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A special identity between three ${}_2F_1(a, b; c; 4)$ hypergeometric series

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Abstract

As a by-product of the calculation of eigenvalues of a particular SO(3) invariant operator in the enveloping algebra of SO(5), a three terms relation between three hypergeometric series emerges.

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1. Introduction

In a series of papers (for example [4–7]) shift operators were constructed out of the enveloping algebra of the generators $l_i (i = 0, \pm)$ of SO(3) and the components $T(j, \mu), \mu = -j, \dots, j$ of an arbitrary $(2j+1)$-dimensional tensor representation of SO(3). The constructed shift operators, denoted O_l^k, $k = -j, \dots, j$, change the l value, where $l(l + 1)$ is the eigenvalue of the SO(3) Casimir operator L^2, by k, without altering the eigenvalue of l_0. A general analysis of the operator O_l^k for arbitrary j values has been given by Hughes and Yadegar [5]. The following general definition is given:

$$O_l^k = \gamma_0^l(l, m)Q_0 + \sum_{\mu=1}^{j} [\gamma_\mu^k(l, m)Q_{+\mu} + (-1)^{j+k}\gamma_\mu^k(l, -m)Q_{-\mu}], \tag{1}$$

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where $\mu = 0, \ldots, j, \quad k \geqslant 0,$

$$\gamma_{\mu}^{k}(l,m) = \langle j, \mu, l, -\mu - m \mid l + k, -m \rangle \times \left\{ \frac{(j+k)!(j-k)!(2l+j+k+1)!(l-m-\mu)!(l+m+k)!(l-m+k)!}{(2l+2k+1)(2j)!(2l-j+k)!(l+m+\mu)![(l-m)!]^2} \right\}^{1/2} \tag{2}$$

$$Q_{\pm\mu} = T(j, \mp\mu) l_{\pm}^{\mu} \tag{3}$$

and where $\langle j_1 m_1 j_2 m_2 \mid jm \rangle$ represents the well-known Clebsch–Gordan coefficient [9]. In [6] we have applied Hughes' general theory to the orthogonal group SO(5), whose symmetric representations play an important role in the classification of the nuclear quadrupole-phonon states. The SO(5) group generators can be expressed in terms of boson creation ($b_{2,\mu}^{+}$) and annihilation ($(-1)^{\mu} b_{2,-\mu}$) operators of the quadrupole-phonon states as follows:

$$l_0 = \sqrt{10}(b_2^{+} b_2)_0^1, \qquad l_{\pm} = \mp 2\sqrt{5}(b_2^{+} b_2)_{\pm 1}^1,$$
$$T(j=3,\mu) = q_{\mu} = (b_2^{+} b_2)_{\mu}^3, \text{ with } \mu = -3, \ldots, 3. \tag{4}$$

By this definition these generators fulfill the following commutator relations:

$$[l_{\pm}, q_{\mu}] = [(3 \mp \mu)(3 \pm \mu + 1)]^{1/2} q_{\mu \pm 1},$$
$$[l_0, q_{\mu}] = \mu q_{\mu}.$$

The Lie algebra SO(5) possesses two Casimir operators, of second and fourth degree in the generators. The quadratic one provides us with the seniority quantum number v. For the symmetric irreducible representations of interest the quartic invariant is not linearly independent of the quadratic one and hence of no use for the labeling of degenerated states. The O_l^0 shift operator however, can be used as an additional labeling operator for these representations. For that purpose, its eigenvalue $\alpha_{v,l}$ with respect to the $|v, l, m\rangle$ quadrupole-phonon states of seniority v, angular momentum l and projection m, is important. Closed formulae for $l = 2v$, $2v-2$, $2v-3$, $2v-4$, $2v-5$ and $2v-7$ have been derived by a technique which makes use of relations between shift operators and their products [2], and by a self-consistent single step algorithm starting from the highest angular momentum state [3]

$$\alpha_{v,2v} = \frac{2\sqrt{2}}{5} v(v+1)(2v+1)(4v+3) \quad (v > 1),$$

$$\alpha_{v,2v-2} = \frac{2\sqrt{2}}{5} v(2v-1)(4v^2 - 5v - 14) \quad (v > 2),$$

$$\alpha_{v,2v-3} = \frac{2\sqrt{2}}{5} (4v-3)(2v^3 - v^2 - 17v + 1) \quad (v > 3),$$

$$\alpha_{v,2v-4} = \frac{2\sqrt{2}}{5} (v-1)(8v^3 - 38v^2 - v - 60) \quad (v > 4),$$

$$\alpha_{v,2v-5} = \frac{2\sqrt{2}}{5}(8v^4 - 42v^3 - 77v^2 + 258v - 150) \quad (v > 5),$$

$$\alpha_{v,2v-7} = \frac{2\sqrt{2}}{5}(8v^4 - 74v^3 + 7v^2 + 209v + 15) \quad (v > 6). \tag{5}$$

Note that the only nondegenerate states occur for the mentioned l-values when v is large enough. Using the explicit expression (1) for O_l^0, applying the Wigher–Eckart theorem and introducing the analytical expression for the occurring 3-j symbol [9], it is easy to verify that the eigenvalue $\alpha_{v,l}$ is proportional to the reduced matrix element $\langle vl\|q\|vl\rangle$ of the tensor q, namely

$$\alpha_{v,l} = \langle v,l,m \mid O_l^0 \mid v,l,m\rangle = \frac{8}{\sqrt{5}}\langle v,l\|q\|v,l\rangle B_{30}(l) \times \frac{l(l+1)(l-1)(l+2)(2l-1)(2l+3)}{\sqrt{2l+1}}, \tag{6}$$

where $B_{jk}(l)$ is defined, as in [9], by:

$$B_{jk}(l) = \left[\frac{(2l+j+k+1)^{(2j+1)}}{2l+2k+1}\right]^{-1/2}$$

with $U^{(x)} = U(U-1)\cdots(U-x+1)$, the lowering factorial.

2. Determination of $\langle vl\|q\|vl\rangle$

There exist several techniques for the evaluation of an expression for the reduced matrix element in (6). We make use of the $SO(5) \downarrow SU(2) \otimes SU(2)$ reduction mechanism of Williams and Pursey [8]. These authors introduce functions $\psi(v,\nu,l,m)$, which span the entire representation space of the irreducible symmetric representation of SO(5) with seniority v, also denoted as the $(v,0)$ representation of SO(5):

$$\psi(v,\nu,l,m) = \int D_{m,K}^{l*}(\Omega)\chi_\omega(v,\nu)\,\mathrm{d}\Omega, \tag{7}$$

where the label ν is introduced to resolve the degeneracy of l. It takes the values

$$\nu = 0,1,2,\ldots,\lfloor v/3\rfloor \tag{8}$$

where $\lfloor v/3\rfloor$ denotes the integral part of $v/3$. The $D_{m,K}^{l}(\Omega)$ is a rotation matrix, while $\chi(v,\nu)$ denotes the "intrinsic states", which form a small subset of the natural basis functions. Williams and Pursey [8] clearly prove that K, the l-projection of the intrinsic states, can take the values

$$K = v - 3\nu, \tag{9}$$

where

$$l = 2K, 2K-2, 2K-3, \ldots, K. \tag{10}$$

The functions $\psi(v,\nu,l,m)$ defined by (7) are not normalized, and if two of them differ only by the value of ν, they are not orthogonal. For the purpose of normalisation one defines the inner product

$$A_l^v(\nu',\nu) = \langle\psi(v,\nu',l,m), \psi(v,\nu,l,m)\rangle. \tag{11}$$

When $v' = v$, $A_l^{\upsilon}(v, v)$ is the square of the normalization constant and we have adopted the convention of taking the positive square root. For $v' \neq v$ (11) is the overlap integral for states of common l but different v belonging to the representation $(\upsilon, 0)$. Williams and Pursey [8] derived several analytical expressions for the $A_l^{\upsilon}(v', v)$ of which we require the following:

$$A_l^{\upsilon}(v', v) = \frac{2^{v'-v}}{(2l+1)(K-K')!}\left[\frac{(\upsilon-v)!(\upsilon-v')!v!v'!(l-K')!(l+K)!}{(l+K')!(l-K)!}\right]^{1/2}$$
$$\times \sum_{\beta} \frac{(-4)^{v-\beta}}{(\upsilon-v-v'+\beta)!(v'-\beta)!(v-\beta)!\beta!}$$
$$\times \int_0^1 \mathrm{d}z(1-z)^{2(\upsilon-v-v'+\beta)} z^{3(v'-\beta)}(1-4z)^{\upsilon-v-v'+\beta}$$
$$\times {}_2F_1(K-l, K+l+1; K-K'+1; z). \tag{12}$$

The quantity is evaluated under the condition $v' \geqslant v$. If v' is smaller than v, (12) still applies, but v' and v have to be interchanged. Williams and Pursey [8] also presented an expression for the reduced matrix elements of q, with respect to the basis functions (7). A corrected form of this result was given in [7] and reads:

$$\langle \upsilon, v', l' \| q \| \upsilon, v, l\rangle = \left[\frac{(2l+1)}{10}\right]^{1/2}$$
$$\times \Bigg(\left[-5v(\upsilon - v + 1)^{1/2}\langle lK33 \mid l'K+3\rangle - \left[\frac{5}{3(\upsilon - v+1)} vl'(l'+1)\right]^{1/2} \right.$$
$$\left. \langle l'K+31-1 \mid l'K+2\rangle\langle lK32 \mid l'K+2\rangle \right] A_{l'}^{\upsilon}(v', v-1)$$
$$+[5(\upsilon - v)(v+1)]^{1/2}\langle lK3-3 \mid l'K-3\rangle A_{l'}^{\upsilon}(v', v+1)$$
$$+\Bigg\{ \left[\frac{3}{2} l'(l'+1)\right]^{1/2} \langle l'K11 \mid l'K+1\rangle\langle lK31 \mid l'K+1\rangle$$
$$-\left[\frac{2}{3} l'(l'+1)\right]^{1/2} \langle l'K1-1 \mid l'K-1\rangle\langle lK3-1 \mid l'K-1\rangle$$
$$-(2\upsilon - v)\langle lK30 \mid l'K\rangle \Bigg\} A_{l'}^{\upsilon}(v', v) \Bigg), \tag{13}$$

where $\langle j_1 m_1 j_2 m_2 \mid jm\rangle$ denotes again a Clebsch–Gordan coefficient. From the conditions (8)–(10) one finds for large enough υ-values that all states with $l = 2\upsilon - k$, ($k = 0, 2, 3, 4, 5$ and 7) exist and are nondegenerate, that all states with $l' = 2\upsilon - k'$, ($k' = 6, 8, 9, 10, 11$ and 13) exist and are doubly degenerate, etc. In the following chapter the class of nondegenerate states are studied and compared to the known results mentioned in (5). As a by-product of this study a special relation between ${}_2F_1$'s emerges.

3. The special case of $v = 0$

For the case $v=0$, which implies due to (9), $K=v$, it follows that the normalized wave functions for the occurring nondegenerate states with angular momentum $l=2v-k$ ($k=0,2,3,4,5,$ and 7) can be defined as

$$|v,l,m\rangle = [A_l^v(0,0)]^{-1/2}\psi(v,0,l,m) \tag{14}$$

and the resulting reduced matrix element of q reads

$$\begin{aligned}\langle v,l\|q\|v,l\rangle &= \frac{1}{A_l^v(0,0)}\langle v,0,l\|q\|v,0,l\rangle\\ &= \left[\frac{(2l+1)}{10}\right]^{1/2}\left(\sqrt{5v}\langle lv3-3\mid lv-3\rangle\frac{A_l^v(0,1)}{A_l^v(0,0)}\right.\\ &\quad+\left[\frac{3}{2}\,l(l+1)\right]^{1/2}\langle lv11\mid lv+1\rangle\langle lv31\mid lv+1\rangle\\ &\quad-\left[\frac{2}{3}\,l(l+1)\right]^{1/2}\langle lv1-1\mid lv-1\rangle\langle lv3-1\mid lv-1\rangle\\ &\quad\left.-2v\langle lv30\mid lv\rangle\right).\end{aligned} \tag{15}$$

The required Clebsch–Gordan coefficients can all be written down analytically from [9]. The expressions for the $A_l^v(v',v)$ directly follow from (12). For $A_l^v(v',v)$ one finds

$$A_l^v(0,0) = \frac{1}{2l+1}\int_0^1 \mathrm{d}z(1-z)^{2v}(1-4z)^v\,{}_2F_1(v-l,v+l+1;1;z). \tag{16}$$

By using properties of the hypergeometric functions (see [1]) it is easy to show that for $v \leqslant l$

$$(1-z)^{2v}\,{}_2F_1(v-l,v+l+1;1;z) = \frac{1}{(l-v)!}\frac{\mathrm{d}^{l-v}}{\mathrm{d}z^{l-v}}\left[z^{l-v}(1-z)^{l+v}\right]$$

and that

$$\begin{aligned}&\int_0^1 (1-v)^{2v}(1-4z)^v\,{}_2F_1(v-l,v+l+1;1;z)\,\mathrm{d}z\\ &\quad=\frac{1}{(l-v)!}\int_0^1 (1-4z)^v\frac{\mathrm{d}^{l-v}}{\mathrm{d}z^{l-v}}\left[z^{l-v}(1-z)^{l+v}\right]\,\mathrm{d}z.\end{aligned}$$

By applying $(l-v)$-times partial integration under the restriction $v \leqslant l$ the occurring integral transforms to

$$A_l^v(0,0) = \frac{4^{l-v}}{2l+1}\,\frac{v!}{(l-v)!(2v-l)!}\int_0^1 (1-4z)^{2v-l}z^{l-v}(1-z)^{l+v}\,\mathrm{d}z. \tag{17}$$

On the right-hand side one recognizes an integral representation of the hypergeometric function, so that $A_l^v(0,0)$ finally takes the form

$$A_l^v(0,0) = \frac{4^{l-v}}{2l+1}\,\frac{v!(l+v)!}{(2l+1)!(2v-l)!}\,{}_2F_1(l-2v,l-v+1;2l+2;4). \tag{18}$$

One can easily verify that $A_l^v(0,0)$ becomes identically zero for $l=2v-1$ and $l>2v$, which shows that the $|v,l,m\rangle$ states with these l-values do not exist. The determination of $A_l^v(0,1)=A_l^v(1,0)$ progresses along the same lines as was the case for the $A_l^v(0,0)$. From (9) with $K=v$ and $K'=v-3$, one obtains

$$A_l^v(1,0)=\frac{2}{3!(2l+1)}\left[\frac{v(l-v+3)!(l+v)!}{(l+v-3)!(l-v)!}\right]^{1/2}$$
$$\times\int_0^1 \mathrm{d}z(1-z)^{2(v-1)}z^3(1-4v)^{v-1}\,{}_2F_1(v-l,v+l+1;4;z).$$

By using again properties of hypergeometric functions [9], one can show that for $l\leqslant v$:

$$z^3(1-z)^{2v-3}\,{}_2F_1(v-l,v+l+1;4;z)=\frac{3!}{(l-v+3)!}\frac{\mathrm{d}^{l-v}}{\mathrm{d}z^{l-v}}\left[z^{l-v+3}(1-z)^{l+v-3}\right],$$

so that

$$\int_0^1 \mathrm{d}z(1-z)^{2v-2}z^3(1-4z)^{v-1}\,{}_2F_1(v-l,v+l+1;4;z)$$
$$=\frac{3!}{(l-v+3)!}\int_0^1 \mathrm{d}z(1-z)(1-4z)^{v-1}\frac{\mathrm{d}^{l-v}}{\mathrm{d}z^{l-v}}\left[z^{l-v+3}(1-z)^{l+v-3}\right]$$
$$=\frac{3!}{(l-v+3)!4}\int_0^1 \mathrm{d}z[(1-4z)^v+3(1-4z)^{v-1}]\frac{\mathrm{d}^{l-v}}{\mathrm{d}z^{l-v}}\left[z^{l-v+3}(1-z)^{l+v-3}\right]$$

which after applying $(l-v)$-times partial integration, reduces to

$$\frac{3!4^{l-v-1}(v-1)!(l+v-3)!}{(2v-l)!(2l+1)!}[v\,{}_2F_1(l-2v,l-v+4;2l+2;4)$$
$$+3(2v-l)\,{}_2F_1(l-2v+1,l-v+4;2l+2;4)].$$

Hence

$$A_l^v(1,0)=\frac{2\,4^{l-v-1}}{(2l+1)^2}\,\frac{(v-1)!(v+l-3)!}{(2l)!(2v-l)!}\left[\frac{v(l-v+3)!(l+v)!}{(l+v-3)!(l-v)!}\right]^{1/2}$$
$$[v_2F_1(l-2v,l-v+4;2l+2;4)$$
$$+3(2v-1)\,{}_2F_1(l-2v+1,l-v+4;2l+2;4)]. \tag{19}$$

Remark. Expressions (17) for $A_l^v(0,0)$ and (19) for $A_l^v(1,0)$ reduce each to a polynomial form in v for $l=2v-k$, $k=0,2,3,4,5,7$, such that in the first instance one can expect that the reduced matrix elements of q (and equivalently also the expressions for the corresponding eigenvalues) are rational forms in v due to the factor $(A_l^v(0,1)/A_l^v(0,0))$ in (15). However, the results reported in (5) demonstrate that these reduced matrix elements are polynomials in v and therefore one can conjecture that

$$v\,{}_2F_1(l-2v,l-v+4;2l+2;4)+3(2v-l)\,{}_2F_1(l-2v+1,l-v+4;2l+2;4)$$

is proportional to ${}_2F_1(l-2v,l-v+1;2l+2;4)$ for the considered l-values.

4. Verification of the relation between three specific $_2F_1(a, b; 2l + 2; 4)$ hypergeometric functions

Introducing in (6) the analytical expressions for the required Clebsch–Gordan coefficients [9] one obtains the following results:

$$\alpha_{v,l} = \frac{1}{10\sqrt{2}} 5(l-v+3)^{(3)} X(v,l)$$
$$-(l(l+1))^2 + 2l(l+1)[15v^2+15v+1] - 5v(v+1)(9v^2+v+2)] \quad (20)$$

with

$$X(v,l) = \frac{1}{{}_2F_1(l-2v, l-v+1; 2l+2; 4)} \times [v\,{}_2F_1(l-2v, l-v+4; 2l+2; 4) + 3(2v-l)\,{}_2F_1(l-2v+1, l-v+4; 2l+2; 4)].$$

For the $l = 2v - k$, $(k = 0, 2, 3, 4, 5, 7)$ we give in the following table the $X(v, l)$ as calculated using (5) and (20).

l	$2v$	$2v-2$	$2v-3$	$2v-4$	$2v-5$	$2v-7$
$X(v,l)$	v	$v-10$	$v+17$	$v-20$	$v+7$	$v-3$

These results suggest that $X(v, l)$ can be writen as a linear form in l and v for the l-values considered. A careful analysis shows that this form is different for even l- and odd l-values, i.e. $5l - 9v$ and $5l - 9v + 32$, respectively. Taking this into account leads to a general relation between the three $_2F_1$'s involved

$$v\,{}_2F_1(l-2v, l-v+4; 2l+2; 4)$$
$$+3(2v-l)\,{}_2F_1(l-2v+1, l-v+4; 2l+2; 4)$$
$$=(5l-9v+16(1-(-1)^l))\,{}_2F_1(l-2v, l-v+1; 2l+2; 4) \quad (21)$$

for $2v - l = 0, 2, 3, 4, 5, 7$. The relation is also valid for $2v - l = 1$, since then both sides are 0. Physically this does not correspond to an acceptable wave function, since the norm of such a state is zero.

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