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Integrable Systems in the realm of Algebraic Geometry

Second Edition



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Preface to the second edition

The present edition of this book, five years after the first edition, has been spiced with several recent results which fit naturally in the point of view that had been adapted in the original text and with some new examples and constructions that will help the reader to appreciate better our approach to integrable systems.

On this occasion I wish to thank my collaborators from the last five years, to wit Christina Birkenhake, Peter Bueken, Rui Fernandes, Masoto Kimura, Vadim Kuznetsov, Marco Pedroni, Michael Penkava, Luis Piován and Claude Roger for a fruitful interaction and for their warm friendship. Most of the results that have been added are taken from, or are inspired by, joint work with some of them; I acknowledge their permission to add these, sometimes unpublished, results.

The colleagues at my newest working environment, the University of Poitiers (France), created for me a pleasant and stimulating working environment. I wish to acknowledge the support of all of them. Special thanks go to Marc van Leeuwen, Claude Quitté and Patrice Tauvel for sharing their insights with me, which usually led to a real improvement of parts of the text.

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