



Using Four Values for Computerized Reasoning

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1 Introduction

In [Be77a, Be77b] Belnap introduced a logic intended to deal in a useful way with inconsistent and incomplete information. This logic is based on a structure called *FOUR*, which has four truth values: the classical ones, t and f , and two new ones: \perp that intuitively denotes lack of information (no knowledge), and \top that indicates inconsistency (“over”-knowledge). Belnap gave quite convincing arguments why “the way a computer should think” should be based on these four value.

In this paper we further examine the logical role of *FOUR* and its adequacy for computerized reasoning. For that we incorporate Shoham’s approach [Sh87, Sh88] and consider four-valued logics that are *preferential*, i.e.: they are based on the idea that inferences should be taken not according to all models of a given theory, but only w.r.t. a subset of them, determined according to certain preference criteria. Here we shall particularly be interested in two main guidelines for making such preferences: (a) prefer models that assume as much consistency as possible, and (b) prefer models that assume a minimal amount of knowledge.

We consider the main properties of the resulting inference relations, and demonstrate their usefulness in some practical examples. We also show that the four-valued setting might be used to overcome some of the drawbacks of classical logic. For instance, it is possible to define within this setting a nonmonotonic inference relation that is the same as the classical logic on consistent theories, but allows nontrivial reasoning with inconsistent data.¹

2 The language and its four-valued semantics

The truth values of Belnap’s logic mentioned above have two natural orderings: One, \leq_t , intuitively reflects differences in the “measure of truth” that every value represents. According to this order, f is the minimal element, t is the maximal one, and \perp, \top are two intermediate values that are incomparable. $(\{t, f, \top, \perp\}, \leq_t)$ is a distributive lattice with an order reversing involution \neg_t for which $\neg_t \top = \perp$ and $\neg_t \perp = \top$. We shall denote the meet and the join of this lattice by \wedge and \vee , respectively. The other partial order, \leq_k , is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. Again, $(\{t, f, \top, \perp\}, \leq_k)$ is a lattice where \perp is its minimal element, \top – the maximal element, and t, f are incomparable.

¹Due to a lack of space some of the proofs of the propositions are omitted or shortened. They will be provided in the full version of this paper.

Following Fitting [Fi89] we shall denote the meet and the join of the \leq_k -lattice by \otimes and \oplus , respectively.

The two lattice orderings are closely related. The knowledge operators \otimes and \oplus are monotone w.r.t. the truth ordering \leq_t , and the truth operators \wedge , \vee , and \neg (as well, of course, as \otimes and \oplus) are monotone w.r.t. \leq_k . Moreover, all the 12 distributive laws hold, as well as De-Morgan's laws. The structure that consists of these four elements and the five basic operators $(\wedge, \vee, \neg, \otimes, \oplus)$ is usually called *FOUR*.

The next step in using *FOUR* for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{t, \top\}$, since both values intuitively represent formulae known to be true. The set \mathcal{D} has the property that $a \wedge b \in \mathcal{D}$ iff $a \otimes b \in \mathcal{D}$ iff both a and b are in \mathcal{D} , while $a \vee b \in \mathcal{D}$ iff $a \oplus b \in \mathcal{D}$ iff either a or b is in \mathcal{D} . From this point the various semantic notions are defined on *FOUR* as natural generalizations of similar classical notions: A *valuation* M is a function that assigns a truth value from *FOUR* to each atomic formula. Any valuation is extended to complex formulae in the obvious way. We will sometimes write $\psi : b \in M$ instead of $M(\psi) = b$. A valuation M *satisfies* ψ iff $M(\psi) \in \mathcal{D}$. A valuation that satisfies every formula in a given set Γ of formulae is a *model* of Γ . The set of all models of Γ is denoted $\text{mod}(\Gamma)$. The structure *FOUR* together with \mathcal{D} as the set of the designated elements will be denoted in the sequel by $\langle \text{FOUR} \rangle$.

Unlike in the classical calculus, Belnap's logic has no tautologies. Thus, excluded middle is not valid in it. This implies that the definition of the material implication $p \mapsto q$ as $\neg p \vee q$ is not adequate there for representing entailments. We introduce instead the following implication operation on *FOUR*:

Definition 2.1 [Av91, AA94] $a \supset b = b$ if $a \in \mathcal{D}$, $a \supset b = t$ otherwise.²

Note that on $\{t, f\}$ the material implication (\mapsto) and the new implication (\supset) are identical, so \supset is a generalizations of the classical implication.

Example 2.2 (Tweety dilemma) Consider the following well-known puzzle:

$$\Gamma = \{ \text{bird}(\text{Tweety}) \mapsto \text{fly}(\text{Tweety}), \text{penguin}(\text{Tweety}) \supset \text{bird}(\text{Tweety}), \\ \text{penguin}(\text{Tweety}) \supset \neg \text{fly}(\text{Tweety}), \text{bird}(\text{Tweety}), \text{penguin}(\text{Tweety}) \}.$$

The first assertion of Γ is formulated by a weaker "implication" than the other two, since it is an instance of a rule that has exceptions. The six four-valued models of Γ are given in Figure 1.

Model No.	bird(Tweety)	fly(Tweety)	penguin(Tweety)
M1 – M2	\top	\top	\top, t
M3 – M4	\top	f	\top, t
M5 – M6	t	\top	\top, t

Figure 1: The models of Γ

²The sense in which \supset is a true implication will be clarified in Proposition 3.3 below.

Unlike the other connectives discussed above, \supset is *not* monotone w.r.t. \leq_k . In the sequel we distinguish between the k -monotomic languages and the nonmonotonic ones, and between languages in which all the connectives have two-valued interpretations and languages that contain \leq_k -connectives as well.

Notation 2.3

L_{scl}	$= \{\neg, \wedge, \vee, \supset\}$	(the strict classical language)
L_{mcl}	$= \{\neg, \wedge, \vee, t, f\}$	(the monotonic classical language)
L_{cl}	$= \{\neg, \wedge, \vee, \supset, t, f\}$	(the classical language)
L_{mon}	$= \{\neg, \wedge, \vee, \otimes, \oplus, t, f, \top, \perp\}$	(the monotonic language)
L_{full}	$= \{\neg, \wedge, \vee, \otimes, \oplus, \supset, t, f, \top, \perp\}$	(the full language)

Notation 2.4 $\mathcal{A}(\Gamma)$ = the set of atomic formulae that appear in some formula of Γ . $\mathcal{L}(\Gamma)$ = the set of literals that appear in some formula of Γ .

We conclude this section by examining the expressive power of the language we introduced above. Theorem 2.6 below provides a strong evidence for its adequacy:

Definition 2.5 An operation $g : \text{FOUR}^n \rightarrow \text{FOUR}$ is *represented* by a formula ψ s.t. $\mathcal{A}(\psi) \subseteq \{p_1, \dots, p_n\}$, if for every valuation M we have $M(\psi) = g(M(p_1), \dots, M(p_n))$.

Theorem 2.6 The language L_{full} is functionally complete for FOUR (i.e.: every function from FOUR^n to FOUR is representable by some formula in L_{full}).

Note: The ten connectives of L_{full} are not independent. Obviously, $t = \neg f$, $a \vee b = \neg(\neg a \wedge \neg b)$. Also, $\top = (a \supset a) \oplus \neg(a \supset a)$, $a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$, $\perp = f \otimes \neg f$, $f = \neg(\perp \supset \perp)$, and $a \otimes b = (a \wedge \perp) \vee (b \wedge \perp) \vee (a \wedge b)$. Using these identities and Theorem 2.6, it follows that the language of $\{\neg, \wedge, \supset, \perp, \top\}$, e.g., is functionally complete for FOUR .

Our next theorem characterize the expressive power of the various fragments of the full language which include at least the connectives of L_{scl} . It turns out that there is a correspondence between these fragments and the combinations of the following conditions:

$$\text{[I]} \ g(\vec{\top}) = \top \quad \text{[II]} \ g(\vec{x}) = \top \Rightarrow \exists 1 \leq i \leq n \ x_i = \top \quad \text{[III]} \ g(\vec{x}) = \perp \Rightarrow \exists 1 \leq i \leq n \ x_i = \perp$$

Theorem 2.7 Let $g : \text{FOUR}^n \rightarrow \text{FOUR}$. Then:

1. g is representable in $\{\neg, \wedge, \supset\}$ iff it satisfies I, II, and III.
2. g is representable in $\{\neg, \wedge, \supset, f\}$ iff it satisfies II and III.
3. g is representable in $\{\neg, \wedge, \supset, \oplus\}$ iff it satisfies I and III.
4. g is representable in $\{\neg, \wedge, \supset, \otimes\}$ iff it satisfies I and II.
5. g is representable in $\{\neg, \wedge, \supset, \otimes, f\}$ iff it satisfies II.
6. g is representable in $\{\neg, \wedge, \supset, \oplus, \otimes\}$ iff it satisfies I.
7. g is representable in $\{\neg, \wedge, \supset, \oplus, f\}$ iff it satisfies III.
8. g is representable in $\{\neg, \wedge, \supset, \oplus, \otimes, f\}$.

3 Reasoning in *FOUR*

3.1 The basic consequence relation

We start with the simplest consequence relation which naturally corresponds to *FOUR*.

Definition 3.1 $\Gamma \models^4 \Delta$ if every model of Γ in $\langle \text{FOUR} \rangle$ satisfies some formula of Δ .

Example 3.2 Consider again the set Γ of Example 2.2. Although Γ is classically inconsistent, nontrivial conclusions about Tweety can be obtained by \models^4 : Tweety is a penguin, a bird, and it cannot fly. The complementary conclusions *cannot* be obtained by \models^4 , as expected.

Note: With respect to L_{mcl} , \models^4 is identical to the set of “first degree entailments” in relevance logic (see [AB75, Du86]). The exact connection is that $\psi_1, \dots, \psi_n \models^4 \phi_1, \dots, \phi_m$ iff $\psi_1 \wedge \dots \wedge \psi_n \rightarrow \phi_1 \vee \dots \vee \phi_m$ is a first degree entailment.

Proposition 3.3 [AA96] $\Gamma, \psi \models^4 \phi, \Delta$ iff $\Gamma \models^4 \psi \supset \phi, \Delta$.

Proposition 3.4 [AA96] \models^4 is monotonic, compact, and paraconsistent ([dC74]).

The consequence relation \models^4 has also a corresponding sound and complete cut-free Gentzen-type proof system. See [AA94, AA96] for a detailed discussion of \models^4 and this system.

As we have seen, \models^4 has a lot of nice properties. Still, it has some serious drawbacks as well: It is too restrictive and “overcautious”. Thus it is strictly weaker than classical logic even for *consistent* theories (a case in which one might prefer to use classical logic). Moreover, it *totally* rejects some very useful (and intuitively justified) inference rules, like the Disjunctive Syllogism: From $\neg p$ and $p \vee q$ one can *never* infer q by using \models^4 .

3.2 Taking advantage of the other partial order

A natural approach for reducing the set of models which are used for drawing conclusions is to consider only the k -minimal ones. The idea behind this approach is that one should not assume anything that is not *really* known.

Definition 3.5 Let M_1, M_2 be two four-valued valuations, and Γ – a set of formulae.

- a) M_1 is k -smaller than M_2 ($M_1 \leq_k M_2$) if for every atomic p , $M_1(p) \leq_k M_2(p)$.
- b) M is a k -minimal model of Γ if M is a \leq_k -minimal element of $\text{mod}(\Gamma)$.

Definition 3.6 $\Gamma \models_k^4 \Delta$ iff every k -minimal model of Γ in $\langle \text{FOUR} \rangle$ satisfies some $\delta \in \Delta$.

Note: Obviously, if $\Gamma \models^4 \Delta$ then $\Gamma \models_k^4 \Delta$.

Example 3.7 Consider again Examples 2.2 and 3.2. Among the six models of Γ two are k -minimal: M_4 and M_6 (see Figure 1). Using these models we reach the same conclusions as in \models^4 :

$$\begin{aligned} \Gamma \models_k^4 \text{bird}(\text{Tweety}), & \quad \Gamma \models_k^4 \text{penguin}(\text{Tweety}), & \quad \Gamma \models_k^4 \neg \text{fly}(\text{Tweety}), \\ \Gamma \not\models_k^4 \neg \text{bird}(\text{Tweety}), & \quad \Gamma \not\models_k^4 \neg \text{penguin}(\text{Tweety}), & \quad \Gamma \not\models_k^4 \text{fly}(\text{Tweety}). \end{aligned}$$

The fact that in the last example we reached the same conclusions (at least with respect to the literals) as in \models^4 is not accidental. It is an instance of the following general proposition:

Proposition 3.8 If Δ does not include \supset , then $\Gamma \models^4 \Delta$ iff $\Gamma \models_k^4 \Delta$.

Sketch of proof: The “only if” direction is trivial. For the other direction, suppose that $\Gamma \models_k^4 \Delta$, and let M be some model of Γ . Using Zorn lemma and the fact that *FOUR* has a finite number of elements it can be shown that there is a k -minimal model N of Γ s.t. $M \geq_k N$. Thus there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{D}$. Since all the operators that correspond to the connectives of Δ are monotone w.r.t. \leq_k , $M(\delta) \geq_k N(\delta)$. But \mathcal{D} is upwards-closed w.r.t. \leq_k , therefore $M(\delta) \in \mathcal{D}$ as well. \square

Corollary 3.9 The logics \models^4 and \models_k^4 are identical w.r.t. L_{mon} .

Proposition 3.8 shows that as long as we are interested in inferring formulae that do not include \supset , we can indeed limit ourselves to k -minimal models without any loss of generality. Example 3.7 shows that this approach may lead to a considerable reduction in the number of models that should be checked.

The situation is completely different when we do allow the implication connective to appear on the right-hand side of \models_k^4 :

Example 3.10 In Example 2.2, $\Gamma \models_k^4 \neg \text{penguin}(\text{Tweety}) \supset f$, although $\Gamma \not\models^4 \neg \text{penguin}(\text{Tweety}) \supset f$.³ It follows that in the *full* language $\models_k^4 \neq \models^4$. This can be strengthened as follows:

Proposition 3.11 \models_k^4 is nonmonotonic.

Proof: $q \models_k^4 \neg q \supset p$, since $\{p: \perp, q: t\}$ is the only k -minimal model of q . On the other hand, $q, \neg q \not\models_k^4 \neg q \supset p$, since $\{p: \perp, q: \top\}$ is the only k -minimal model of $\{q, \neg q\}$. \square

Proposition 3.12 \models_k^4 is a *plausibility logic* in the sense of Lehmann ([Le92]). I.e., it satisfies *inclusion* ($\Gamma, \psi \models_k^4 \psi$), *right monotonicity* (if $\Gamma \models_k^4 \Delta$, then $\Gamma \models_k^4 \psi, \Delta$), *cautious left monotonicity* (if $\Gamma \models_k^4 \psi$ and $\Gamma \models_k^4 \Delta$, then $\Gamma, \psi \models_k^4 \Delta$)⁴, and *cautious cut* (if $\Gamma, \psi \models_k^4 \Delta$ and $\Gamma \models_k^4 \psi, \Delta$, then $\Gamma \models_k^4 \Delta$).

3.3 A consequence relation for preferring consistency

Recall that the basic idea in taking the k -minimal models was to avoid meaningless (or redundant) information. A “by-product” of this approach is a reduction in the level of inconsistency of our set of assumptions; When we assume less, the tendency of getting into conflicts decreases. In what follows we shall use a more direct approach of preserving consistency: Given a (possibly inconsistent) theory Γ , the idea is to give precedence to those models of Γ that minimize the amount of inconsistent beliefs in Γ .

Notation 3.13 Let M be a four-valued valuation. Denote: $\mathcal{I}_1 = \{\top\}$ and let $I(M, \mathcal{I}_1) = \{p \mid p \text{ is atomic and } M(p) \in \mathcal{I}_1\}$.

Intuitively, \mathcal{I}_1 is the set of inconsistent values of $\langle \text{FOUR} \rangle$, and $I(M, \mathcal{I}_1)$ is the inconsistent assignments of M w.r.t. \mathcal{I}_1 .

³The meaning of $\psi \supset f$ is that ψ *cannot* be true. This, of course, is stronger than saying that ψ is not a theorem, or even that $\neg \psi$ is a consequence of the assumptions.

⁴This rule was first proposed in [Ga85]

Definition 3.14 Let Γ be a set of formulae, and M, N — models of Γ .

a) M is *more consistent* than N w.r.t. \mathcal{I}_1 ($M >_{\mathcal{I}_1} N$) if $I(M, \mathcal{I}_1) \subset I(N, \mathcal{I}_1)$.

b) M is a *most consistent* model of Γ w.r.t. \mathcal{I}_1 (\mathcal{I}_1 -mcm, in short), if there is no other model of Γ which is more consistent than M w.r.t. \mathcal{I}_1 . The set of all the \mathcal{I}_1 -mcms of Γ is denoted $mcm(\Gamma, \mathcal{I}_1)$.

Definition 3.15 $\Gamma \models_{\mathcal{I}_1}^4 \Delta$ if every \mathcal{I}_1 -mcm of Γ satisfies some formula of Δ .

Example 3.16 Consider again Examples 2.2, 3.2, 3.7, and 3.10. Denote by Γ' the knowledge-base before Tweety is known to be a penguin, i.e.:

$$\Gamma' = \{ \text{bird}(\text{Tweety}) \multimap \text{fly}(\text{Tweety}), \text{penguin}(\text{Tweety}) \supset \text{bird}(\text{Tweety}), \\ \text{penguin}(\text{Tweety}) \supset \neg \text{fly}(\text{Tweety}), \text{bird}(\text{Tweety}) \}.$$

Γ' has 18 models altogether. They are listed in Figure 2.

Model No.	bird(Tweety)	fly(Tweety)	penguin(Tweety)
M1 – M8	\top	\top, f	\top, t, f, \perp
M9 – M12	\top	t, \perp	f, \perp
M13 – M16	t	\top	\top, t, f, \perp
M17 – M18	t	t	f, \perp

Figure 2: The models of Γ'

Here $mcm(\Gamma', \mathcal{I}_1) = \{M17, M18\}$. Thus, using $\models_{\mathcal{I}_1}^4$ one can infer that $\text{bird}(\text{Tweety})$ (but $\neg \text{bird}(\text{Tweety})$ is not true), and $\text{fly}(\text{Tweety})$ (while $\neg \text{fly}(\text{Tweety})$ is not true). Also, nothing is yet known about Tweety being a penguin.

Suppose now that a new data arrives: $\text{penguin}(\text{Tweety})$. The models of the modified knowledge-base, Γ , are listed in Figure 1. The mcms of Γ w.r.t. \mathcal{I}_1 are denoted there by M4 and M6. Therefore, according to the new information one should alter his beliefs and infer the intuitive conclusions, that $\text{bird}(\text{Tweety})$, $\text{penguin}(\text{Tweety})$, and $\neg \text{fly}(\text{Tweety})$. The complements of these assertions cannot be inferred by $\models_{\mathcal{I}_1}^4$, as one expects.

Proposition 3.17 $\models_{\mathcal{I}_1}^4$ is: (a) paraconsistent, (b) nonmonotonic.

Proof: For part (a) note that $p, \neg p \not\models_{\mathcal{I}_1}^4 q$. A countermodel assigns \top to p and f to q . For part (b) consider, e.g., $\Gamma = \{p, \neg p \vee \neg q\}$. Then $\Gamma \models_{\mathcal{I}_1}^4 \neg q$ while $\Gamma \cup \{q\} \not\models_{\mathcal{I}_1}^4 \neg q$. \square

Proposition 3.18 $\models_{\mathcal{I}_1}^4$ is a plausibility logic (see Proposition 3.12).

As we have already noted, one of the advantages of $\models_{\mathcal{I}_1}^4$ w.r.t. \models^4 is that the set of models needed for drawing conclusions from the formers is never bigger than that of the latter. In the following proposition we consider cases in which it is possible to reduce the amount of the relevant models even further, without changing the logic:

Proposition 3.19 Suppose that the formulae of Δ are in the language L_{mon} (i.e., without \supset). Then $\Gamma \models_{\mathcal{I}_1}^4 \Delta$ iff every k -minimal element of $mcm(\Gamma, \mathcal{I}_1)$ is a model of some $\delta \in \Delta$.⁵

⁵This result is a generalization of [AA98, Theorem 4.3] to the case that Γ is infinite and may contain implications.

Note: Proposition 3.19 is no longer true when \supset occurs in the conclusions. For a counter-example consider, e.g., $\Gamma = \{p, p \vee q\}$. The k -minimal element of $mcm(\Gamma, \mathcal{I}_1)$ assigns t to p and \perp to q , therefore $q \supset \neg q$ is true in it. However, $p, p \vee q \not\models_{\mathcal{I}_1}^4 q \supset \neg q$.

We conclude this subsection by comparing $\models_{\mathcal{I}_1}^4$ with \models^4 , \models_k^4 , and with the two-valued consequence relation:

Proposition 3.20

- a) If $\Gamma \models^4 \Delta$ then $\Gamma \models_{\mathcal{I}_1}^4 \Delta$.
- b) If $\Gamma \models_k^4 \Delta$ then $\Gamma \models_{\mathcal{I}_1}^4 \Delta$, provided that the formulae of Δ do not contain \supset .
- c) $\models_{\mathcal{I}_1}^4 \neq \models^4$ and $\models_{\mathcal{I}_1}^4 \neq \models_k^4$.

Proof: Part (a) is immediate from the definition of $\models_{\mathcal{I}_1}^4$. Part (b) follows from part (a) and Proposition 3.8. Finally, part (c) follows from Proposition 3.17(b) and its proof, since both \models^4 and \models_k^4 are monotonic w.r.t. the language of $\{\neg, \vee\}$. \square

Proposition 3.21 If Γ, ψ are in the classical language L_{cl} and $\Gamma \models_{\mathcal{I}_1}^4 \psi$, then ψ classically follows from Γ .

In the *monotonic* classical language L_{mcl} we have a partial converse:

Proposition 3.22 Let Γ be a classically consistent set in L_{mcl} . Suppose also that ψ is a formula in CNF, none of its conjuncts is a tautology.⁶ Then ψ classically follows from Γ iff $\Gamma \models_{\mathcal{I}_1}^4 \psi$.

Proposition 3.22 together with Proposition 3.19 entail that for checking whether a formula in L_{mcl} classically follows from a consistent set Γ , it is sufficient to perform the following steps: (a) convert the formula to a conjunctive normal form, (b) drop all the conjuncts which are tautologies, and (c) check the remaining formula only w.r.t. the k -minimal \mathcal{I}_1 -mcms of Γ .⁷

3.4 A consequence relation for preferring classical assignments

The approach presented in this subsection is similar to that of the previous one. The difference is that this time we prefer definite knowledge to an uncertain one. In particular, the approach taken here prefers classical inferences whenever their use is possible.

Notation 3.23 Let M be a four-valued valuation. Denote $\mathcal{I}_2 = \{\top, \perp\}$ and let $I(M, \mathcal{I}_2) = \{p \mid p \text{ is atomic and } M(p) \in \mathcal{I}_2\}$.

This time \mathcal{I}_2 is the set of the nonclassical values of *FOUR*, and $I(M, \mathcal{I}_2)$ corresponds to the nonclassical assignments of the valuation M .

Definition 3.24 Let Γ be a set of formulae, and M, N – models of Γ .

- a) M is *more consistent* than N w.r.t. \mathcal{I}_2 ($M >_{\mathcal{I}_2} N$) if $I(M, \mathcal{I}_2) \subset I(N, \mathcal{I}_2)$.
- b) M is a *most consistent* model of Γ w.r.t. \mathcal{I}_2 (\mathcal{I}_2 -mcm, in short), if there is no other model of Γ which is more consistent than M w.r.t. \mathcal{I}_2 . The \mathcal{I}_2 -mcms of Γ are denoted by $mcm(\Gamma, \mathcal{I}_2)$.

⁶Classically, every formulae which is not a tautology is equivalent to some formula of this form.

⁷This process might be useful in case Γ is a *fixed* theory, but the check should be made for many different potential conclusions. Note that if Γ is consistent then the number of its k -minimal \mathcal{I}_1 -mcms is never greater than the number of its classical models, and is frequently smaller.

Definition 3.25 $\Gamma \models_{\mathcal{I}_2}^4 \Delta$ if every \mathcal{I}_2 -mcm of Γ satisfies some formula of Δ .

Example 3.26 Consider again Example 3.16 and Figure 2. When taking \mathcal{I}_2 as the set of the “inconsistent” values, M17 — the only classical model — is also the only \mathcal{I}_2 -mcm of Γ' . It follows that according to $\models_{\mathcal{I}_2}^4$ one can infer that $\text{bird}(\text{Tweety})$, $\text{fly}(\text{Tweety})$, and $\neg \text{penguin}(\text{Tweety})$. The inverse assertions are not true, as expected.

Now, let $\Gamma = \Gamma' \cup \{\text{penguin}(\text{Tweety})\}$. As in the case of $\models_{\mathcal{I}_1}^4$, $mcm(\Gamma, \mathcal{I}_2)$ consists of the valuations denoted $M4$ and $M6$ in Figure 1. The new conclusions are, therefore, $\text{bird}(\text{Tweety})$, $\text{penguin}(\text{Tweety})$, and $\neg \text{fly}(\text{Tweety})$. Again, the complements of these assertions cannot be inferred by $\models_{\mathcal{I}_2}^4$, as expected.

As in the case of $\models_{\mathcal{I}_1}^4$ we have the following propositions:

Proposition 3.27 $\models_{\mathcal{I}_2}^4$ is: (a) paraconsistent, (b) nonmonotonic.

Proposition 3.28 $\models_{\mathcal{I}_2}^4$ is a plausibility logic (see Proposition 3.12).

Proposition 3.29

- a) If $\Gamma \models^4 \Delta$ then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$.
- b) If $\Gamma \models_k^4 \Delta$ then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$, provided that the formulae of Δ do not contain \supset .
- c) $\models_{\mathcal{I}_2}^4 \neq \models^4$ and $\models_{\mathcal{I}_2}^4 \neq \models_k^4$.

As in the case of $\models_{\mathcal{I}_1}^4$, in certain cases it is possible to reduce the amount of models that have to be considered for making conclusions in $\models_{\mathcal{I}_2}^4$:

Proposition 3.30 Suppose that Γ is *finite*, and the formulae of Δ are in L_{mon} . Then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$ iff each k -minimal element of $mcm(\Gamma, \mathcal{I}_2)$ satisfies some $\delta \in \Delta$.

Notes:

(1) As in Proposition 3.19, the condition about Δ is necessary in Proposition 3.30 as well: Consider, e.g., $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The k -minimal element of $mcm(\Gamma, \mathcal{I}_2)$ assigns \perp to p , and so $p \supset f$ is true in it. On the other hand, $\Gamma \not\models_{\mathcal{I}_2}^4 p \supset f$.

(2) The condition in Proposition 3.30 that Γ should be finite is also necessary: If Γ is not finite, it is *not always sufficient* to consider only the k -minimal elements of $mcm(\Gamma, \mathcal{I}_2)$ for inferring $\Gamma \models_{\mathcal{I}_2}^4 \Delta$, even if the formulae in Δ are all in the language without \supset . To see that, consider the following infinite set: $\Gamma = \{p_i \vee \neg p_i \supset p_{i+1} \wedge \neg p_{i+1} \mid i \geq 1\}$. It is easy to verify that $mcm(\Gamma, \mathcal{I}_2) = \{M_1^t, M_1^f, M_2^t, M_2^f, \dots\}$, where for every $j \geq 1$, M_j^t assigns \perp to $\{p_1, \dots, p_{j-1}\}$, t to p_j , and \top to $\{p_{j+1}, p_{j+2}, \dots\}$. M_j^f is the same valuation as M_j^t , except that p_j is assigned f instead of t . Therefore $\Gamma \not\models_{\mathcal{I}_2}^4 p_1$. On the other hand, $mcm(\Gamma, \mathcal{I}_2)$ has no k -minimal element (since for every $j \geq 1$, $M_{j+1}^t <_k M_j^t$ and $M_{j+1}^f <_k M_j^f$), therefore everything would have followed from this set (in particular p_1), had we used only the k -minimal \mathcal{I}_2 -mcms of Γ for drawing conclusions.

(3) In the monotonic classical language, Γ may be infinite:

Proposition 3.31 Suppose that the formulae of Γ are in L_{mcl} . Then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$ iff each k -minimal element of $mcm(\Gamma, \mathcal{I}_2)$ satisfies some $\delta \in \Delta$.

Proof: By Proposition 3.34 below, in this case every \mathcal{I}_2 -mcm of Γ is also k -minimal in $mcm(\Gamma, \mathcal{I}_2)$, and so the claim follows. \square

We conclude this subsection by comparing $\models_{\mathcal{I}_2}^4$ with the other inference relations considered here:

Proposition 3.32

- a) If $\Gamma \models^4 \Delta$ then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$.
- b) If $\Gamma \models_k^4 \Delta$ then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$, provided that the formulae of Δ do not contain \supset .
- c) $\models_{\mathcal{I}_2}^4 \neq \models^4$ and $\models_{\mathcal{I}_2}^4 \neq \models_k^4$.

Proof: The proof is the same as that of Proposition 3.20, using $\models_{\mathcal{I}_2}^4$ instead of $\models_{\mathcal{I}_1}^4$. \square

Proposition 3.33 Suppose that Γ, ψ are in L_{cl} .

- a) If $\Gamma \models_{\mathcal{I}_2}^4 \psi$, then ψ classically follows from Γ .
- b) Suppose that Γ is classically consistent. Then ψ classically follows from Γ iff $\Gamma \models_{\mathcal{I}_2}^4 \psi$.

Proof: The proof of part (a) is the same as that Proposition 3.21. Part (b) follows from the fact that if Γ is classically consistent then the set of its classical models is the same as the set of the \mathcal{I}_2 -mcms of Γ in *FOUR*. \square

It follows that $\models_{\mathcal{I}_2}^4$ is a nonmonotonic consequence relation that is equivalent to classical logic on consistent theories, and is nontrivial w.r.t. inconsistent theories.

Another important observation is that relative to the monotonic classical language, $\models_{\mathcal{I}_2}^4$ is actually a three valued logic:

Proposition 3.34 Suppose that the formulae of Γ are in L_{mcl} and that M is an \mathcal{I}_2 -mcm of Γ . Then there is no formula ψ s.t. $M(\psi) = \perp$.

Proof: Since $\{t, f, \top\}$ is closed under \neg, \vee and \wedge , it is sufficient to show the proposition only for atomic formulae. Define a transformation $g : \text{FOUR} \rightarrow \{t, f, \top\}$ as follows: $g(\perp) = t$, $g(b) = b$ otherwise. Obviously, for every atom p , $g \circ M(p) \geq_k M(p)$. Since every connective in the language of Γ is k -monotone, $\forall \gamma \in \Gamma$ $g \circ M(\gamma) \geq_k M(\gamma)$. Now, \mathcal{D} is upward-closed w.r.t. \leq_k , and so $\forall \gamma \in \Gamma$ $g \circ M(\gamma) \in \mathcal{D}$. Thus $g \circ M$ is also a model of Γ . Since $g \circ M \geq_{\mathcal{I}_2} M$, necessarily $g \circ M = M$. \square

Finally, we compare $\models_{\mathcal{I}_1}^4$ and $\models_{\mathcal{I}_2}^4$: In general, neither of these consequence relations is stronger than the other. Consider, for instance, $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The only \mathcal{I}_1 -mcm of Γ assigns \perp to p , while this valuation as well as the one in which p is assigned \top are the \mathcal{I}_2 -mcms of Γ . Therefore, $\Gamma \models_{\mathcal{I}_1}^4 p \supset q$ while $\Gamma \not\models_{\mathcal{I}_2}^4 p \supset q$. On the other hand, $\models_{\mathcal{I}_2}^4 p \vee \neg p$ but $\not\models_{\mathcal{I}_1}^4 p \vee \neg p$.

Proposition 3.35 Suppose that $\mathcal{A}(\Gamma, \psi) = \{p_1, p_2, \dots\}$. Then $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \dots \models_{\mathcal{I}_1}^4 \psi$ iff $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \dots \models_{\mathcal{I}_2}^4 \psi$.

Proof: Denote: $\Gamma' = \Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \dots\}$. Then $\text{mcm}(\Gamma', \mathcal{I}_1) = \text{mcm}(\Gamma', \mathcal{I}_2)$, since each model of Γ' assigns to the formulae in $\mathcal{A}(\Gamma, \psi)$ values from $\{t, f, \top\}$. \square

Proposition 3.36 Let Γ, Δ, ψ be in L_{mcl} .

- a) If $\Gamma \models_{\mathcal{I}_1}^4 \Delta$ then $\Gamma \models_{\mathcal{I}_2}^4 \Delta$.
- b) If ψ is a CNF-formula, none of its conjuncts is a tautology, then $\Gamma \models_{\mathcal{I}_1}^4 \psi$ iff $\Gamma \models_{\mathcal{I}_2}^4 \psi$.

Outline of proof: Part (a) follows from the fact that in L_{mcl} every \mathcal{I}_2 -mcm of Γ is also an \mathcal{I}_1 -mcm of Γ . For showing part (b) it suffices to show the claim for a disjunction ψ of literals that does not contain an atomic formula and its negation. So assume that $\Gamma \not\models_{\mathcal{I}_1}^4 \psi$. Then there is an \mathcal{I}_1 -mcm

M of Γ s.t. $M(\psi) \notin \mathcal{D}$. Consider the valuation M' , defined as follows: $M'(p) = t$ if $M(p) = \perp$ for $p \notin \mathcal{L}(\psi)$; $M'(p) = f$ if $M(p) = \perp$ for $p \in \mathcal{L}(\psi)$; and $M'(p) = M(p)$ otherwise. M' is an \mathcal{I}_2 -mcm of Γ and $M'(\psi) \notin \mathcal{D}$, thus $\Gamma \not\models_{\mathcal{I}_2}^4 \psi$. \square

Note: The converse of part (a) of Proposition 3.36 is not true in general. For instance, $\models_{\mathcal{I}_2}^4 p \vee \neg p$ while $\not\models_{\mathcal{I}_1}^4 p \vee \neg p$.

3.5 An example: Four-valued diagnosis

The example below demonstrates a usage of the formalisms considered here in a practical problem: Figure 3 depicts a circuit that consists of six components: two and-gates $A1$ and $A2$, two xor-gates $X1$ and $X2$, and two or-gates $O1$ and $O2$. It shows also the results of an experiment which was done with this circuit. According to this experiment the circuit is faulty; the values of the output lines of $X2$ and $O1$ are not the expected ones. The third output line (that of $O2$) does have the expected value, although one of its inputs is not known. Our goal is to find some minimal set of components the collective failure of which can explain the observed malfunction.

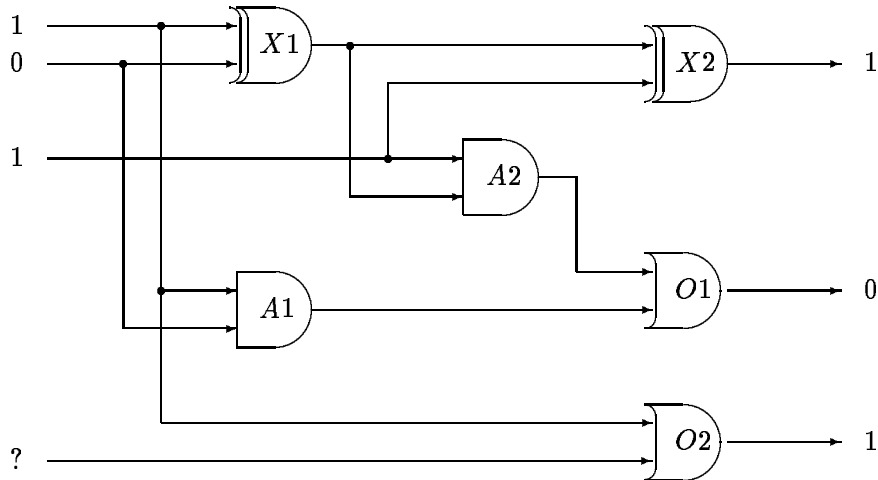


Figure 3: A faulty circuit

A description of this circuit, together with the results of the experiment, is given below. Here, every non-grounded formula represents its set of ground instances. We write $\psi \leftrightarrow \phi$ instead of $(\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi)$, where $\psi \rightarrow \phi = (\psi \supset \phi) \wedge (\neg \phi \supset \neg \psi)$. Note that in *FOUR*, $M(\psi \leftrightarrow \phi) \in D$ iff $M(\psi) = M(\phi)$. Also, $\square \psi$ abbreviates the formula $\psi \wedge (\neg \psi \supset f)$. Its intuitive meaning is that ψ is absolutely true. Since we know in advance the values of three input wires and of all the output wires, as well as the kind of each gate in the system, we attached this certainty operator (\square) to the corresponding predicates. The correct behavior of each gate, on the other hand, is only a default assumption. Therefore the predicate *ok* is not preceded by the \square -operator. Note that the resulting knowledge-base is classically inconsistent.

$$\text{andG}(x) \wedge \text{ok}(x) \mapsto (\text{out}(x) \leftrightarrow (\text{in1}(x) \wedge \text{in2}(x))),^8$$

⁸Recall that \mapsto denotes the material implication.

$\text{xorG}(\mathbf{x}) \wedge \text{ok}(\mathbf{x}) \mapsto (\text{out}(\mathbf{x}) \leftrightarrow (\text{in1}(\mathbf{x}) + \text{in2}(\mathbf{x}))),^9$
 $\text{orG}(\mathbf{x}) \wedge \text{ok}(\mathbf{x}) \mapsto (\text{out}(\mathbf{x}) \leftrightarrow (\text{in1}(\mathbf{x}) \vee \text{in2}(\mathbf{x}))),$
 $\neg(\text{andG}(\mathbf{x}) \wedge \text{orG}(\mathbf{x})) \wedge \neg(\text{andG}(\mathbf{x}) \wedge \text{xorG}(\mathbf{x})) \wedge \neg(\text{xorG}(\mathbf{x}) \wedge \text{orG}(\mathbf{x}))$
 $\text{in1}(X1) \leftrightarrow \text{in1}(A1), \text{in1}(X1) \leftrightarrow \text{in1}(O2), \text{in2}(X1) \leftrightarrow \text{in2}(A1),$
 $\text{in1}(A2) \leftrightarrow \text{in2}(X2), \text{out}(X1) \leftrightarrow \text{in2}(A2), \text{out}(X1) \leftrightarrow \text{in1}(X2),$
 $\text{out}(A1) \leftrightarrow \text{in2}(O1), \text{out}(A2) \leftrightarrow \text{in1}(O1),$
 $\Box\text{andG}(A1), \Box\text{andG}(A2), \Box\text{xorG}(X1), \Box\text{xorG}(X2), \Box\text{orG}(O1), \Box\text{orG}(O2),$
 $\Box\text{in1}(X1), \Box\neg\text{in2}(X1), \Box\text{in1}(A2), \Box\text{out}(X2), \Box\neg\text{out}(O1), \Box\text{out}(O2),$
 $\text{ok}(A1), \text{ok}(A2), \text{ok}(X1), \text{ok}(X2), \text{ok}(O1), \text{ok}(O2).$

Denote the above knowledge-base by Γ . In $\langle FOUR \rangle$, Γ has 232 models, but just three k -minimal ones (which in this case are also the \mathcal{I}_1 -mcms of Γ). The table of Figure 4 lists these models. We have omitted from it predicates that have the same value in all the models of Γ , and any predicate that always has the same value as some other predicate that already appears in the table.

Model No.	in2 A2	in1 O1	in2 O1	in2 O2	ok A1	ok A2	ok X1	ok X2	ok O1	ok O2
<i>M1</i>	<i>f</i>	<i>f</i>	<i>f</i>	\perp	<i>t</i>	<i>t</i>	\top	<i>t</i>	<i>t</i>	<i>t</i>
<i>M2</i>	<i>t</i>	<i>f</i>	<i>f</i>	\perp	<i>t</i>	\top	<i>t</i>	\top	<i>t</i>	<i>t</i>
<i>M3</i>	<i>t</i>	<i>t</i>	<i>f</i>	\perp	<i>t</i>	<i>t</i>	<i>t</i>	\top	\top	<i>t</i>

Figure 4: The k -minimal models of Γ

From the table it follows that:

$$\Gamma \models_k^4 \neg\text{ok}(X1) \vee (\neg\text{ok}(X2) \wedge \neg\text{ok}(A2)) \vee (\neg\text{ok}(X2) \wedge \neg\text{ok}(O1))$$

This exactly corresponds to the diagnoses for the possible causes of the malfunction of a similar (but simpler) circuit in [Re87, Example 2.2], [Gi88, Sections 15,16], and [AA98, Section 6.2].

4 Conclusion

In this work we have examined the applicability of the four-valued structure *FOUR* to computerized reasoning. After defining a language which is functionally complete for *FOUR*, we have considered several consequence relations and examined their main properties. We have seen that these relations have many desirable properties, which are particularly suitable for commonsense reasoning. We have shown that these relations might be used for overcoming some of the drawbacks of classical logic. Finally, we demonstrated the usefulness of the formalisms presented here through a practical example in the area of model-based diagnosis.

The outcome of this paper is, so we believe, a vindication of Belnap's thesis that "the way a computer should think" should be based on the four basic values.

⁹To avoid overloading, we use here $+$ (rather than \oplus) for the xor operation.

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