|\overline{z}^{m-1}}\right\}\geq 0.$$

This condition must hold for all values of z, such that $|z| = r < 1$. Upon choosing ϕ according to (15) and noting that $Re(-e^{i\psi}) \geq -|e^{i\psi}| = -1$, the above inequality reduces to

$$\frac{(1-\gamma)-(3+\gamma)b_1-\sum_{m=2}^{\infty}\left[(2m-1-\gamma)\omega_m^\tau(\alpha_1;\lambda;p;q)|a_m|r^{m-1}+(2m+1+\gamma)\omega_m^\tau(\alpha_1;\lambda;p;q)|b_m|r^{m-1}\right]}{1-\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|a_m|r^{m-1}+\sum_{m=1}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)|b_m|r^{m-1}} \geq 0. \tag{19}$$

If (18) does not hold, then the numerator in (19) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in (0,1) for which the quotient in (19) is negative. This contradicts our assumption that $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (18) holds true when $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$. This completes the proof of Theorem 2. □

If we put $\phi = 2\pi/k$ in (15), then Theorem 2 gives the following corollary.

Corollary 1. *A necessary and sufficient condition for $f = h + \overline{g}$ satisfying (18) to be starlike is that*

$$\arg(a_m) = \pi - 2(m-1)\pi/k,$$

and

$$\arg(b_m) = 2\pi - 2(m-1)\pi/k, (k = 1, 2, 3, \dots).$$

3. DISTORTION AND EXTREME POINTS

In this section we obtain the distortion bounds for the functions $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$ that lead to a covering result for the family $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$.

Theorem 3. *If $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$ then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right)r^2.$$

Proof. We will only prove the right- hand inequality of the above theorem. The arguments for the left- hand inequality are similar and so we omit it. Let $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$ taking the absolute value of f, we obtain

$$|f(z)| \leq (1+|b_1|)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \leq (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|).$$

This implies that

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}\left(\frac{1-\gamma}{3-\gamma}\right)$$
$$\times \sum_{m=2}^{\infty}\left[\left(\frac{3-\gamma}{1-\gamma}\right)\frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}|a_m| + \left(\frac{3-\gamma}{1-\gamma}\right)\frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}|b_m|\right] r^2$$
$$\leq (1+|b_1|)r + \frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}\left(\frac{1-\gamma}{3-\gamma}\right)\left[1 - \frac{3+\gamma}{1-\gamma}|b_1|\right] r^2$$
$$\leq (1+|b_1|)r + \frac{1}{\omega_2^\tau(\alpha_1;\lambda;p;q)}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right) r^2,$$

which establishes the desired inequality. □

As a consequence of the above theorem and Corollary (1), we state the following covering lemma.

Corollary *Let $f = h + \overline{g}$ and of the form (2) be so that $f \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$. Then*

$$\left\{ w : |w| < \frac{3\omega_2^\tau(\alpha_1;\lambda;p;q) - 1 - [\omega_2^\tau(\alpha_1;\lambda;p;q) - 1]\gamma}{(3-\gamma)\omega_2^\tau(\alpha_1;\lambda;p;q)} - \frac{3\omega_2^\tau(\alpha_1;\lambda;p;q) - 1 - [\omega_2^\tau(\alpha_1;\lambda;p;q) + 1]b_1}{(3-\gamma)\omega_2^\tau(\alpha_1;\lambda;p;q)} \right\} \subset f(\mathcal{U}).$$

For a compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Unlike many other classes, characterized by necessary and sufficient coefficient conditions, the family $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ is not a convex family. Nevertheless, we may still apply the coefficient characterization of the $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ to determine the extreme points.

Theorem 4. *The closed convex hull of $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ (denoted by $clco\mathcal{VL}_H(\tau;\lambda;k,\gamma)$) is*

$$\left\{ f(z) = z + \sum_{m=2}^{\infty}|a_m|z^m + \overline{\sum_{m=1}^{\infty}|b_m|z^m}, : \sum_{m=2}^{\infty} m[|a_m| + |b_m|] < 1 - b_1 \right\}$$

By setting $\rho_m = \dfrac{(1-\gamma)}{(2m-1-\gamma)\omega_m^\tau(\alpha_1;\lambda;p;q)}$ and $\mu_m = \dfrac{(1+\gamma)}{(2m+1+\gamma)\omega_m^\tau(\alpha_1;\lambda;p;q)}$, then for b_1 fixed, the extreme points for clco $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ are

$$\{z + \rho_m x z^m + \overline{b_1 z}\} \cup \{z + \overline{b_1 z + \mu_m x z^m}\} \tag{20}$$

where $m \geq 2$ and $|x| = 1 - |b_1|$.

Proof. Any function f in $clco\mathcal{VL}_H(\tau;\lambda;k,\gamma)$, be expressed as

$$f(z) = z + \sum_{m=2}^{\infty}|a_m|e^{i\eta_m}z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty}|b_m|e^{i\delta_m}z^m},$$

where the coefficients satisfy the inequality (16). Set

$h_1(z) = z, \quad g_1(z) = b_1 z, \quad h_m(z) = z + \rho_m e^{i\eta_m}z^m, \quad g_m(z) = b_1 z + \mu_m e^{i\delta_m}z^m$ for $m = 2, 3, \ldots$

Writing $X_m = \dfrac{|a_m|}{\rho_m}$, $Y_m = \dfrac{|b_m|}{\mu_m}$, $m = 2, 3, \ldots$ and $X_1 = 1 - \sum_{m=2}^{\infty} X_m$; $Y_1 = 1 - \sum_{m=2}^{\infty} Y_m$, we get

$$f(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m \overline{g_m(z)}).$$

In particular, putting

$$f_1(z) = z + \overline{b_1 z}\text{and } f_m(z) = z + \rho_m x z^m + \overline{b_1 z} + \overline{\mu_m y z^m}, \qquad (m \geq 2, |x| + |y| = 1 - |b_1|)$$

we see that extreme points of $clco\mathcal{VL}_H(\tau;\lambda;k,\gamma) \subset \{f_m(z)\}$.
To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \rho_2(1-|b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \rho_2(1-|b_1|)z^2\},$$

a convex linear combination of functions in $clco\ \mathcal{VL}_H(\tau;\lambda;k,\gamma)$.To see that f_m is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in $clco\ \mathcal{VL}_H(\tau;\lambda;k,\gamma)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1+\epsilon$ and $B = 1 - \left|\frac{\epsilon x}{y}\right|$. We then see that both

$$t_1(z) = z + \rho_m Axz^m + \overline{b_1 z + \mu_m yBz^m}$$

and

$$t_2(z) = z + \rho_m(2-A)xz^m + \overline{b_1 z + \mu_m y(2-B)z^m}$$

are in $clco\ \mathcal{VL}_H(\tau;\lambda;k,\gamma)$, and that

$$f_m(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bounds show that functions of the form (20) are the extreme points for $clco$ $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$, and so the proof is complete. □

Now, we consider the closure property of the class $\mathcal{VL}_H(\tau;\lambda;k,\gamma)$ under the generalized Bernardi–Libera–Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt (c > -1).$$

We prove the following result.

Theorem *Let $f(z) \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$, then $\mathcal{L}_c(f(z)) \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$*

Proof. Using (2) and (15), we get

$$\mathcal{L}_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t) + \overline{g(t)}\right]dt.$$

$$= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t - \sum_{m=2}^{\infty}|a_m|t^m\right)dt + \overline{\int_0^z t^{c-1}\left(\sum_{m=1}^{\infty}|b_m|t^m\right)dt}\right)$$

$$= z - \sum_{m=2}^{\infty}A_m z^m + \sum_{m=1}^{\infty}B_m z^m$$

where

$$A_m = \frac{c+1}{c+m}|a_m|; B_m = \frac{c+1}{c+m}|b_m|.$$

Hence

$$\sum_{m=1}^{\infty}\left[\frac{2m-1-\gamma}{1-\gamma}\left(\frac{c+1}{c+m}|a_m|\right) + \frac{2m+1+\gamma}{1-\gamma}\left(\frac{c+1}{c+m}|b_m|\right)\right]\omega_m^{\tau}(\alpha_1;\lambda;p;q)$$

$$\leq \sum_{n=1}^{\infty}\left[\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\right]\omega_m^{\tau}(\alpha_1;\lambda;p;q) \leq 2,$$

and since $f(z) \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$, therefore by Theorem 2, $\mathcal{L}_c(f(z)) \in \mathcal{VL}_H(\tau;\lambda;k,\gamma)$. □

4. INCLUSION RELATION

Following Avici and Zlotkiewicz [1] (see also Ruscheweyh [12]), we refer to the the $\delta-$ neighborhood of the function $f(z)$ defined by (2) to be the set of functions F for which

$$N_\delta(f)=\left\{F(z)=z+\sum_{m=2}^{\infty}A_mz^m+\sum_{m=1}^{\infty}\overline{B_mz^m},\sum_{m=2}^{\infty}m(|a_m-A_m|)+|b_m-B_m|+|b_1-B_1|\le\delta\right\}. \tag{21}$$

In our case, let us define the generalized δ-neighborhood of f to be the set

$$N_\delta(f)=\left\{F:\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)[(2m-1-\gamma)(|a_m-A_m|)+(2m+1+\gamma)|b_m-B_m|]\right. \tag{22}$$
$$\left.+(1-\gamma)|b_1-B_1|\le(1-\gamma)\delta\right\}.$$

Theorem 6. *Let f be given by (2). If f satisfies the conditions*

$$\sum_{m=2}^{\infty}m(2m-1-\gamma)|a_m|\omega_m^\tau(\alpha_1;\lambda;p;q)+\sum_{m=1}^{\infty}m(2m+1+\gamma)|b_m|\omega_m^\tau(\alpha_1;\lambda;p;q)\le(1-\gamma), \tag{23}$$

$0\le\gamma<1$ and

$$\delta=\frac{1-\gamma}{3-\gamma}\left(1-\frac{3+\gamma}{1-\gamma}|b_1|\right), \tag{24}$$

then $N(f)\subset\mathcal{GL}_H(\tau;\lambda;k,\gamma)$.

Proof. Let f satisfy (23) and $F(z)$ be given by

$$F(z)=z+\overline{B_1z}+\sum_{m=2}^{\infty}\left(A_mz^m+\overline{B_mz^m}\right)$$

which belongs to $N(f)$. We obtain

$$(3+\gamma)|B_1|+\sum_{m=2}^{\infty}((2m-1-\gamma)|A_m|+(2m+1+\gamma)|B_m|)\,\omega_m^\tau(\alpha_1;\lambda;p;q)\le(3+\gamma)|B_1-b_1|$$
$$+(3+\gamma)|b_1|+\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)\,[(2m-1-\gamma)|A_m-a_m|+(2m+1+\gamma)|B_m-b_m|]$$
$$+\sum_{m=2}^{\infty}\omega_m^\tau(\alpha_1;\lambda;p;q)\,[(2m-1-\gamma)|a_m|+(2m+1+\gamma)|b_m|]$$
$$\le(1-\gamma)\delta+(3+\gamma)|b_1|+\frac{1}{3-\gamma}\sum_{m=2}^{\infty}m\omega_m^\tau(\alpha_1;\lambda;p;q)\,((2m-1-\gamma)|a_m|+(2m+1+\gamma)|b_m|)$$
$$\le(1-\gamma)\delta+(3+\gamma)|b_1|+\frac{1}{3-\gamma}\,[(1-\gamma)-(3+\gamma)|b_1|]\le1-\gamma.$$

Hence for $\delta=\dfrac{1-\gamma}{3-\gamma}\left(1-\dfrac{3+\gamma}{1-\gamma}|b_1|\right)$, we infer that $F(z)\in\mathcal{GL}_H(\tau;\lambda;k,\gamma)$ which concludes the proof of Theorem 5. □

Concluding Remarks:

If $\tau=1$ and $\lambda=0$ we note that the generalized hypergeometric function contains, as its further special cases, such other linear operators the Hohlov operator, the Carlson–Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi–Libera–Livingston operator, the fractional derivative operator, and so on. The various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes (see [7–10]). The details involved in the derivations of such specializations of the results presented in this paper are fairly straight-forward.

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