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SHARPNESS RESULTS OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

BY

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Abstract

The main aim of this paper is to use the concept of finite Blaschke product to prove sharpness of some of the known results. Geometric properties of a class of functions $\mathcal{U}(\lambda)$ were discussed in [8, 11, 12]. Also, starlikeness of $\mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$ for $\mu \leq n$ was obtained in [9, 10]. In this paper, we prove the sharpness of those results using the technique of R. Fournier [3] which was later revised by R. Fournier and S. Ponnusamy [5].

1. Introduction

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the set of all functions analytic in Δ with the usual normalization $f(0) = 0 = f'(0) - 1$, and let $\mathcal{A}_0 = \{f(z)/z : f \in \mathcal{A}\}$. Also, we let $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \Delta\}$. If $f \in \mathcal{S}$ maps Δ onto a starlike domain (with respect to the origin), i.e. $tw \in f(\Delta)$ whenever $t \in [0, 1]$ and $w \in f(\Delta)$, then we say that f is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. For $0 \leq \alpha < 1$, a function $f \in \mathcal{S}$ is starlike of order α, denoted by $\mathcal{S}^*(\alpha)$, if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \Delta. \tag{1.1}$$

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It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* α, $0 < \alpha \leq 1$ if and only if f satisfies the analytic condition

$$\frac{zf'(z)}{f(z)} \prec \Big(\frac{1+z}{1-z}\Big)^{\alpha}, \quad z \in \Delta,$$

where $\prec$ denotes the usual subordination (see eg. [2]). The class of all functions which are strongly starlike of order α is denoted by $\mathcal{S}_\alpha$. Clearly, $\mathcal{S}_1 = \mathcal{S}^*$. Let $\mathcal{R}_\alpha$ be the set of all functions in $\mathcal{A}$ such that

$$f'(z) \prec \Big(\frac{1+z}{1-z}\Big)^{\alpha}, \quad z \in \Delta.$$

It is well-known that $\mathcal{R}_1 = \mathcal{R} \subsetneq \mathcal{S}$. For $\mu \leq n$, $n \geq 1$ and $\lambda > 0$, let $\mathcal{U}_n(\lambda, \mu)$ denote the class of all functions $f \in \mathcal{A}_n$ satisfying

$$\frac{f(z)}{z} \neq 0 \quad \text{and} \quad \left| f'(z)\Big(\frac{z}{f(z)}\Big)^{\mu+1} - 1 \right| < \lambda, \quad z \in \Delta.$$

Also, let $\mathcal{U}_1(\lambda, \mu) := \mathcal{U}(\lambda, \mu)$. Geometric properties of the class $\mathcal{U}(\lambda, \mu)$ has been studied in detail in [5]. As usual, we set $\mathcal{U}(\lambda, 1) = \mathcal{U}(\lambda)$ and $\mathcal{U}(1) = \mathcal{U}$. It is well-known that $\mathcal{U}(\lambda) \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ (see [1, 7]). We also introduce

$$\mathcal{B}_n = \{w \in \mathcal{H}(\Delta) : |w(z)| < 1 \ \text{ and } \ w^{(k)}(0) = 0 \text{ for } k = 0, 1, 2, \ldots, n-1\}.$$

By the Schwarz lemma, one has $|w(z)| \leq |z|^n$. Here $\mathcal{H}(\Delta)$ denotes the class of functions analytic in Δ.

In [8, 11, 12], certain sufficient conditions in terms of λ (> 0), α and n (≥ 1) were obtained, so that $\mathcal{U}(\lambda) \cap \mathcal{A}_n$ is a subset of $\mathcal{S}^*(\alpha)$ or $\mathcal{S}_\alpha$ or $\mathcal{R}_\alpha$. Similarly, certain sufficiency conditions for functions in $\mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$ to be in $\mathcal{S}^*(\alpha)$ or $\mathcal{S}_\alpha$ were obtained by S. Ponnusamy and P. Sahoo in [9, 10]. In all these cases the sharpness of the results were left open. Now using the technique of R. Fournier and a recently revised version of R. Fournier and S. Ponnusamy [5], we prove the sharpness part.

The proofs mainly rely on the following Lemmas.

Lemmma 1.2. *Given φ and ψ in $\mathbb{R}$, there exists a sequence $\{b_n\}$ of finite Blaschke products such that $b_n(1) = e^{i\varphi}$, $b_n(0) = 0$ and $b_n(z) \to e^{i\psi}z$ in the sense of convergence in $\mathcal{H}(\Delta)$.*

Here a finite Blaschke product is a function of the type

$$b(z) = e^{i\gamma} \prod_{j=1}^{m} \frac{z-a}{1-\overline{a}_j z}, \quad \{a_j\}_{j=1}^{n} \subset \Delta, \quad \gamma \in \mathbb{R}.$$

This result is due to R. Fournier [3]. A slightly extended version of the above lemma was proved in [13]. We also have a stronger version of the above lemma which is obtained from a result due to W. B. Jones and St. Ruscheweyh [6].

Lemma 1.3. *There exists an infinite sequence* $\{w_n\}$ *of finite Blaschke products with the following property: given a function* $w \in \mathcal{H}(\Delta)$ *with* $w(\Delta) \subseteq \Delta$ *and two sets of nodes* $\{\varphi_k\}_{k=1}^{m}$ *and* $\{\psi_k\}_{k=1}^{m}$ *in* $\mathbb{R}$ *where* φ_k*'s are assumed to be pairwise distinct* $(\mathrm{mod}\, 2\pi)$*, there exists a subsequence* $\{w_{n_j}\}$ *of* $\{w_n\}$ *such that*

$$w_{n_j}(e^{i\varphi_k}) \ = \ e^{i\psi_k}, \quad 1 \le k \le m, \quad j \ge 1$$

and

$$\lim_{j\to\infty} w_{n_j} \ = \ w \quad \text{ in } \ \mathcal{H}(\Delta).$$

We also require the following

Lemma 1.4. [4] *Let* $\theta \in \mathbb{R}$ *and* $\mathrm{Re}(c) < n$*. Then the functional*

$$I(w) = \sum_{k=n}^{\infty} \frac{a_k(w)}{k-c} e^{ik\theta}, \quad w(z) = \sum_{k=n}^{\infty} a_k(w) z^k \in \mathcal{B}_n,$$

is well defined and continuous over $\mathcal{B}_n$.

2. Sharpness results

In this chapter, we restate the sharp version of the theorems stated in [8, 9, 10, 11, 12] and prove the sharpness part.

Theorem 2.1.[8, Theorem 3.1] *Let* $f \in \mathcal{U}(\lambda)$, $0 < \lambda \le 1$ *and* $\gamma \in (0, 1]$.

Define

$$\lambda_\gamma^* = \frac{-|f''(0)|\cos(\pi\gamma/4) + \sin(\pi\gamma/4)\sqrt{16\cos^2(\pi\gamma/4) - |f''(0)|^2}}{2\cos(\pi\gamma/4)}$$

and $\lambda_\gamma^{\mathcal{R}}$ *is given by the inequality*

$$\sin(\pi\gamma/2)\sqrt{4-\lambda^2} \geq (|f''(0)|+\lambda)\sqrt{4-(|f''(0)|+\lambda)^2} + \lambda\cos(\pi\gamma/2).$$

Then

(i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ *if and only if* $0 < \lambda \leq \lambda_\gamma^*/2$,

(ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ *if and only if* $0 < \lambda \leq \lambda_\gamma^{\mathcal{R}}/2$.

In [8], the sharpness part of the last theorem remained unanswered. Now we are in a position to show that each of the bounds $\lambda_\gamma^*/2$ and $\lambda_\gamma^{\mathcal{R}}/2$ cannot be replaced by a larger number without violating the conclusion.

Proof. **Case (i):** Let $f \in \mathcal{U}(\lambda)$. Then, as usual, we have the following

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1 - a_2 z - \lambda \displaystyle\int_0^1 \frac{w(tz)}{t^2}dt} = \frac{1+\lambda w(z)}{1 - a_2 z - \lambda w(z) * F_1(z)} \tag{2.2}$$

where $w \in \mathcal{B}_2$, and

$$F_1(z) = \sum_{n=2}^{\infty} \frac{z^n}{n-1} = -z\mathrm{Log}(1-z)$$

Thus, from (2.2), Lemma 1.4 and maximum modulus principle, we see that for every $f \in \mathcal{U}(\lambda)$ and $z \in \Delta$, there exists a ψ and φ in $\mathbb{R}$ such that

$$\mathrm{Arg}\left(\frac{zf'(z)}{f(z)}\right) \leq \mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{1 - a_2 - \lambda e^{i\varphi}}\right). \tag{2.3}$$

Here, we observe that the above relation is possible due to of the fact that $I(w)$ is continuous on $\mathcal{B}_2$ (from Lemma 1.4). Indeed, By Lemma 1.2, given a ψ, φ in $\mathbb{R}$, there exists a sequence of finite Blaschke products such that

$$w_n(1) = e^{i\psi} \quad \text{and} \quad w_n(z) \to e^{i\varphi}z^2 \quad \text{in} \quad \mathcal{H}(\Delta).$$

Define f_n's in $\mathcal{U}(\lambda)$ such that

$$\lim_{n\to\infty}\frac{f_n'(1)}{f_n(1)}=\frac{1+\lambda e^{i\psi}}{1-a_2-\lambda e^{i\varphi}}.$$

In fact, from the above equation, we have equality in (2.3) for some $f\in\mathcal{U}(\lambda)$. Thus we have obtianed sharpness of the result. Now, since $|a_2|+\lambda\le 1$, taking $\varphi=\mathrm{Arg}a_2$ we have

$$\mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{1-a_2-\lambda e^{i\varphi}}\right)\le\arcsin(\lambda)+\arcsin(|a_2|+\lambda)\le\frac{\gamma\pi}{2}.$$

From the above relation, we get the required sharpened condition for functions in $\mathcal{U}(\lambda)$ to be in $\mathcal{S}_\gamma$.

Case (ii): Since $f\in\mathcal{U}(\lambda)$ and for some $w\in\mathcal{B}_2$, we have the following

$$f'(z)=\frac{1+\lambda w(z)}{\left(1-a_2z-\lambda\displaystyle\int_0^1\frac{w(tz)}{t^2}dt\right)^2}=\frac{1+\lambda w(z)}{(1-a_2z-\lambda w(z)*F_1(z))^2}.$$

Repeating the steps as in Case (i), it follows easily that

$$\mathrm{Arg}f'(z)\le\mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{(1-a_2-\lambda e^{i\varphi})^2}\right). \tag{2.4}$$

Here, we observe that the above relation is possible because of the fact that $I(w)$ is continuous on $\mathcal{B}_2$ (from Lemma 1.4). Defining f_n's in $\mathcal{U}(\lambda)$ with

$$\lim_{n\to\infty}f_n'(1)=\frac{1+\lambda e^{i\psi}}{(1-a_2-\lambda e^{i\varphi})^2}$$

we have equality in (2.4) for some $f\in\mathcal{U}(\lambda)$. Now, since $|a_2|+\lambda\le 1$, we have

$$\mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{(1-a_2-\lambda e^{i\varphi})^2}\right)\le\arcsin(\lambda)+2\arcsin(|a_2|+\lambda)\le\frac{\gamma\pi}{2}.$$

From the above relation, we get the required sharp result for functions to be in $\mathcal{R}_\gamma$. □

Using the above arguments, we can also prove that the following result is sharp.

Theorem 2.5. [10, Theorem 3.1] *Let* $\gamma \in (0,1]$, $n \geq 1$, *and*

$$\lambda^*(\gamma,n) = \Big\{ -n(n+\cos(\pi\gamma/2))|a_{n+1}| + \sin(\gamma\pi/2) \times \sqrt{1+n^2(1-|a_{n+1}|^2)+2n\cos(\gamma\pi/2)}\Big\} \Big/ \Big[1+2n\cos(\gamma\pi/2)+n^2\Big].$$

If $f \in \mathcal{U}_n(\lambda,n)$, *then* $f \in \mathcal{S}_\gamma$ *if and only if* $0 < \lambda \leq \lambda^*(\gamma,n)$.

Now, let us prove the sharpness of the result for functions having missing Taylor coefficients to be in the class $\mathcal{S}_\gamma$ and $\mathcal{R}_\gamma$.

Theorem 2.6. [11] *Let* $\gamma \in (0,1]$ *and* $n \geq 2$ *be fixed. Let* $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$,

$$\lambda^*(\gamma,n) = \frac{(n-1)\sin(\pi\gamma/2)}{\sqrt{n^2-4(n-1)\sin^2(\pi\gamma/4)}}$$

and $\lambda^{\mathcal{R}}(\gamma,n)$ *be the largest positive* $\lambda > 0$ *satisfying the equation*

$$\sqrt{1-\lambda^2}\sin(\pi\gamma/2) = 2\Big(\frac{\lambda}{n-1}\Big)\sqrt{1-\Big(\frac{\lambda}{n-1}\Big)^2} + \lambda\cos(\pi\gamma/2).$$

Then

(i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \leq \lambda^*(\gamma,n)$.

(ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ *for* $0 < \lambda \leq \lambda^{\mathcal{R}}(\gamma,n)$.

The above bounds for $\lambda^*(\gamma,n)$ *and* $\lambda^{\mathcal{R}}(\gamma,n)$ *are sharp.*

Proof. **Case (i):** Since $f \in \mathcal{U}(\lambda) \cap \mathcal{A}_n$, we have the following

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^2}dt} = \frac{1+\lambda w(z)}{1-\lambda w(z) * F_1(z)},$$

for some $w \in \mathcal{B}_n$. Thus, from the above representation for functions in $\mathcal{U}(\lambda)$ with missing Taylor's coefficients, Lemma 1.4 and maximum modulus principle, we see that for all $f \in \mathcal{U}(\lambda) \cap \mathcal{A}_n$, there exists a ψ, $\varphi \in \mathbb{R}$ such that

$$\mathrm{Arg}\Big(\frac{zf'(z)}{f(z)}\Big) \leq \mathrm{Arg}\Big(\frac{1+\lambda e^{i\psi}}{1-\lambda e^{i\varphi}/(n-1)}\Big). \tag{2.7}$$

Here, we observe that the above relation is possible because of the fact that $I(w)$ is continuous on $\mathcal{B}_n$ (from Lemma 1.4). By Lemma 1.2, given a ψ, φ in $\mathbb{R}$, there exists a sequence of finite Blaschke products such that

$$w_k(1) = e^{i\psi} \quad \text{and} \quad w_k(z) \to e^{i\varphi} z^n \quad \text{in} \quad \mathcal{H}(\Delta).$$

Defining f_k's in $\mathcal{U}(\lambda)$ such that

$$\lim_{k\to\infty} \frac{f_k'(1)}{f_k(1)} = \frac{1+\lambda e^{i\psi}}{1-\lambda e^{i\varphi}/(n-1)}.$$

In fact, from the above equation, we have equality in (2.7) for some $f \in \mathcal{U}(\lambda) \cap \mathcal{A}_n$. Thus the result is sharp. Indeed, for each k, fixing $\theta = 0$ in definition of I,

$$w_k(z) * F_1(z) = \int_0^1 \frac{w_k(tz)}{t^2} dt \to I(w_k) \quad \text{as} \quad z \to 1.$$

Since I is continuous in $\mathcal{B}_n$, we see that

$$I(w_k) \to e^{i\varphi}/(n-1) \quad \text{as} \quad w_k(z) \to e^{i\varphi} z^n.$$

Now, since $\lambda \leq 1$, we have

$$\mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{1-\lambda e^{i\varphi}/(n-1)}\right) \leq \arcsin(\lambda) + \arcsin(\lambda/(n-1)) \leq \frac{\gamma\pi}{2}.$$

From the above relation, we get the required result for $\mathcal{S}_\gamma$.

Case (ii): Since $f \in \mathcal{U}(\lambda) \cap \mathcal{A}_n$, we have the following

$$f'(z) = \frac{1+\lambda w(z)}{\Big(1-\lambda \displaystyle\int_0^1 \frac{w(tz)}{t^2} dt\Big)^2} = \frac{1+\lambda w(z)}{(1-\lambda w(z) * F_1(z))^2}$$

where $w \in \mathcal{B}_n$. By Lemma 1.2, given a ψ, φ in $\mathbb{R}$, there exists a sequence of finite Blaschke products such that

$$w_k(1) = e^{i\psi} \quad \text{and} \quad w_k(z) \to e^{i\varphi} z^n \quad \text{in} \quad \mathcal{H}(\Delta).$$

Thus, from the above representation for $\mathcal{U}(\lambda)$, Lemma 1.4 and maximum modulus principle, we see that

$$\mathrm{Arg}(f'(z)) \leq \mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{(1-\lambda e^{i\varphi}/(n-1))^2}\right). \tag{2.8}$$

Here, we observe that the above relation is possible because of the fact that $I(w)$ is continuous on $\mathcal{B}_n$ (from Lemma 1.4). Defining f_k's in $\mathcal{U}(\lambda)$ such that

$$\lim_{k\to\infty} f_k'(1) = \frac{1+\lambda e^{i\psi}}{(1-\lambda e^{i\varphi}/(n-1))^2}.$$

In fact, from the above equation, we have equality in 2.8. Now, since $\lambda \leq 1$, we have

$$\mathrm{Arg}\left(\frac{1+\lambda e^{i\psi}}{(1-\lambda e^{i\varphi}/(n-1))^2}\right) \leq \arcsin(\lambda) + 2\arcsin(\lambda/(n-1)) \leq \frac{\gamma\pi}{2}.$$

From the above relation, we get the required sharp result for functions to be in $\mathcal{R}_\gamma$. □

Repeating the above proof for $f \in \mathcal{U}_n(\lambda,\mu)$, we have the sharpness of the following

Theorem 2.9.[9, Theorem 3.1] *Let $\gamma \in (0,1]$, $n \geq 1$, $\mu \in (0,n)$ and*

$$\lambda_*(\gamma,\mu,n) = \frac{(n-\mu)\sin(\gamma\pi/2)}{\sqrt{(n-\mu)^2+\mu^2+2\mu(n-\mu)\cos(\gamma\pi/2)}}.$$

If $f \in \mathcal{U}_n(\lambda,\mu)$, then $f \in \mathcal{S}_\gamma$ for $0 < \lambda \leq \lambda_*(\gamma,\mu,n)$. This result is sharp.

Our next result is to find sharpness of the result for functions in $\mathcal{U}_n(\lambda, n)$ to be starlike of order δ.

Theorem 2.10.[10, Theorem 5.1] *If $f(z) \in \mathcal{U}_n(\lambda,n)$ and $b = |a_{n+1}| \leq$*

$1/n$, then $f\in\mathcal{S}^*(\delta)$ if and only if $0<\lambda\le\lambda_0(\delta)$, where

$$\lambda_0(\delta)=\begin{cases}\dfrac{\sqrt{(1-2\delta)(1+n^2(1-2\delta-b^2))}-n^2b(1-2\delta)}{1+n^2(1-2\delta)} & \text{for } 0\le\delta\le\dfrac{n(b+1)}{n(b+2)+1}\\[2ex] \dfrac{1-\delta(1+nb)}{1+n\delta} & \text{for } \dfrac{n(b+1)}{n(b+2)+1}<\delta<1.\end{cases}$$

Proof. Since we know that

$$\frac{zf'(z)}{f(z)}=\frac{1+\lambda w(z)}{1-na_{n+1}z-n\lambda\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt},$$

where $w\in\mathcal{B}_{n+1}$, we can easily see that

$$\frac{1}{1-\delta}\left(\frac{zf'(z)}{f(z)}-\delta\right)=\frac{1+\dfrac{\lambda w(z)}{1-\delta}+\dfrac{n\delta}{1-\delta}\left[a_{n+1}z^n+\lambda\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt\right]}{1-na_{n+1}z^n-n\lambda\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt}.$$

Now, we have to show that $\mathrm{Re}\left(\dfrac{zf'(z)}{f(z)}\right)>\delta$. To do this, according to a well-known result [14] and the last equation, it suffices to show that

$$\frac{1+\dfrac{\lambda w(z)}{1-\delta}+\dfrac{n\delta}{1-\delta}\left[a_{n+1}z^n+\lambda\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt\right]}{1-na_{n+1}z^n-n\lambda\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt}\neq -iT,\quad T\in\mathbb{R},$$

which is easily seen to be equivalent to

$$\lambda\left[\frac{w(z)+n(\delta-i(1-\delta)T)\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt}{(1-\delta)(1+iT)+na_{n+1}z(\delta-iT(1-\delta))}\right]\neq -1,\quad T\in\mathbb{R}.$$

If we let

$$M=\sup_{z\in\Delta,w\in\mathcal{B}_{n+1},T\in\mathbb{R}}\left|\frac{w(z)+n(\delta-i(1-\delta)T)\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt}{(1-\delta)(1+iT)+na_{n+1}z(\delta-iT(1-\delta))}\right|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_{n+1}$, we obtain that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta \quad \text{if} \quad \lambda M \le 1.$$

This observation shows that it suffices to find M. First we notice that

$$M \le \sup_{T\in\mathbb{R}}\left\{\frac{1+n\sqrt{\delta^2+(1-\delta)^2T^2}}{|(1-\delta)\sqrt{1+T^2}-nb\sqrt{\delta^2+(1-\delta)^2T^2}|}\right\},$$

where, for convenience, we use the notation $b=|a_{n+1}|$. In fact, in the sequel, we prove that equality holds in the above relation, hence the sharpness is exhibited.

From Lemma 1.3, for ψ, φ in $\mathbb{R}$, there exists a sequence of finite Blaschke products $\{w_k\}$ such that

$$w_k(e^{i\theta}) = e^{i\psi} \quad \text{and} \quad w_k(z) \to e^{i\varphi}z^{n+1} \quad \text{in} \quad \mathcal{H}(\Delta).$$

Here

$$\theta = -\mathrm{Arg}[(\delta-(1-\delta)iT)a_{n+1}] + \mathrm{Arg}(1+iT).$$

Therefore, as in the proof of the previous theorem, we have the following relation for each $T\in\mathbb{R}$

$$\begin{aligned}&\sup_{\substack{w\in\mathcal{B}_{n+1}\\ z\in\Delta}}\left|\frac{w(z)+n(\delta-i(1-\delta)T)\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}dt}{(1-\delta)(1+iT)+na_{n+1}z(\delta-iT(1-\delta))}\right|\\ &\le \sup_{\psi,\varphi\in\mathbb{R}}\frac{\left|e^{i\psi}+n\sqrt{\delta^2+(1-\delta)^2T^2}e^{i(\varphi+(n+1)\theta+\theta_1)}\right|}{|(1-\delta)\sqrt{1+T^2}-nb\sqrt{\delta^2+(1-\delta)^2T^2}|}\end{aligned}$$

where $\theta_1 = \mathrm{Arg}(\delta-(1-\delta)iT)$. Fixing φ and choosing $\psi=\varphi+(n+1)\theta+\theta_1$, we get the required equality. Thus the bound for M is sharp as a function of T. Bound for M is then obtained as in [10, Theorem 5.1]. □

Taking $n=1$ in the above theorem, we have the following

Theorem 2.11.[12, Theorem 1.2] *If* $f\in\mathcal{U}(\lambda)$ *and* $a=|f''(0)|/2\le 1$, *then* $f \in \mathcal{S}^*(\delta)$ *if and only if* $0 < \lambda \leq \lambda(\delta)$, *where*

$$\lambda(\delta) = \begin{cases} \dfrac{\sqrt{(1-2\delta)(2-a^2-2\delta)} - a(1-2\delta)}{2(1-\delta)} & \text{if } 0 \leq \delta < \dfrac{1+a}{3+a}, \\ \dfrac{1-\delta(1+a)}{1+\delta} & \text{if } \dfrac{1+a}{3+a} \leq \delta < \dfrac{1}{1+a}. \end{cases}$$

Finally we prove the sharpness result for functions with missing Taylor coefficients to be starlike of order α.

Theorem 2.12.[9, Theorem 3.3] *Let* $\alpha \in [0,1)$, $n \geq 1$ *and* $\mu \in (0,n)$. *If* $f(z) \in \mathcal{U}_n(\lambda,\mu)$, *then* $f \in \mathcal{S}^*(\alpha)$ *for* $0 < \lambda \leq \lambda^*(\alpha,\mu,n)$, *where*

$$\lambda^*(\alpha,\mu,n) = \begin{cases} \dfrac{(n-\mu)\sqrt{1-2\alpha}}{\sqrt{(n-\mu)^2+\mu^2(1-2\alpha)}} & \text{for } 0 \leq \alpha \leq \dfrac{\mu}{n+\mu} \\ \dfrac{(n-\mu)(1-\alpha)}{n-\mu+\mu\alpha} & \text{for } \dfrac{\mu}{n+\mu} < \alpha < 1. \end{cases}$$

The bounds for $\lambda^*(\alpha,\mu,n)$ *is the best possible. That is, we cannot improve the bound for* $\lambda^*(\alpha,\mu,n)$ *without violating the hypothesis.*

Proof. Suppose that $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}_n(\lambda,\mu)$. Then, it is a simple exercise to see that

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-\lambda\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}dt}$$

and therefore,

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) = \frac{1+\frac{\lambda w(z)}{1-\alpha} + \frac{\alpha\lambda}{1-\alpha}\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}dt}{1-\lambda\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}dt}.$$

We need to show that $f \in \mathcal{S}^*(\alpha)$.

$$\frac{1+\dfrac{\lambda w(z)}{1-\alpha} + \dfrac{\alpha\lambda}{1-\alpha}\displaystyle\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}dt}{1-\lambda\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}dt} \neq -iT, \quad T \in \mathbb{R},$$

which is equivalent to

$$\lambda\left[\frac{w(z)+(\alpha-i(1-\alpha)T)\displaystyle\int_0^1\frac{w(t^{1/\mu}z)}{t^2}dt}{(1-\alpha)(1+iT)}\right]\neq -1,\quad T\in\mathbb{R}.$$

If we let

$$M=\sup_{z\in\Delta,w\in\mathcal{B}_n,T\in\mathbb{R}}\left|\frac{w(z)+(\alpha-i(1-\alpha)T)\displaystyle\int_0^1\frac{w(t^{1/\mu}z)}{t^2}dt}{(1-\alpha)(1+iT)}\right|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_n$, we obtain that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad\text{if}\quad\lambda M\leq 1.$$

This observation shows that it suffices to find M. First we notice that

$$M\leq\sup_{T\in\mathbb{R}}\left\{\frac{(n-\mu)+\mu\sqrt{\alpha^2+(1-\alpha)^2T^2}}{(n-\mu)(1-\alpha)\sqrt{1+T^2}}\right\}.$$

Here we prove that this inequality is sharp, in particular, the bound for M is the best possible.

From Lemma 1.2, ψ, φ in $\mathbb{R}$, there exists a sequence of finite Blaschke products $\{w_k\}$ such that $w_k(1)=e^{i\psi}$ and $w_k(z)\to e^{i\varphi}z^n$ in $\mathcal{H}(\Delta)$. Therefore, we have the following relation for each $T\in\mathbb{R}$

$$\sup_{\substack{w\in\mathcal{B}_{n+1}\\ z\in\Delta}}\left|\frac{w(z)+(\alpha-i(1-\alpha)T)\displaystyle\int_0^1\frac{w(t^{1/\mu}z)}{t^2}dt}{(1-\alpha)(1+iT)}\right|$$
$$\leq\sup_{\psi,\varphi\in\mathbb{R}}\frac{\left|e^{i\psi}+\dfrac{\mu}{n-\mu}\sqrt{\alpha^2+(1-\alpha)^2T^2}e^{i(\varphi+\theta_1)}\right|}{(1-\alpha)\sqrt{1+T^2}}$$

where $\theta_1=\mathrm{Arg}(\alpha-(1-\alpha)iT)$. Fixing φ and choosing $\psi=\varphi+\theta_1$, we get the required relation. Thus the bound for M is sharp as a function of T. □

Taking $\mu=1$ in the above theorem, we have the following

Theorem 2.13.[11] *If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belongs to $\mathcal{U}(\lambda)$ for some $n \geq 2$, then $f \in \mathcal{S}^*(\alpha)$ if and only if $0 < \lambda \leq \lambda(\alpha, n)$, where*

$$\lambda(\alpha,n) = \begin{cases} \dfrac{(n-1)\sqrt{(1-2\alpha)[(n-1)^2+1-2\alpha]}}{(n-1)^2+1-2\alpha} & \text{if } \ 0 \leq \alpha \leq 1/(n+1) \\ \dfrac{(n-1)(1-\alpha)}{n+\alpha-1} & \text{if } \ 1/(n+1) < \alpha < 1. \end{cases}$$

3. Conclusion

Geometric properties of a class of functions $\mathcal{U}(\lambda)$ were discussed in [8, 11, 12] where the question of sharpness of the result was left open. Also, sufficient conditions for starlikeness of $\mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$ for $\mu \leq n$ obtained in [9, 10] where the not sharp. In this paper, sharpness of those results are proved using finite Blaschke product.

In conclusion, we have the following

Remark.

(1) In all the above discussions on $\mathcal{U}_n(\lambda, \mu)$, μ is considered to be real. Similar results on sharpness of the bounds can be obtained when μ is complex. For example, when $n = 1$ and μ a complex number in $\mathcal{U}_n(\lambda, \mu)$, we have the following interesting lemma by R. Fournier and S. Ponnusamy [5] in which the sharpness for this special case is obtained.

Lemma 3.1. *Let $\mu \in \mathbb{C}$ with* Re$\mu < 1$. *Then*

$$\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^* \quad \text{if and only if} \quad 0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2 + |\mu|^2}}.$$

Further, from the above lemma, it is clear that $\mathcal{U}(1, \mu) \subset \mathcal{S}^*$ if and only if $\mu = 0$.

(2) Moreover, from the discussion on sufficient conditions for starlikeness of $\mathcal{U}_n(\lambda, \mu)$ for $\mu \leq n$ in the previous section (Theorems 2.10 and 2.12), we can observe that λ as a function of μ is discontinuous at the point $\mu = n$. More precisely, we can see that in Theorem 2.12 taking $\alpha = 0$, $\lambda^*(\alpha, \mu, n) \to 0$ as $\mu \to n$ whereas in Theorem 2.10 taking $\delta = 0$, we see

that $\lambda_0(\delta) = (\sqrt{1+n^2-n^2b^2} - n^2b)/(1+n^2)$ which is nonzero unless $b = 1/n$.

References

1. L. A. Aksentiev, Sufficient conditions for univalence of regular functions. (Russian) *Izv. Vysš. Učebn. Zaved. Matematika*, **3**(1958), No. 4, 3-7.

2. P. L. Duren, Univalent functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

3. R. Fournier, On integrals of bounded analytic functions in the unit disk, *Complex Variables Theory Appl.* **11**(1989), 125–133.

4. R. Fournier and P. T. Mocanu, Differential inequalities and starlikeness, *Complex Variables Theory Appl.*, **48**(2003), No. 4, 283-292.

5. R. Fournier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Equ.*, 2007, To appear.

6. W. B. Jones and St. Ruscheweyh, Blaschke product interpolation and its application to the design of digital filters, *Constr. Approx.*, **3**(1987), 405-409.

7. S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.*, **33**(1972), No. 2, 392-394.

8. M. Obradović, S. Ponnusamy, V. Singh and P. Vasundhra, Univalency, starlikesess and convexity applied to certain classes of rational functions, *Analysis (Munich)*, **22**(2002), No. 3, 225-242.

9. S. Ponnusamy and P. Sahoo, Geometric properties of certain linear integral transforms, *Bull. Belg. Math. Soc.*, **12**(2005), 95-108.

10. S. Ponnusamy and P. Sahoo, Special classes of univalent functions with missing coefficients and integral transforms, *Bull. Malays. Math. Sci. Soc.*, **12**(2005), No. 2, 141-156.

11. S. Ponnusamy and P. Vasundhra, Univalent functions with missing Taylor coefficients, *Hokkaido Math. J.*, **33**(2004), No. 2, 341-355.

12. S. Ponnusamy and P. Vasundhra, Criteria for univalence, starlikeness and convexity, *Ann. Polon. Math.*, **85**(2005), No. 2, 121-133.

13. F. Rønning, St. Ruscheweyh and N. Samaris, Sharp estimates conditions for analytic functions with bounded derivative, *J. Austral. Math. Soc.*, **69**(2000), 303-315.

14. St. Ruscheweyh, Convolutions in geometric function theory, *Les Presses de l'Université de Montréal, Montréal*, 1982.

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