

# Order of Convexity of Integral Transforms and Duality

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## Abstract

Recently, Ali et al [2] defined the class  $\mathcal{W}_\beta(\alpha, \gamma)$  consisting of functions  $f$  which satisfy

$$\Re e^{i\phi} \left( (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) > 0,$$

for all  $z \in E = \{z : |z| < 1\}$  and for  $\alpha, \gamma \geq 0$  and  $\beta < 1$ ,  $\phi \in \mathbb{R}$  (the set of reals). For  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ , they discussed the convexity of the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where  $\lambda$  is a non-negative real-valued integrable function satisfying the condition  $\int_0^1 \lambda(t) dt = 1$ . The aim of present paper is to find conditions on  $\lambda(t)$  such that  $V_\lambda(f)$  is convex of order  $\delta$  ( $0 \leq \delta \leq 1/2$ ) whenever  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ . As applications, we study various choices of  $\lambda(t)$ , related to classical integral transforms.

Key Words: Starlike function, Convex function, Hadamard product, Duality.

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## 1 Introduction

Let  $\mathcal{A}$  denote the class of analytic functions  $f$  defined in the open unit disc  $E = \{z : |z| < 1\}$  with the normalization  $f(0) = f'(0) - 1 = 0$ . Let  $\mathcal{A}_0 = \{g : g(z) = f(z)/z, f \in \mathcal{A}\}$ . Let  $S$  be the subclass of  $\mathcal{A}$  consisting of univalent functions in  $E$ . A function  $f \in S$  is said to be starlike or convex, if  $f$  maps  $E$  conformally onto the domains, respectively, starlike with respect to the origin and convex. The generalization of these two classes are given by the following analytic characterizations :

$$S^*(\beta) = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad 0 \leq \beta < 1 \right\}$$

$$K(\beta) = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \quad 0 \leq \beta < 1 \right\}.$$

For  $\beta = 0$ , we usually set  $S^*(0) = S^*$  and  $K(0) = K$ .

For two functions  $f(z) = z + a_2z^2 + a_3z^3 + \dots$  and  $g(z) = z + b_2z^2 + b_3z^3 + \dots$  in  $\mathcal{A}$ , their Hadamard product (or convolution) is the function  $f * g$  defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For  $f \in \mathcal{A}$ , Fournier and Ruscheweyh [8] introduced the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad (1.1)$$

where  $\lambda$  is a non-negative real-valued integrable function satisfying the condition  $\int_0^1 \lambda(t) dt = 1$ . This operator contains some of the well-known operators such as Libera, Bernardi and Komatu as its special cases. This operator has been studied by a number of authors for various choices of  $\lambda(t)$  (for example see [1], [4], [6], [8]). Fournier and Ruscheweyh [8] applied the duality theory ([10, 11]) to prove the starlikeness of the linear integral transform  $V_\lambda(f)$  when  $f$  varies in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} (f'(z) - \beta) > 0, z \in E \right\}.$$

In 1995, Ali and Singh [3] discussed the convexity properties of the integral transform (1.1) for functions  $f$  in the class  $\mathcal{P}(\beta)$ . In 2002, Choi et al. [7] investigated convexity properties of the integral transform (1.1) for functions  $f$  in the class

$$\mathcal{P}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left( (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right) > 0, z \in E \right\}.$$

It is evident that the class  $\mathcal{P}_\gamma(\beta)$  is closely related to the class  $\mathcal{R}_\gamma(\beta)$  defined by

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} (f'(z) + \gamma z f''(z) - \beta) > 0, z \in E \right\}.$$

Clearly,  $f \in \mathcal{R}_\gamma(\beta)$  if and only if  $zf'$  belongs to  $\mathcal{P}_\gamma(\beta)$ .

In a very recent paper, R.M.ali et al [2] discussed the convexity of the integral transform (1.1) for the functions  $f$  in a more general class  $\mathcal{W}_\beta(\alpha, \gamma)$

$$\left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left( (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) > 0, z \in E \right\}. \quad (1.2)$$

Note that  $\mathcal{W}_\beta(1, 0) \equiv \mathcal{P}(\beta)$ ,  $\mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$  and  $\mathcal{W}_\beta(1 + 2\gamma, \gamma) \equiv \mathcal{R}_\gamma(\beta)$ .

In the present paper, we shall mainly tackle the problem of finding a sharp estimate of the parameter  $\beta$  that ensures  $V_\lambda(f)$  to be convex of order  $\delta$  for  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ . To prove our result, we shall need the duality theory for convolutions, so we include here some basic concepts and results from this theory. For a subset  $\mathcal{B} \subset \mathcal{A}_0$ , we define

$$\mathcal{B}^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0, z \in E, \text{ for all } f \in \mathcal{B}\}$$

The set  $\mathcal{B}^*$  is called the dual of  $\mathcal{B}$ . Further, the second dual of  $\mathcal{B}$  is defined as  $\mathcal{B}^{**} = (\mathcal{B}^*)^*$ . We state below a fundamental result.

**Theorem 1.1.** Let

$$\mathcal{B} = \left\{ \beta + (1 - \beta) \left( \frac{1 + xz}{1 + yz} \right) : |x| = |y| = 1 \right\}, \quad \beta \in \mathbb{R}, \beta \neq 1.$$

Then, we have

$$(1) \mathcal{B}^{**} = \{g \in \mathcal{A}_0 : \exists \phi \in \mathbb{R} \text{ such that } \Re\{e^{i\phi}(g(z) - \beta)\} > 0, z \in E\}.$$

(2) If  $\Gamma_1$  and  $\Gamma_2$  are two continuous linear functionals on  $\mathcal{B}$  with  $0 \notin \Gamma_2$ , then for every  $g \in \mathcal{B}^{**}$  we can find  $v \in \mathcal{B}$  such that

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.$$

The basic reference to this theory is the book by Ruscheweyh [10] (see also [11]).

## 2 Preliminaries

We follow the notations used in [1]. Let  $\mu \geq 0$  and  $\nu \geq 0$  satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \quad (2.1)$$

When  $\gamma = 0$ , then  $\mu$  is chosen to be 0, in which case,  $\nu = \alpha \geq 0$ . When  $\alpha = 1 + 2\gamma$ , (2.1) yields  $\mu + \nu = 1 + \gamma = 1 + \mu\nu$ , or  $(\mu - 1)(1 - \nu) = 0$ .

- (i) For  $\gamma > 0$ , then choosing  $\mu = 1$  gives  $\nu = \gamma$ .
- (ii) For  $\gamma = 0$ , then  $\mu = 0$  and  $\nu = \alpha = 1$ .

Whenever the particular case  $\alpha = 1 + 2\gamma$  will be considered, the values of  $\mu$  and  $\nu$  for  $\gamma > 0$  will be taken as  $\mu = 1$  and  $\nu = \gamma$  respectively, while  $\mu = 0$  and  $\nu = 1 = \alpha$  in the case when  $\gamma = 0$ .

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n + 1} z^n, \quad (2.2)$$

and

$$\begin{aligned} \psi_{\mu,\nu}(z) &= \phi_{\mu,\nu}^{-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{(n\nu + 1)(n\mu + 1)} z^n \\ &= \int_0^1 \int_0^1 \frac{ds dt}{(1 - t^\nu s^\mu z)^2}. \end{aligned} \quad (2.3)$$

Here  $\phi_{\mu,\nu}^{-1}$  denotes the convolution inverse of  $\phi_{\mu,\nu}$  such that  $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = z/(1 - z)$ . If  $\gamma = 0$ , then  $\mu = 0$ ,  $\nu = \alpha$ , and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{n\alpha + 1} z^n = \int_0^1 \frac{dt}{(1 - t^\alpha z)^2}.$$

If  $\gamma > 0$ , then  $\nu > 0$ ,  $\mu > 0$ , and making the change of variables  $u = t^\nu$ ,  $v = s^\mu$  results in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv.$$

Thus the function  $\psi_{\mu,\nu}$  can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv, & \gamma > 0; \\ \int_0^1 \frac{dt}{(1-t^\alpha z)^2}, & \gamma = 0, \alpha > 0. \end{cases} \quad (2.4)$$

Let  $q$  be the solution of the initial value problem

$$\frac{d}{dt} \left( t^{1/\nu} q(t) \right) = \begin{cases} \frac{1}{\mu\nu} t^{1/\nu-1} \int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds, & \gamma > 0, \\ \frac{1}{\alpha} \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} t^{1/\alpha-1}, & \gamma = 0, \alpha > 0, \end{cases} \quad (2.5)$$

satisfying  $q(0) = 1$ .

Solving the differential equation (2.5), we have

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta)-(1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw. \quad (2.6)$$

In particular,

$$q_\alpha(t) = \frac{1}{\alpha} \int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha-1} ds, \quad \gamma = 0, \alpha > 0. \quad (2.7)$$

Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx, \quad \nu > 0, \quad (2.8)$$

and

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu} dx, & \gamma > 0, \\ \Lambda_\alpha(t), & \gamma = 0, (\mu = 0, \nu = \alpha > 0). \end{cases} \quad (2.9)$$

For the function  $\Pi_{\mu,\nu}(t)$ , we define

$$\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) = \begin{cases} \Re \int_0^1 t^{1/\mu-1} \Pi_{\mu,\nu}(t) \left[ h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt, & \gamma > 0, \\ \Re \int_0^1 t^{1/\alpha-1} \Pi_{0,\alpha}(t) \left[ h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt, & \gamma = 0, \end{cases} \quad (2.10)$$

where  $h_\delta(z)$  is defined as

$$h_\delta(z) = \frac{z \left( 1 + \frac{\epsilon+2\delta-1}{2-2\delta} z \right)}{(1-z)^2}, \quad |\epsilon| = 1. \quad (2.11)$$

With these notations, we are now in a position to state our first result, which generalizes many earlier results in this direction.

### 3 Main results

**Theorem 3.1.** Let  $\mu \geq 0$ ,  $\nu \geq 0$  satisfy (2.1) . Define  $\beta < 1$  by

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t)q(t)dt, \quad (3.1)$$

where  $q(t)$  is the solution of the initial-value problem (2.5). Further for  $\Lambda_\nu(t)$  and  $\Pi_{\mu,\nu}(t)$  defined by (2.8) and (2.9) respectively, assume that  $t^{1/\nu}\Lambda_\nu(t) \rightarrow 0$ , and  $t^{1/\nu}\Pi_{\mu,\nu}(t) \rightarrow 0$  as  $t \rightarrow 0^+$ . Then for  $\delta \in [0, \frac{1}{2}]$ ,  $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset K(\delta)$  if and only if  $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$ , where  $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta)$  and  $h_\delta$  are defined by equations (2.10) and (2.11) respectively.

**Proof.** As the case  $\gamma = 0$  ( $\mu = 0$ ,  $\nu = \alpha$ ) corresponds to the Theorem 2.3 in [5], so we will prove the result only when  $\gamma > 0$ .

Let

$$H(z) = (1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z).$$

Since  $\mu + \nu = \alpha - \gamma$  and  $\mu\nu = \gamma$ , therefore

$$\begin{aligned} H(z) &= (1 + \gamma - (\alpha - \gamma))\frac{f(z)}{z} + (\alpha - \gamma - \gamma)f'(z) + \gamma z f''(z) \\ &= (1 + \mu\nu - \mu - \nu)\frac{f(z)}{z} + (\mu + \nu - \mu\nu)f'(z) + \mu\nu z f''(z). \end{aligned}$$

Writing  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ , we obtain from (2.2)

$$H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}(n\nu + 1)(n\mu + 1)z^n = f'(z) * \phi_{\mu,\nu}(z), \quad (3.2)$$

and (2.3) gives that

$$f'(z) = H(z) * \psi_{\mu,\nu}(z). \quad (3.3)$$

Now, for  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ , we have

$$\Re \left\{ e^{i\phi} \frac{H(z) - \beta}{1 - \beta} \right\} > 0.$$

Thus, in the view of the Theorem 1.1, we may confine ourselves to functions  $f \in \mathcal{W}_\beta(\alpha, \gamma)$  for which

$$H(z) = \beta + (1 - \beta) \left( \frac{1 + xz}{1 + yz} \right), \quad |x| = |y| = 1.$$

Thus (3.3) gives

$$f'(z) = \left( (1 - \beta) \frac{1 + xz}{1 + yz} + \beta \right) * \psi_{\mu,\nu}(z), \quad (3.4)$$

and therefore

$$\frac{f(z)}{z} = \frac{1}{z} \int_0^z \left( (1 - \beta) \frac{1 + xw}{1 + yw} + \beta \right) dw * \psi(z). \quad (3.5)$$

Here  $\psi := \psi_{\mu,\nu}$ .

A well-known result from the theory of convolutions [9, Pg 94] (also see [11]) states that

$$F \in K(\delta) \Leftrightarrow \frac{1}{z}(zF' * h_\delta)(z) \neq 0, \quad z \in E,$$

where

$$h_\delta(z) = \frac{z \left(1 + \frac{\epsilon+2\delta-1}{2-2\delta}z\right)}{(1-z)^2}, \quad |\epsilon| = 1.$$

Hence  $F \in K(\delta)$  if and only if

$$0 \neq \frac{1}{z}(V_\lambda(f)(z) * zh'_\delta(z)) = \frac{1}{z} \left[ \int_0^1 \lambda(t) \frac{f(tz)}{t} dt * zh'_\delta(z) \right] = \int_0^1 \frac{\lambda(t)}{1-tz} dt * \frac{f(z)}{z} * h'_\delta(z)$$

Using (3.5), we have

$$\begin{aligned} 0 &\neq \int_0^1 \frac{\lambda(t)}{1-tz} dt * \left[ \frac{1}{z} \int_0^z \left( (1-\beta) \frac{1+xw}{1+yw} + \beta \right) dw * \psi(z) \right] * h'_\delta(z) \\ &= \int_0^1 \frac{\lambda(t)}{1-tz} dt * h'_\delta(z) * \left[ \frac{1}{z} \int_0^z \left( (1-\beta) \frac{1+xw}{1+yw} + \beta \right) dw \right] * \psi(z) \\ &= \int_0^1 \lambda(t) h'_\delta(tz) dt * (1-\beta) \left[ \frac{1}{z} \int_0^z \left( \frac{1+xw}{1+yw} + \frac{\beta}{(1-\beta)} \right) dw \right] * \psi(z) \\ &= (1-\beta) \left[ \int_0^1 \lambda(t) h'_\delta(tz) dt + \frac{\beta}{(1-\beta)} \right] * \frac{1}{z} \int_0^z \frac{1+xw}{1+yw} dw * \psi(z) \\ &= (1-\beta) \left[ \int_0^1 \lambda(t) \left( \frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} \right] * \frac{1+xz}{1+yz} * \psi(z). \end{aligned}$$

This holds if and only if [11, p. 23]

$$\begin{aligned} &\Re(1-\beta) \left[ \int_0^1 \lambda(t) \left( \frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} \right] * \psi(z) \geq \frac{1}{2}, \\ \Leftrightarrow &\Re(1-\beta) \left[ \int_0^1 \lambda(t) \left( \frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)} \right] * \psi(z) \geq 0, \\ \Leftrightarrow &\Re \left[ \int_0^1 \lambda(t) \left( \frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta - \frac{1}{2}}{(1-\beta)} \right] * \psi(z) \geq 0, \\ \Leftrightarrow &\Re \left[ \int_0^1 \lambda(t) \left( \frac{1}{z} \int_0^z h'_\delta(tw) dw - q(t) \right) dt \right] * \psi(z) \geq 0, \quad (\text{using (3.1)}) \\ \\ \Leftrightarrow &\Re \left[ \int_0^1 \lambda(t) (h'_\delta(tz) - q(t)) dt \right] * \frac{1}{z} \int_0^z \psi(w) dw \geq 0, \\ \Leftrightarrow &\Re \left[ \int_0^1 \lambda(t) (h'_\delta(tz) - q(t)) dt \right] * \sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} \geq 0, \quad (\text{using (2.3)}) \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left( \sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * h'_\delta(tz) - q(t) \right) dt \geq 0, \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{d\eta d\zeta}{(1-\eta^\nu \zeta^\mu z)} * h'_\delta(tz) - q(t) \right) dt \geq 0, \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 h'_\delta(tz \eta^\nu \zeta^\mu) d\eta d\zeta - q(t) \right) dt \geq 0, \end{aligned}$$

which can also be written as

$$\Re \int_0^1 \lambda(t) \left( \int_0^1 \int_0^1 \frac{1}{\mu\nu} h'_\delta(tzuv) u^{1/\nu-1} v^{1/\mu-1} dv du - q(t) \right) dt \geq 0.$$

Writing  $w = tu$ , we get

$$\Re \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left[ \int_0^t \int_0^1 h'_\delta(wzv) w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} q(t) \right] dt \geq 0.$$

An integration by parts with respect to  $t$  and (2.5) gives

$$\Re \int_0^1 \Lambda_\nu(t) \left[ \int_0^1 h'_\delta(tzv) t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 \frac{1-\delta-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds \right] dt \geq 0.$$

Again writing  $w = vt$  and  $\eta = st$  above inequality reduces to

$$\Re \int_0^1 \Lambda_\nu(t) t^{1/\nu-1/\mu-1} \left[ \int_0^t h'_\delta(wz) w^{1/\mu-1} dw - \int_0^t \frac{1-\delta-(1+\delta)\eta}{(1-\delta)(1+\eta)^3} \eta^{1/\mu-1} d\eta \right] dt \geq 0,$$

which after integration by parts with respect to  $t$  yields

$$\Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left[ h'_\delta(tz) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0.$$

Thus  $F \in K(\delta)$  if and only if  $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$ .

Finally, to prove the sharpness, let  $f \in \mathcal{W}_\beta(\alpha, \gamma)$  be of the form for which

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we obtain that

$$f(z) = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{1}{(n\nu+1)(n\mu+1)} z^{n+1}.$$

Thus

$$F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where  $\tau_n = \int_0^1 \lambda(t) t^n dt$ . From (2.5), it is a simple exercise to write  $q(t)$  in a series expansion as

$$q(t) = 1 + \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n. \quad (3.6)$$

Now, by (3.1) and (3.6), we have

$$\begin{aligned} \frac{\beta - \frac{1}{2}}{1-\beta} &= - \int_0^1 \lambda(t) q(t) dt \\ &= - \int_0^1 \lambda(t) \left[ 1 + \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n \right] dt \\ &= -1 - \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} \int_0^1 \lambda(t) t^n dt. \end{aligned}$$

Therefore

$$\frac{1}{2(1-\beta)} = -\frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}. \quad (3.7)$$

Finally, we see that

$$F'(z) = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(n+1)\tau_n}{(n\nu+1)(n\mu+1)} z^n.$$

Therefore

$$(zF'(z))' = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(n+1)^2\tau_n}{(n\nu+1)(n\mu+1)} z^n.$$

For  $z = -1$ , we have

$$\begin{aligned} (zF'(z))'(-1) &= 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^2\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)} + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n \delta (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1 - (1-\delta) + \delta 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \quad (\text{Using (3.7)}) \\ &= \delta \left( 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \right) \\ &= \delta F'(-1). \end{aligned}$$

Thus  $(zF'(z))'/F'(z)$  at  $z = -1$  equals  $\delta$ . This implies that the result is sharp for the order of convexity.

## 4 Consequences of Theorem 3.1

To obtain a sufficient condition for the convexity of order  $\delta$  of the integral transform (1.1) by a much easier method, we present the following theorem.

**Theorem 4.1.** Let  $\Lambda_\nu(t)$ ,  $\Pi_{\mu,\nu}(t)$  be integrable on  $[0,1]$  and positive on  $(0,1)$ . Also, suppose that  $t^{1/\nu}\Lambda_\nu(t) \rightarrow 0$ , and  $t^{1/\nu}\Pi_{\mu,\nu}(t) \rightarrow 0$  as  $t \rightarrow 0^+$ . Assume further that  $\mu \geq 1$  and

$$\frac{\left(-t\Pi'_{\mu,\nu}(t) + \left(1 - \frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)}{(1+t)(1-t)^{1+2\delta}} \text{ is decreasing on } (0,1). \quad (4.1)$$

For  $\delta \in [0, 1/2]$ , if  $\beta$  satisfies (3.1), then  $V_\lambda(f) \in K(\delta)$  for  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ .

**Proof.** For  $\gamma > 0$ , integration by parts with respect to  $t$  yields

$$\begin{aligned} &\int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \left( \Re(h'_\delta(tz)) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right) dt \\ &= \int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \frac{d}{dt} \left( \Re \frac{h_\delta(tz)}{z} - \frac{t(1-\delta(1+t))}{(1-\delta)(1+t)^2} \right) dt \\ &= \int_0^1 t^{\frac{1}{\mu}-1} \left( -t\Pi'_{\mu,\nu}(t) + \left(1 - \frac{1}{\mu}\right)\Pi_{\mu,\nu}(t) \right) \left( \Re \frac{h_\delta(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right) dt. \end{aligned}$$

Also for  $\mu \geq 1$ , the function  $t^{1/\mu-1}$  is decreasing on  $(0,1)$ . Thus, the condition (4.1) along with Theorem 1 from [8] yields

$$\int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \left( \Re(h'_\delta(tz)) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right) dt > 0.$$

Thus, an application of Theorem 3.1 evidently leads to the desired result.  $\square$

Below, we obtain the conditions to ensure convexity of  $V_\lambda(f)$ . As defined in (2.8) and (2.9), for  $\gamma > 0$ ,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu} dx, \quad \text{and} \quad \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx.$$

In order to apply Theorem 4.1, we have to prove that the function

$$k(t) = \frac{\left( t^{\frac{1}{\nu}-\frac{1}{\mu}} \Lambda_\nu(t) + \left( 1 - \frac{1}{\mu} \right) \Pi_{\mu,\nu}(t) \right)}{(1+t)(1-t)^{1+2\delta}} := \frac{p(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing in  $(0,1)$ . Since  $k(t) > 0$  and

$$\begin{aligned} \frac{k'(t)}{k(t)} &= \frac{p'(t)}{p(t)} + \frac{2(t+\delta(1+t))}{1-t^2} \\ &= \frac{2(t+\delta(1+t))}{(1-t^2)p(t)} \left[ \frac{(1-t^2)p'(t)}{2(t+\delta(1+t))} + p(t) \right] = \frac{2(t+\delta(1+t))}{(1-t^2)p(t)} [q(t)] \quad (\text{say}). \end{aligned}$$

Thus to prove that  $k'(t) \leq 0$ , it is enough to prove that  $q(t) \leq 0$ . Since  $q(1) = 0$ , so it remains to show that  $q(t)$  is increasing over  $(0,1)$ . Now

$$q'(t) = \frac{(1+t)}{2(t+\delta(1+t))^2} \left[ (1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t) \right].$$

So,  $q'(t) \geq 0$  for  $t \in (0,1)$  is equivalent to the inequality  $r(t) \geq 0$ , where

$$r(t) = (1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t)$$

By using the idea similar to the one used to prove Theorem 3.1 in [6], we can write

$$r(t) = -\lambda(t)t^{1-\frac{1}{\mu}} \left[ \left( \frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)} X(t) \right] + \left[ \left( \frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) \right] \left( \frac{1}{\nu} - 1 \right) t^{\frac{1}{\nu}-\frac{1}{\mu}-1} \int_t^1 A(s) ds \quad (4.2)$$

where,

$$\begin{aligned} A(t) &= \lambda(t)t^{-1/\nu}, \\ X(t) &= (1-t)(t+\delta(1+t)), \\ Z(t) &= -t(1-t-\delta(1+t))(1+2\delta). \end{aligned} \quad (4.3)$$

Clearly,  $A(t) > 0$  and  $X(t) > 0$  for all  $t \in (0,1)$ .

Thus,  $r(t)$  is non-negative if

$$\left( \frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)} X(t) \leq 0 \quad \text{and} \quad \left[ \left( \frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) \right] \left( \frac{1}{\nu} - 1 \right) \geq 0. \quad (4.4)$$

Since  $\nu \geq 1$ , we can rewrite the condition (4.4) as follows :

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} - \left( \frac{X(t) + Z(t)}{X(t)} \right) \quad \text{and} \quad \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq - \left( \frac{X(t) + Z(t)}{X(t)} \right). \quad (4.5)$$

In view of the fact that  $X(t) + Z(t)$  and  $X(t)$  are non-negative on  $(0,1)$ , the above inequality further reduces to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad \text{and} \quad \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq 0. \quad (4.6)$$

For  $\mu \geq 1$ , condition (2.1) implies  $\nu \geq \mu \geq 1$ . Thus, condition (4.6) implies that  $r(t)$  is non-negative if

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (4.7)$$

These conditions leads to the following theorem.

**Theorem 4.2.** Assume that both  $\Lambda_\nu(t)$ ,  $\Pi_{\mu,\nu}(t)$  are integrable on  $[0,1]$  and positive on  $(0,1)$ . Let  $\lambda(t)$  be a non-negative real-valued integrable function on  $[0,1]$  and satisfy the condition

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (4.8)$$

Let  $f \in \mathcal{W}_\beta(\alpha, \gamma)$  and  $\beta < 1$  with

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t)q(t)dt,$$

where  $q(t)$  is defined by (2.6). Then  $F(z) = V_\lambda(f)(z) \in K(\delta)$  for  $\delta \in [0, 1/2]$ . The conclusion does not hold for smaller values of  $\beta$ .

On the other hand, when  $\gamma = 0$  ( $\mu = 0$ ,  $\nu = \alpha > 0$ ), so we get the following result.

**Theorem 4.3.** Let  $\lambda(t)$  be a non-negative real-valued integrable function on  $[0,1]$ . Assume that both  $\Lambda_\alpha(t)$ ,  $\Pi_{0,\alpha}(t)$  are integrable on  $[0,1]$  and positive on  $(0,1)$ . Let  $\lambda(1) = 0$  and  $\lambda$  satisfies the condition

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \quad \alpha \geq 1. \quad (4.9)$$

Let  $f \in \mathcal{W}_\beta(\alpha, \gamma)$  and  $\beta < 1$  with

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t)q_\alpha(t)dt,$$

where  $q_\alpha(t)$  is defined by (2.7) with  $\delta \in [0, 1/2]$ . Then  $F(z) = V_\lambda(f)(z) \in K(\delta)$ . The conclusion does not hold for smaller values of  $\beta$ .

**Proof.** As in Theorem 3.1, for  $\gamma = 0$  and  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ , we have  $V_\lambda(f)(z) \in K(\delta)$  if

$$\int_0^1 t^{\frac{1}{\alpha}-1} \Pi_{0,\alpha}(t) \left( \Re(h'_\delta(tz)) - \frac{1 - \delta - (1 + \delta)t}{(1 - \delta)(1 + t)^3} \right) dt > 0,$$

which is equivalent to

$$\int_0^1 t^{\frac{1}{\alpha}-1} \left( t^{1-\frac{1}{\alpha}} \lambda(t) + \left( 1 - \frac{1}{\alpha} \right) \Lambda_\alpha(t) \right) \left( \Re \frac{h_\delta(tz)}{tz} - \frac{1 - \delta(1 + t)}{(1 - \delta)(1 + t)^2} \right) dt > 0.$$

Since  $t^{\frac{1}{\alpha}-1}$  is decreasing on  $(0,1)$  for  $\alpha \geq 1$ , thus to apply Theorem 1 in [8], it is enough to show that

$$p(t) = \frac{t^{1-\frac{1}{\alpha}}\lambda(t) + (1 - \frac{1}{\alpha})\Lambda_\alpha(t)}{(1+t)(1-t)^{1+2\delta}} := \frac{k(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing on  $(0,1)$ . Here, logarithmic differentiation implies that

$$\frac{p'(t)}{p(t)} = \frac{2(t + \delta(1+t))}{(1-t^2)k(t)} \left[ \frac{(1-t^2)k'(t)}{2(t + \delta(1+t))} + k(t) \right].$$

Since  $p(t) > 0$  for  $\alpha \geq 1$ , thus to prove that  $p'(t) \leq 0$  on  $(0,1)$  it remains to show that

$$r(t) = k(t) + \frac{(1-t^2)k'(t)}{2(t + \delta(1+t))} \leq 0.$$

Since  $r(1) = 0$ , so  $r(t) \leq 0$  if  $r(t)$  is increasing on  $(0,1)$ . Thus,  $r'(t)$  is non-negative if

$$\frac{t^{\frac{-1}{\alpha}}(1+t)}{2(t + \delta(1+t))} \left\{ X(t)t\lambda''(t) + \left[ \left(1 - \frac{1}{\alpha}\right)X(t) + Z(t) \right] \lambda'(t) \right\} \geq 0,$$

where  $X(t)$  and  $Z(t)$  are as defined in (4.3). Further simplification yields that

$$t\lambda''(t) + \left( \frac{X(t) + Z(t)}{X(t)} - \frac{1}{\alpha} \right) \lambda'(t) \geq 0.$$

Since,  $X(t)$  and  $X(t) + Z(t)$  are non-negative in  $(0,1)$ , thus  $r'(t) \geq 0$  is equivalent to

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \quad \alpha \geq 1,$$

which completes the proof.

**Remarks 4.4.** Observe that results in [2] can be obtained from our results by setting  $\delta = 0$ .

## 5 Applications

In this section, we apply Theorem 4.2 and Theorem 4.3 to obtain certain results regarding convexity of well-known integral operators. The proofs of the following results run on the same lines as given in [2] and hence omitted.

Consider  $\lambda$  to be defined as

$$\lambda(t) = (1+c)t^c, \quad c > -1.$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1+c) \int_0^1 t^{c-1} f(tz) dt, \quad c > -1, \quad (5.1)$$

is the well-known Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (5.1) with  $c = 0$  and  $c = 1$  respectively. For this special case of  $\lambda$ , the following result holds.

**Theorem 5.1.** Let  $c > -1$  and  $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -(1 + c) \int_0^1 t^c q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1 - \delta) - (1 + \delta)swt}{(1 - \delta)(1 + swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

Then for  $\delta \in [0, 1/2]$ , we have  $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset K(\delta)$  provided  $c$  satisfies the condition :

$$c \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.2)$$

The value of  $\beta$  is sharp.

Writing  $\alpha = 1 + 2\gamma$ ,  $\gamma > 0$  and  $\mu = 1$  in Theorem 5.1 gives the following criteria of convexity :

**Corollary 5.2.** Let  $-1 < c \leq 3 - 1/\gamma$  and  $\gamma \geq 1$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -(1 + c) \int_0^1 t^c q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1 - \delta) - (1 + \delta)swt}{(1 - \delta)(1 + swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

Then for  $\delta \in [0, 1/2]$ , we have  $V_\lambda(\mathcal{R}_\beta(\gamma)) \subset K(\delta)$ . The value of  $\beta$  is sharp.

Further, letting  $\gamma = 1$  and  $c = 0$  in Corollary 5.2, we have

**Corollary 5.3.** Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = \frac{1}{1 - \delta} \left( \delta \frac{\pi^2}{12} - \log 2 \right)$$

If  $f \in \mathcal{R}_\beta(1)$ , then Alexander transform  $F_0(z) \equiv A[f](z) = \int_0^1 \frac{f(tz)}{t} dt$  is convex of order  $\delta$  where  $\delta \in [0, 1/2]$ . The value of  $\beta$  is sharp.

**Remark 5.4.** 1. For  $\delta = 0$ ,

$$\beta_0 = \frac{1 - 2 \log 2}{2 - 2 \log 2} = -0.629 \dots$$

Then, for  $f$  satisfying

$$\Re e^{i\phi} (f'(z) + zf''(z) - \beta) > 0, \quad z \in E,$$

Alexander transform  $A[f]$  is convex. It has been shown in [8] that  $\beta_0$  is the best possible bound here.

2. We note that for  $\delta = 1/2$ ,  $\beta_{1/2} = 0.590 \dots$ . Then, for  $f$  satisfying

$$\Re e^{i\phi} (f'(z) + zf''(z) - \beta) > 0, \quad z \in E,$$

Alexander transform  $A[f]$  is convex of order  $\frac{1}{2}$ .

While, the case  $c = 0$  in Theorem 5.1 yields yet another interesting result, which we state as a theorem.

**Theorem 5.5.** Let  $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ . If  $F \in \mathcal{A}$  satisfies

$$\Re(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)) > \beta, \quad z \in E,$$

and  $\beta < 1$  satisfies

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw,$$

then for  $\delta \in [0, 1/2]$ ,  $F$  belongs to  $K(\delta)$ . The value of  $\beta$  is sharp.

To state our next theorem, we define

$$\lambda(t) = \begin{cases} (a+1)(b+1) \frac{t^a(1-t^{b-a})}{b-a}, & b \neq a; \\ (a+1)^2 t^a \log(1/t), & b = a, \end{cases} \quad (5.3)$$

where  $b > -1$  and  $a > -1$ .

Then,

$$V_\lambda(f)(z) = G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1} (1-t^{b-a}) f(tz) dt, & b \neq a; \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz) dt, & b = a. \end{cases}$$

**Theorem 5.6.** Let  $b > -1$ ,  $a > -1$  and  $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

and  $\lambda(t)$  is defined by (5.3). If  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ , then the convolution operator  $G_f(a, b; z)$  belongs to  $K(\delta)$  with  $\delta \in [0, 1/2]$  if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.4)$$

The value of  $\beta$  is sharp.

Substituting  $\alpha = 1 + 2\gamma$ ,  $\gamma > 0$  and  $\mu = 1$  in Theorem 5.1, gives the following result :

**Corollary 5.7.** Let  $b > -1$ ,  $-1 < a \leq 3 - 1/\gamma$  and  $\gamma \geq 1$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

and  $\lambda(t)$  is defined by (5.3). If  $f \in \mathcal{R}_\beta(\gamma)$ , then the convolution operator  $G_f(a, b; z)$  belongs to  $K(\delta)$  with  $\delta \in [0, 1/2]$ . The value of  $\beta$  is sharp.

While for  $\gamma = 0$ , with an application of Theorem 4.3, we get the following result :

**Theorem 5.8.** Let  $b > -1$ ,  $a > -1$  and  $\alpha \geq 1$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q_\alpha(t) dt,$$

where  $q_\alpha$  is given by

$$q_\alpha(t) = \frac{1}{\alpha} \int_0^1 \frac{(1-\delta) - (1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha-1} ds$$

and  $\lambda(t)$  is defined by (5.3). If  $f \in \mathcal{P}_\beta(\alpha)$ , then the convolution operator  $G_f(a, b; z)$  belongs to  $K(\delta)$  with  $\delta \in [0, 1/2]$  if one of the following conditions holds :

- (i)  $-1 < a \leq 0$  and  $a = b$ , or
- (ii)  $-1 < a \leq 0$  and  $-1 < a < b \leq 1 + 1/\alpha$ .

The value of  $\beta$  is sharp.

Now, we define

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (\log(1/t))^{p-1}, \quad a > -1, \quad p \geq 0.$$

In this case,  $V_\lambda$  reduces to the Komatu operator [9]

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left( \log \left( \frac{1}{t} \right) \right)^{p-1} t^{a-1} f(tz) dt, \quad a > -1, \quad p \geq 0.$$

For  $p = 1$  Komatu operator gives the Bernardi integral operator. For this  $\lambda$ , the following result holds.

**Theorem 5.9.** Let  $a > p - 2 > -1$  and  $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ . Let  $\beta < 1$  satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where  $q$  is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

If  $f \in \mathcal{W}_\beta(\alpha, \gamma)$ , then the function

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left( \log \left( \frac{1}{t} \right) \right)^{p-1} t^{a-1} f(tz) dt$$

belongs to  $K(\delta)$  with  $\delta \in [0, 1/2]$  if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.5)$$

The value of  $\beta$  is sharp.

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