APPROXIMATE DISCRETIZATION METHOD FOR LINEAR DELAYED SYSTEMS [1]

M. Velasco-Villa * B. del-Muro-Cuellar **

* *CINVESTAV-IPN*
Departamento de Ingeniería Eléctrica
Sección de Mecatrónica
A.P. 14-740, 07000, México DF, México
Tel: (+52) 55 50 61 38 00
E-mail: velasco@mail.cinvestav.mx.
** *Instituto Mexicano del Petróleo*
Programa de Investigación en Matemáticas Aplicadas y Computación
Tel: (+52) 55 91 75 75 71
E-mail: bdelmuro@imp.mx

Abstract: In this work it is analyzed the discrete time approximation problem for a class of linear time invariant systems with time delays at the input signal and at the state. It is proposed a methodology intended to obtain a discrete time representation (approximate) of the original continuous time system. The performance of the proposed scheme is shown by means of digital simulations over an academic example where a discrete time controller, designed by considering the discrete time model is proposed to solve the regulation problem.

Keywords: Discrete time modeling, discrete time control, regulation.

1. INTRODUCTION

In this work it is considered a class of invariant time delay systems that involve time delays at the input signal and at the state. This class of systems has been studied in the literature under several perspectives (Malek-Zavarei and Jamshidi, 1987; Górecki and Fuksa, 1989).

From a theoretical point of view the stability of linear time invariant systems with time delays at the input signal and the state has been widely studied producing several methodologies to evaluate their stability properties. In particular in (Mori and Kokame, 1989; Wang, 1992) the stability properties of the considered class it is analyzed producing different sufficient conditions based on the value of the time delays. In (Arunsawatwong, 1996) necessary and sufficient conditions are provided to assure the stability based on practical tests over specific regions on the complex plane.

Some other works has focus theirs attention to the analysis of problems like disturbance decoupling; for example in (Conte and Perdon, 1995), the authors present conditions for the solution of the problem by means of a geometrical approach based on the consideration of delay operators. Also, the use of operators has allowed the analysis of the input-output decoupling problem (Iwai *et al.*, 1978; Tzafestas and Paraskevopoulos, 1973).

[1] Partially Supported by CONACyT-México under Grant 42093.

The study of classical control problems like regulation or path tracking have shown the necessity, in a general framework, of the use of future values of the state of the system in their respective solution. This causality problem has had as a consequence the characterization of the conditions, under which, a specific control problem admit a solution by means of a non anticipative feedback law (see for example, (Conte and Perdon, 1995) in the case of disturbance decoupling).

Several works has been devoted to the estimation of future values of the state of a time delay system to be used in the solution of specific problems that can not be overcome by the use of causal feedback laws. In this sense, the use of the well known Smith predictor (Mori and Smith, 1957) has produced several control schemes. Under this approach, linear systems with time delay only at the input (Astrom *et al.*, 1994; Watanabe and Ito, 1981) or delays at the input and at the state (Maza-Casas *et al.*, 1999) have been considered. This latter case, in general, leads to approximate control schemes that are based on causal feedback control law produced by considering an approximate finite dimensional model free of delay of the original time-delay system.

In this work, it is proposed a new methodology to obtain an approximate discrete time model for the original continuous time input and state delayed system. The work is organized as follows: In Section 2 it is presented the class of time delay systems under analysis. In Section 3 it is described the approximate discretization strategy and the equivalent discrete time model is obtained. The performance of the approximation strategy is shown by means of an academic example in Section 4 and finally in Section 5 some conclusions are presented.

2. CLASS OF TIME-DELAY SYSTEMS

Consider the class of linear time delay systems of the form,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + Bu(t-\tau) \\ y(t) &= Cx(t) \\ x(\varphi) &= \phi(\varphi), \quad \varphi \in [-h, 0] \end{aligned} \tag{1}$$

where $h \geq 0$ is a constant time delay associated to the state, $\tau \geq 0$ is the constant time-delay associated to the input, $\phi(\varphi)$ is a continuous function of initial conditions with $-h \leq \varphi \leq 0$. The state $x \in R^n$, the input $u \in R$ and the output $y \in R$. Finally, the matrices A, $A_1 \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$ are supposed to be known.

In the considered class of systems it is assumed a single-input single output system for simplicity of presentation only, taking into account instead a system with delays at the state and input.

System (1) can be described as a transfer function by direct application of Laplace transform, producing,

$$\begin{aligned} sX(s) &= \left(A + A_1 e^{-hs}\right) X(s) + Be^{-\tau s} U(s) \\ Y(s) &= CX(s) \end{aligned}$$

from where, it is obtained,

$$\frac{Y(s)}{U(s)} = C\left[sI - \left(A + A_1 e^{-hs}\right)\right]^{-1} Be^{-\tau s}$$

This transfer function produces a characteristic equation described by a quasipolynomial of the form,

$$\det\left(sI - A - A_1 e^{-hs}\right) = 0 \tag{2}$$

where there exists a dependence on the time delay associated with the state.

These systems are described by differential-difference equations (Malek-Zavarei and Jamshidi, 1987). This fact represents a great difference with respect to systems with time-delays only at the input signal. It should be notice that equation (2) has an infinite number of solutions that avoid, for example, the consideration of a pole placement strategy, at least in a usual manner.

In general terms, the design of controllers for the considered class of systems it is not a trivial issue. The main purpose of this work is to propose a simple and effective methodology to obtain an approximate discrete time representation that can be used in the design of discrete time controllers for classical control problems for the original continuous time system (1).

3. DISCRETIZATION STRATEGY

In this section it will be provided a description of the discretization methodology proposed to obtain an approximate discrete time representation of the original system (1). It is intended to find an approximate discrete model of system (1) due to the fact that is not possible to obtain an exact discrete time representation for the considered class of time delay systems.

To obtain an exact discrete time model for a linear continuous system it is necessary, in general, the knowledge of the solution of the system (fundamental matrix). Since system (1) considers delays at the state, the computation of its fundamental matrix it is not a feasible task (Malek-Zavarei and Jamshidi, 1987; Górecki and Fuksa, 1989).

In the case of linear systems involving delays only at the input signal (or at the output), obtaining an exact discrete time representation is an easy task

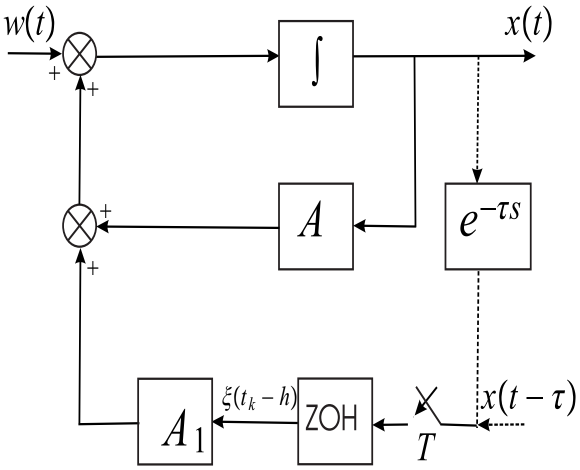

Fig. 1. General scheme of approximation

(due to the availability of its fundamental matrix) by considering the *real translation theorem* which allows the manipulation of the associated time delays τ in the case that there exists an integer k such that $\tau = kT$ for a given sampling time T, this is,

$$\mathcal{Z}(e^{-kTs}F(s)) = z^{-k}F(z)$$

where $\mathcal{Z}(G(s))$ denotes the z transform of the function $G(s)$.

To overcome the unavailability of the transfer function for a system of the form (1) in what follows it is presented a methodology to obtain an equivalent approximate discrete time model.

3.1 Discrete time model

The key assumption on the methodology to obtain an approximate discrete time model is to consider the delay associated to the state as a second (virtual) input to the system that is assumed to be sampled by the action of a zero order hold as is shown in Figure 1. This particular strategy produces a system in state space form described by,

$$\begin{aligned} \dot{x}_r(t) &= Ax_r(t) + A_1\xi(t_k - h) + Bu(t-\tau) \\ y(t) &= Cx_r(t) \\ x(t_0 - h) &= \phi(\varphi) \end{aligned} \tag{3}$$

where $\xi(t_k - h)$ correspond to the second virtual (sampled) input for the original system (1) which is a piecewise constant function.

Theorem 3.1. Consider system (1) and the partially sampled system (3) given by Figure 1. Under this sampling condition, the solution $x_r(t)$ of system (3) is an approximation of the solution $x(t)$ of the original system (1) with respect to the sampling period T.

Proof. Consider the solution of system (1) for $t_0 \le t \le h$ and the initial condition $\xi(t-h)$. Note that for the considered period of time, the delay state $x(t-h)$ is determined by the initial condition function and then, it is possible to consider for $t_0 \le t \le h$ the following systems,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1\xi(t-h) + Bu(t-\tau) \\ \dot{x}_r(t) &= Ax_r(t) + A_1\xi(t_k-h) + Bu(t-\tau). \end{aligned}$$

Defining the error signal,

$$e_x(t) = x(t) - x_r(t)$$

it is obtained the dynamical system,

$$\dot{e}_x(t) = Ae_x(t) + A_1\gamma(t) \tag{4}$$

where $\gamma(t)$ is a function defined as

$$\gamma(t) = \xi(t-h) - \xi(t_k - h),$$

where it is clear that $\gamma(t)$ correspond to the error induced by the consideration of the zero order hold on the delay state feedback loop of system (1). In order to show explicitly the order of approximation of system (3) with respect to (1) consider also the solution of system (4) on the segment $t_0 \le t \le h$, that it is given by,

$$e_x(t) = \Phi(t,t_0)e_x(t_0) + \int_{t_0}^{h} \Phi(t,s)A_1\gamma(s)ds$$

Considering the effect of the zero order hold and the fact that the initial condition is the same for systems (1) and (3) it follows that

$$e_x(t_0) = \xi(t_0) - \xi_r(t_0) = 0$$

producing the error dynamics,

$$e_x(t) = \int_{t_0}^{h} \Phi(t,s)A_1\gamma(s)ds \tag{5}$$

Since $\gamma(t) = \xi(t-h) - \xi(t_k - h)$ is a consequence of the zero order hold, then $\gamma(t)$ is a function of order one (with respect to the sampling period T), this is $\gamma(t) = o(T)$. It is clear that the equation (5) can be seen as an integration error between functions $\xi(t-h)$ and $\xi(t_k-h)$, therefore, considering a rectangular (by the effect of the ZOH) numerical integration method (quadrature formula) (Mathews, 1992), it follows that the integral (5) represents also a function of order one. To conclude the proof it is suffice to consider the step method (Driver, 1977) for the solution of system (4) over the segments $t_k \le t \le t_k + h$ ■

Remark 3.2. From the above developments it is clear that the error $e_x(t)$ can be minimized by considering a sampling and hold device of greater order, for example, a first order hold, that correspond in terms of numerical integration to the use of a trapezoidal method.

Remark 3.3. It is important to note that the error of approximation $e_x(t)$ depends directly over the sampling period T. Therefore, increasing the frequency $T \to 0$, will also improve the approximation.

An approximate discrete time model for the continuous time system (1) can now be obtained by considering its approximate representation (3). In order to obtain this model consider the solution $x_r(t)$ of the (partially sampled) system (3), over the interval $t_0 \le t \le h$, with the sampled initial condition $\xi(t_k - h)$,

$$\begin{aligned} x_r(t) &= \Phi(t,t_0)x(t_0) \\ &+ \int_{t_0}^{h} \Phi(t,s)A_1\xi(t_k-h)ds \\ &+ \int_{t_0}^{h} \Phi(t,s)Bu(s-\tau)ds. \end{aligned}$$

over the considered period of time, $\xi(t_k - h)$ and $u(t-\tau)$ can be viewed as external inputs. By the consideration of a zero order hold over the input $u(t-\tau)$ it is possible to write

$$\begin{aligned} x_r(t_{k+1}) &= \Phi(t_{k+1},t_k)x(t_k) \\ &+ \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s)ds \times \\ &\times [A_1\xi(t_k-h) + Bu(t_k-\tau)] \end{aligned}$$

where

$$\Phi(t_{k+1},t_k) = e^{(t_{k+1}-t_k)A} = e^{TA}.$$

Proposition 3.4. An approximate discrete time model for the continuous time system (1) is given by

$$x_{k+1} = \bar{A}x_k + \bar{A}_1x(t_k-h) + \bar{B}u(t_k-\tau)$$

with

$$\begin{aligned} \bar{A} &= \Phi(t_{k+1},t_k) \\ \bar{A}_1 &= \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s)dsA_1 \\ \bar{B} &= \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},s)dsB \end{aligned}$$

Proof. From the above developments, and considering from Theorem 3.1 that $x_r(t)$ is an approximation for $x(t)$ it is clear that for $t_0 \le t \le h$, it is obtained the approximate discrete time system

$$x_{k+1} = \bar{A}x_k + \bar{A}_1\xi(t_k-h) + \bar{B}u(t_k-\tau) \quad (6)$$

applying the above developments over the segments $t_k \le t \le t_k + h$, then it follows that the initial condition $\xi(t_k - h)$ are equal to the delay state $x(t_k - h)$ obtained on the precedent segment of time. This concludes the proof. ■

Remark 3.5. Considering that the time delays h and τ of system (1) are multiple of the sampling period T, this is $\tau = m_1T$ and $h = m_2T$, (commensurable delays) system (6) can be depicted as in Figure 2.

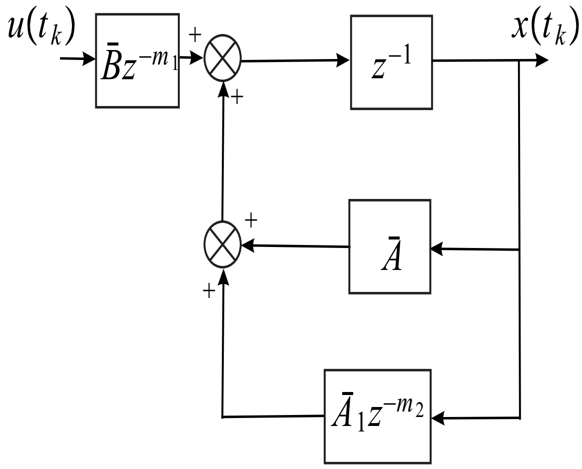

Fig. 2. Approximate discrete time system

4. ACADEMIC EXAMPLE: REGULATION PROBLEM

In this section, the performance of the presented discretization strategy will be analyzed by means of an academic example. It is consider the worst situation, this is, it is assumed that systems (1) is unstable. To show the feasibility of the proposed scheme, the discrete time approximation is used to design a discrete time controller for the continuous time system that solve the regulation problem.

With this purpose, consider the input and state time delayed system of the form (1), with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned} \quad (7)$$

where it is considered a time delay associated with the input $\tau = 0.4$, and a time delay associated with the state $h = 0.2$. It is an easy task to verify that system (7) is unstable (Arunsawatwong, 1996). Applying the methodology described in Section 3 and considering a sampling period $T = 0.2$ it is obtained the discrete time system,

$$\begin{aligned} x(t_{k+1}) &= \bar{A}x(t_k) + \bar{A}_1x(t_k-h) + \bar{B}u(t_k-\tau) \\ y(t_k) &= \bar{C}x(t_k) \end{aligned} \quad (8)$$

with

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 1 & 0.1813 \\ 0 & 0.8187 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0 & 0.0187 \\ 0 & 0.1813 \end{bmatrix}, \\ \bar{B} &= \begin{bmatrix} 0.0187 \\ 0.1813 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

where, from the fact that $\tau = 2T$, and $h = T$, system (8) can be equivalently rewritten in the form,

$$\begin{aligned} x(t_{k+1}) &= \bar{A}x(t_k) + \bar{A}_1x(t_{k-1}) + \bar{B}u(t_{k-2}) \\ y(t_k) &= \bar{C}x(t_k). \end{aligned} \quad (9)$$

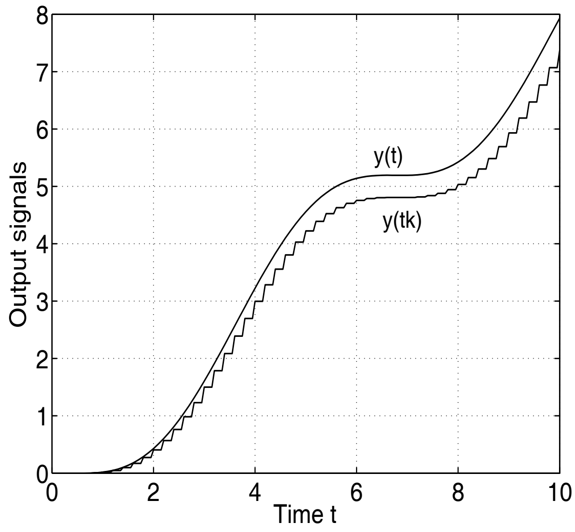

Fig. 3. Continuous and discrete open loop responses

The corresponding transfer function is given as,

$$\frac{Y(z)}{U(z)} = \frac{0.0187z + 0.0175}{z^4 - 1.8187z3 + 0.6374z^2 + 0.1813z} \quad (10)$$

The behavior of the continuous time model and its discrete approximation is shown in Figure 3 where the approximation error it was estimated in the worst case situation of unstable systems.

Based on the obtained approximate discrete time model it is now possible to propose a controller that solves the regulation problem for the continuous time system. The open loop poles of the system are obtained from discrete transfer function (10) as $z = \{0, -1 - 1, -0.1813\}$. A simple controller that solves the regulation problem for system (9,10) is given as

$$H(z) = \theta(z-1)/(z) \quad (11)$$

The root locus of system (10) in cascade with (11) is given in Figure 4, where the system is critically damped for $\theta = 4.2$ and critically stable for $\theta = 18.2$. Figure 5 shows the behavior of the closed loop system for $\theta = 4.2$ and a reference signal equal to one. Figure 6 shows the discrete control signal $u(t_k)$ under the same conditions.

Remark 4.1. Note that system (8,9) can be described in standard way as a system without delay at the state,

$$\begin{aligned} \phi(t_{k+1}) &= A_t\phi(t_k) + B_t u(t_{k-2}) \\ y(t_k) &= C_t\phi(t_k) \end{aligned}$$

with

$$A_t = \begin{bmatrix} 0 & I_{(2x2)} \\ \bar{A}_1 & \bar{A} \end{bmatrix}, \; B_t = \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix},$$
$$C_t = \begin{bmatrix} 0 & \bar{C} \end{bmatrix}.$$

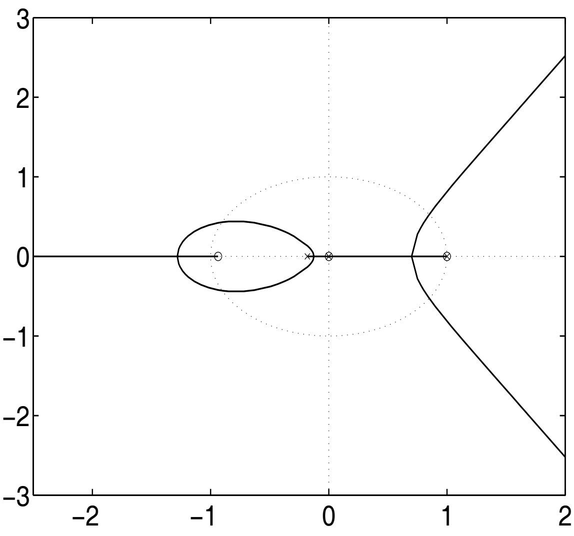

Fig. 4. Root locus for the discrete time system (10)-(11)

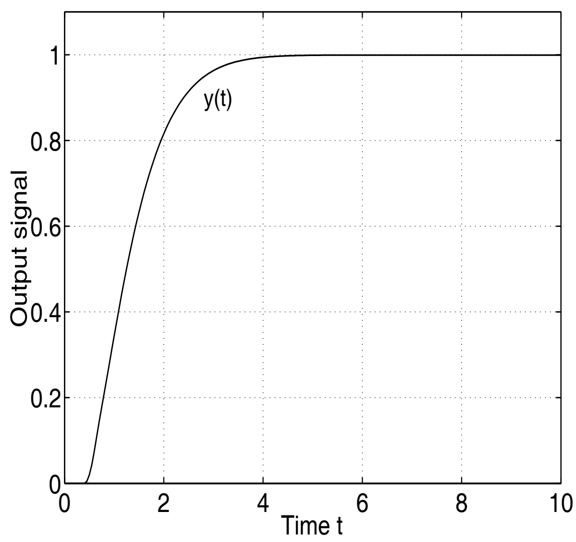

Fig. 5. Output of the controlled system

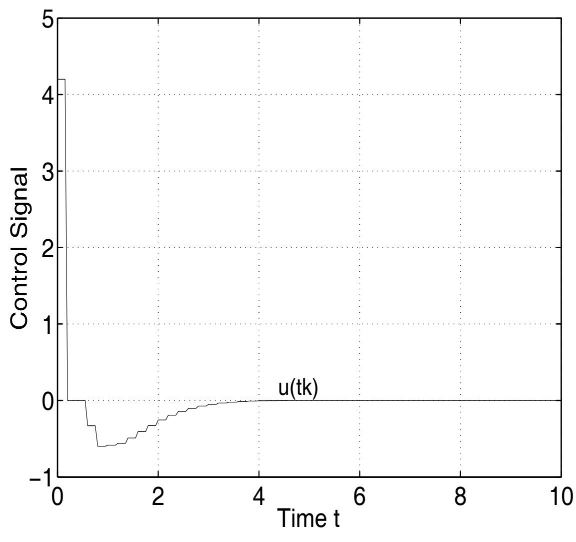

Fig. 6. Control signal $u(t_k)$

5. CONCLUSIONS

This work deals with the discrete time approximation problem for a class of invariant time-delay systems with time delays at the input signal and at the state. It is proposed a new methodology that allows getting an approximate discrete time representation of the continuous time model based on the consideration of a virtual input signal. In order to show the performance of the proposed strategy, numerical simulation are carried out considering an academic example, where the regulation problem for the continuous time system is solve by considering a compensator designed on the obtained approximate model. A formal proof for the closed loop stability of the controlled system is an issue that is considered as a future research topic.

REFERENCES

Arunsawatwong, A. (1996). Stability of retarded delay differential systems. *Int. J. Control* **65**(2), 347–364.

Astrom, K. J., C. C. Hang and B. C. Lim (1994). A new smith predictor for controlling a process with an integrator and long dead time. *IEEE Trans. Aut. Control* **39**(2), 343–345.

Conte, G. and A. M. Perdon (1995). The disturbance decoupling problem for systems over a ring. *SIAM, J. Contr. Optimiz.* **33**, 750–764.

Driver, R. D. (1977). *Ordinary and delay differential equations*. Springer-Verlag. London.

Górecki, A. and S. Fuksa (1989). *Analysis and Synthesis of time delay systems*. John Wiley & Sons. USA.

Iwai, Z., D. E. Seborg, D. G. Fisher and N. Kobayashi (1978). Decoupling of linear time variant systems with time delays in the control variable or state variable. *Int. J. Control* **28**(6), 869–888.

Malek-Zavarei, M. and M. Jamshidi (1987). *Time-Delay systems. Analysis, optimization and applications*. NorthHolland. The Netherlands.

Mathews, J. H. (1992). *Numerical methods for mathematics science and engineering*. Prentice-Hall. New York.

Maza-Casas, L., M. Velasco-Villa and Ja. Alvarez-Gallegos (1999). On the state prediction of linear systems with time-delays in the input and the state. In: *Proc. 38th. IEEE Conf. On Decision and Control*. USA. pp. 239–244.

Mori, T. and H. Kokame (1989). Stability of $\dot{x}(t) = ax(t) + bx(t-\tau)$. *IEEE Transactions on Automatic Control* **34**, 460–462.

Mori, T. and H. Kokame O. J. M. Smith (1957). Closer control of loops with deadtime. *Chem. Eng. Prog.* **53**(5), 217–219.

Tzafestas, S. G. and P. Paraskevopoulos (1973). On the decoupling of multivariable control systems with time delays. *Int. J. Control* **17**(2), 405–415.

Wang, S. S. (1992). Further results on stability of $\dot{x}(t) = ax(t) + bx(t-\tau)$. *System and Control Letters* **19**, 165–168.

Watanabe, K. and M. Ito (1981). Process-model control for linear systems with delay. *IEEE Trans. Aut. Cont.* **26**, 1261–1269.