Uncertainty and dependence in classical and quantum logic — the role of triangular norms [1]

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Abstract

We distinguish three types of uncertainty: the probabilistic, the quantum and the fuzzy uncertainty. In these three contexts, we use different mathematical models to describe the event structures — the Boolean, the quantum and the fuzzy logics. We define states in a uniform way as valuations. Let us ask the following question: If a, b are two elements of a (Boolean, quantum, fuzzy, resp.) logic L and if m is a state on L, what "parameters" are necessary and sufficient to determine the values of m on the sublogic $L_{a,b}$ generated by a, b? In this paper we initiate the study of this question. Surprisingly, triangular norms (investigated usually in the area of fuzzy logic) play an important role here, too. It turns out that the types of uncertainty correspond to types of dependence between events.

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1 Three types of uncertainty

In the statistical description of real systems, we often encounter the following three types of uncertainty.

1. Probabilistic uncertainty. This uncertainty occurs in considerations affected by unknown circumstances.

2. Quantum uncertainty. This type of uncertainty occurs when the observations of the system cause irreversible changes of states. A typical example is a quantum experiment.

3. Fuzzy uncertainty. This uncertainty occurs when we study sets of events whose truth values are not necessarily only *true* or *false*.

Various mathematical structures have been invented and pursued in the study of systems with these three types of uncertainties. In this note, we take up the Boolean logics for the probabilistic uncertainties, the quantum logics for quantum uncertainties, and the fuzzy logics for fuzzy uncertainties. Obviously, the quantum logics and the fuzzy logics are both generalizations of Boolean logics. A common generalization of the quantum and fuzzy logics seems to be desirable and, to our

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knowledge, does not seem to be available in a straightforward way [4, 11, 19]. We raise the question of dependence/independence of two events in the respective logics. The three above-mentioned types of uncertainty correspond to three types of dependence in these structures.

2 Preliminaries — triangular norms

Triangular norms are usually viewed as fuzzy generalizations of the Boolean conjunction. However, they were studied in the early sixties in the area of probabilistic metric spaces (see [22]), and some of them even earlier. Let us recall their basic properties relevant to our considerations.

A *t-norm* (triangular norm) is an operation $T : [0,1]^2 \to [0,1]$ which is commutative, associative, monotone in each component, and which satisfies the boundary condition $T(1,u) = u$ (see e. g. [3, 22]). The function $\neg : u \mapsto 1-u$ is the standard fuzzy negation. The *dual t-conorm* to T is the operation $S : [0,1]^2 \to [0,1]$ defined by the (de Morgan) formula $S(u,v) = \neg T(\neg u, \neg v) = 1 - T(1-u, 1-v)$.

The *Frank family of t-norms* T_s, $s \in [0,\infty]$, will play an essential role in the sequel. For $s \in (0,\infty) \setminus \{1\}$, the Frank t-norms are defined by the formula

$$T_s : (u,v) \mapsto \log_s \left(1 + \frac{(s^u-1)(s^v-1)}{s-1}\right).$$

The limit cases coincide with the most frequently used t-norms:

$$T_0 = T_{\mathbf{M}} : (u,v) \mapsto \min(u,v) \quad \text{(minimum t-norm)},$$

$$T_\infty = T_{\mathbf{Ł}} : (u,v) \mapsto \max(u+v-1,0) \quad \text{(Łukasiewicz t-norm)},$$

$$T_1 = T_{\mathbf{P}} : (u,v) \mapsto u \cdot v \quad \text{(product t-norm)}.$$

The following property, characteristic for the Frank t-norms, was proved in [7].

Theorem 2.1 : *Let T be a t-norm and S its dual t-conorm. The equality*

$$T(u,v) + S(u,v) = u + v$$

is satisfied for all $u,v \in [0,1]$ iff T belongs to the Frank family of t-norms.

We shall need also the following property of Frank t-norms.

Theorem 2.2 : *For each $s \in [0,\infty]$ and each $u,v \in [0,1]$, the following equality holds:*

$$T_{1/s}(1-u,v) = v - T_s(u,v).$$

Remark 2.3 : We also take into account the limit case $\{s, 1/s\} = \{0,\infty\}$. The latter theorem gives also one of the implications in Th. 2.1 as an easy consequence:

$$T_s(u,v) = v - T_{1/s}(1-u,v) = v - (1-u) + T_s(1-u,1-v) = v + u - S_s(u,v),$$

where S_s is the dual t-conorm to T_s.

PROOF: The cases of $s \in \{0,1,\infty\}$ are trivial. In the remaining cases, we obtain:

$$T_{1/s}(1-u,v) = \log_{1/s}\left(1 + \frac{(s^{-(1-u)}-1)(s^{-v}-1)}{s^{-1}-1}\right) =$$

$$= -\log_s\left(1 - \frac{s(s^{u-1}-1)(s^{-v}-1)}{s-1}\right) = -\log_s \frac{s-1-s^{u-v}+s^{1-v}+s^u-s}{s-1} =$$

$$= -\log_s \left(s^{-v} \cdot \frac{-s^v - s^u + s + s^{u+v}}{s-1} \right) = v - \log_s \frac{-s^v - s^u + s + s^{u+v}}{s-1} =$$

$$= v - \log_s \left(1 + \frac{(s^u - 1)(s^v - 1)}{s-1} \right) = v - T_s(u,v)$$

□

The following property of the Frank family of t-norms was proved by Frank (see [7]).

Theorem 2.4 : *For fixed $u, v \in (0,1)$, the function*

$$F_{u,v} : s \mapsto T_s(u,v)$$

is continuous and strictly decreasing. It is a bijection of $[0,\infty]$ and $[T_{\mathbf{Ł}}(u,v), T_{\mathbf{M}}(u,v)]$.

3 Uncertainty in classical logic

In the classical logic, the event structure is described by a Boolean algebra L. (Countable operations will not be of interest in this paper because we will only deal with finite subalgebras.)

A *state* on L is a mapping $m : L \to [0,1]$ such that:

(s1) $m(0) = 0, \quad m(1) = 1$

(s2) $\forall x, y \in L : \; m(x \vee y) + m(x \wedge y) = m(x) + m(y)$ (the valuation property).

Let $a, b \in L$. The (Boolean) subalgebra $L_{a,b}$ of L generated by a, b is a homomorphic image of the free Boolean algebra F_{BA} with 2 free generators (F_{BA} is isomorphic to 2^4). In order to describe the state space of $L_{a,b}$, it is sufficient to characterize the state space of F_{BA}. We denote by x, y the free generators of F_{BA}.

Let m be a state on F_{BA}. An easy calculation shows that m is uniquely determined by its values on x, y, and $x \wedge y$. While $m(x)$, $m(y)$ can be arbitrary numbers from $[0,1]$, the value $m(x \wedge y)$ is constrained by some inequalities.

Proposition 3.1 : *Let F_{BA} be the free Boolean algebra with 2 free generators x, y.*

1. *If m is a state on F_{BA}, then*

$$T_{\mathbf{Ł}}(m(x), m(y)) \leq m(x \wedge y) \leq T_{\mathbf{M}}(m(x), m(y)).$$

2. *If $u, v, w \in [0,1]$ such that*

$$T_{\mathbf{Ł}}(u,v) \leq w \leq T_{\mathbf{M}}(u,v),$$

then there is a state m on F_{BA} such that $m(x) = u$, $m(y) = v$, $m(x \wedge y) = w$.

PROOF: 1. As

$$(x \wedge y) \vee (x \wedge y') = x,$$

$$(x \wedge y) \wedge (x \wedge y') = 0,$$

we obtain

$$m(x \wedge y) + m(x \wedge y') = m(x) + m(0) = m(x)$$

and $m(x \wedge y) \leq m(x)$. Similarly, $m(x \wedge y) \leq m(y)$. Thus

$$m(x \wedge y) \leq \min(m(x), m(y)) = T_{\mathbf{M}}(m(x), m(y)).$$

As

$$m(x) + m(y) - m(x \wedge y) = m(x \vee y) \leq 1,$$

we get

$$m(x \wedge y) \geq m(x) + m(y) - 1.$$

This, together with $m(x \wedge y) \geq 0$, gives

$$m(x \wedge y) \geq T_{\mathbf{Ł}}(m(x), m(y)).$$

2. We define $m(x \wedge y) = w$, $m(x \wedge y') = u - w$, $m(x' \wedge y) = v - w$, $m(x' \wedge y') = 1 - u - v + w$, and extend m uniquely to a state on F_{BA}. □

We see that the minimum t-norm $T_{\mathbf{M}}$ gives the best upper estimate of $m(x \wedge y)$ corresponding to the case of maximal positive dependence between x and y. The Łukasiewicz t-norm $T_{\mathbf{Ł}}$ gives the best lower estimate corresponding to the maximal negative dependence. Also the product t-norm is found to play an important role: The events x, y are stochastically independent iff $m(x \wedge y) = m(x) \cdot m(y) = T_{\mathbf{P}}(m(x), m(y))$.

Suppose that values $m(x), m(y)$ of a state m on F_{BA} are given. If $m(x) \in \{0,1\}$ or $m(y) \in \{0,1\}$, then m is uniquely determined. If $m(x), m(y) \in (0,1)$, there remains one degree of freedom. Obviously, the value $m(x \wedge y)$ is sufficient to determine the state m uniquely. However, it is reasonable to search for a parameter which attains its extreme values exactly at the bounds given by Prop. 3.1 and which changes its sign when an event is replaced with its complement. Let us suggest the following notion.

Definition 3.2 : A *degree of probabilistic dependence*, p, is a mapping which assigns a value $p_m(x,y)$ to each state m and events x, y such that $m(x), m(y) \in (0,1)$, and which is subject to the following axioms:

(p1) $p_m(x,y) = p_m(y,x)$,

(p2) $p_m(x',y) = -p_m(x,y)$,

(p3) if $m(x_1) = m(x_2)$, $m(y_1) = m(y_2)$ and $m(x_1 \wedge y_1) < m(x_2 \wedge y_2)$, then $p_m(x_1,y_1) < p_m(x_2,y_2)$,

(p4) $p_m(x,y) = 0$ iff $m(x \wedge y) = T_{\mathbf{P}}(m(x), m(y))$,

(p5) $p_m(x,y) = 1$ iff $m(x \wedge y) = T_{\mathbf{M}}(m(x), m(y))$,

(p6) $p_m(x,y) = -1$ iff $m(x \wedge y) = T_{\mathbf{Ł}}(m(x), m(y))$.

Let us comment on the above notion. If $m(x) \in \{0,1\}$ or $m(y) \in \{0,1\}$, then $p_m(x,y)$ is undefined. The axioms express the following requirements: The value $p_m(x,y)$ is zero if and only if x, y are stochastically independent. For fixed $m(x), m(y) \in (0,1)$ and $m(x \wedge y)$ ranging in $[T_{\mathbf{Ł}}(m(x), m(y),\ T_{\mathbf{M}}(m(x), m(y)]$, the values $p_m(x,y)$ cover the whole interval $[-1,1]$. Moreover, $p_m(x,y)$ increases with the increase of $m(x \wedge y)$. If one of the arguments is replaced by its complement, the degree of probabilistic dependence changes its sign. The axiom (p2) has the following motivation: Let x_1, x_2, y_1, y_2 satisfy the assumption of (p3) and let

$$m(x_2 \wedge y_2) - m(x_1 \wedge y_1) = \varepsilon.$$

Then

$$m(x_2' \wedge y_2) - m(x_1' \wedge y_1) = -\varepsilon,$$

$$m(x_2 \wedge y_2') - m(x_1 \wedge y_1') = -\varepsilon,$$

$$m(x_2' \wedge y_2') - m(x_1' \wedge y_1') = \varepsilon.$$

Therefore an increase ε of the degree of probabilistic dependence of x, y should be considered as a decrease of the degree of probabilistic dependence of x', y. The axiom (p2) makes the role of x, x' "antisymmetric". As a consequence of (p2), we obtain a weaker condition

(p2–) $p_m(x', y') = p_m(x, y)$.

It seems plausible that these axioms for a degree of probabilistic dependence are well motivated and natural. Obviously, a question arises whether such a mapping always exists.

The most natural idea is to try a correlation-like parameter. The correlation of two random variables ξ, η on L is

$$corr(\xi, \eta) = \frac{E(\xi \cdot \eta) - E\xi \cdot E\eta}{\sqrt{E(\xi - E\xi)^2 \cdot E(\eta - E\eta)^2}},$$

where E denotes the mean value. If ξ, η attain only the values 0,1 and if we identify $E\xi, E\eta, E(\xi \cdot \eta)$ with $m(x), m(y), m(x \wedge y)$, resp., we obtain the formula

$$corr_m(x, y) = \frac{m(x \wedge y) - m(x) \cdot m(y)}{\sqrt{m(x) \cdot m(x') \cdot m(y) \cdot m(y')}}.$$

The parameter *corr* (taken for p) satisfies (p1)–(p4). However, (p5) and (p6) are not satisfied unless $m(x) = m(y)$. Therefore, a modified parameter is desirable.

Another attempt — a linear interpolation of the bounds from Prop. 3.1 — gives us the formula

$$lin_m(x, y) = \frac{2 \cdot m(x \wedge y) - T_{\mathbf{M}}(m(x), m(y)) - T_{\mathbf{Ł}}(m(x), m(y))}{T_{\mathbf{M}}(m(x), m(y)) - T_{\mathbf{Ł}}(m(x), m(y))}.$$

The parameter *lin* satisfies (p1)–(p3) and (p5)–(p6), but it does not satisfy (p4).

We see that a more involved formula for the degree of probabilistic dependence is needed. In view of the fact that the values of m on F_{BA} are uniquely determined by $m(x)$, $m(y)$ and $m(x \wedge y)$, we can express the intended degree of probabilistic dependence in the form

$$p_m(x, y) = f(m(x), m(y), m(x \wedge y)),$$

where f is a suitable function (called the *generating function* of p). The function f is defined for all triples of arguments u, v, w such that $u, v \in (0, 1)$ and $w \in [T_{\mathbf{Ł}}(u, v), T_{\mathbf{M}}(u, v)]$. The axioms (p1)–(p6) for a degree of probabilistic dependence p are then equivalent to the following conditions for f:

(f1) $f(u, v, w) = f(v, u, w)$,

(f2) $f(1 - u, v, v - w) = -f(u, v, w)$,

(f3) $w_1 < w_2 \Longrightarrow f(u, v, w_1) < f(u, v, w_2)$,

(f4) $f(u, v, T_{\mathbf{P}}(u, v)) = 0$,

(f5) $f(u, v, T_{\mathbf{M}}(u, v)) = 1$, (f6) $f(u,v,T_{\mathbf{L}}(u,v)) = -1$.

The property (p2–) corresponds to

(f2–) $f(1-u, 1-v, 1-u-v+w) = f(u,v,w)$.

Let us suggest to construct the degree of probabilistic dependence as follows: Let the contours of the degree of probabilistic dependence be the graphs of Frank t-norms. In other words, let $p_m(x,y)$ attain a constant value $g(s)$ (depending only on s but not on x, y, m) for all arguments satisfying $m(x \wedge y) = T_s(m(x), m(y))$. In terms of the generating function f of p, it is required that $f(u,v,T_s(u,v)) = g(s)$ for all $u, v \in (0,1)$. Using the inverse, $F_{u,v}^{-1}$, of the function $F_{u,v}$ from Th. 2.4, we can substitute $s = F_{u,v}^{-1}(w)$, $w = T_s(u,v)$. We obtain

(F) $f(u,v,w) = g(F_{u,v}^{-1}(w))$.

The following theorem tells us when the latter formula gives rise to (the generating function of) a degree of probabilistic dependence.

Theorem 3.3 : *The function f defined by* (F) *is a generating function of a degree of probabilistic dependence iff $g : [0,\infty] \to [-1,1]$ is a decreasing bijection such that*

(g2) $g(1/s) = -g(s)$

for all $s \in (0,\infty)$. The corresponding degree of probabilistic dependence, d, is defined by setting

$$d_m(x,y) = g(F_{m(x),m(y)}^{-1}(m(x \wedge y))).$$

Alternatively, $d_m(x,y) = g(s_{x,y})$, where $s_{x,y}$ is the unique element of $[0,\infty]$ such that $T_{s_{x,y}}(m(x), m(y)) = m(x \wedge y)$.

PROOF: 1. Let g be a function satisfying the conditions of the theorem and let f be defined by (F). We shall verify conditions (f1)–(f6).

Condition (f1) follows from the commutativity of T_s. As g is a decreasing bijection of $[0,\infty]$ onto $[-1,1]$, $g(0) = 1$ and $g(\infty) = -1$, which ensures (f5), (f6). Taking $s = 1$ in (g2), we obtain $g(1) = -g(1) = 0$, which implies (f4). The functions g, $F_{u,v}$ and $F_{u,v}^{-1}$ are strictly decreasing, so f being a composition of two of them is strictly increasing in its third argument — condition (f3) is verified.

It remains to check the validity of (f2), i. e., it remains to verify the equality

$$g(F_{1-u,v}^{-1}(v-w)) = -g(F_{u,v}^{-1}(w)).$$

In the equality from Th. 2.2, we substitute $w = T_s(u,v)$, $s = F_{u,v}^{-1}(w)$, and obtain

$$T_{1/s}(1-u,v) = v - w,$$

$$F_{1-u,v}^{-1}(v-w) = 1/s,$$

$$g(F_{1-u,v}^{-1}(v-w)) = g(1/s) = -g(s) = -g(F_{u,v}^{-1}(w)),$$

which we were to prove.

2. Let (F) define a generating function of a degree of probabilistic dependence. To prove that the properties of g are necessary, it is sufficient to compare them with the considerations from the first part of the proof. □

Remark 3.4 : Let $h : [0,1] \to [0,1]$ be a decreasing bijection. We may define $g : [0,\infty] \to [-1,1]$ by the formula

$$g(s) = \begin{cases} h(s) & \text{if} \quad s \leq 1, \\ -h(1/s) & \text{if} \quad s > 1. \end{cases}$$

Then g satisfies the conditions of Th. 3.3 and formula (F) defines a generating function of a degree of probabilistic dependence. Thus, Th. 3.3 gives a family of degrees of probabilistic dependence which differ by a nonlinear transformation of the range $[-1,1]$.

However, Th. 3.3 probably does not give all solutions of (p1)–(p6). (No other examples are known to us at this moment.) If a degree of probabilistic dependence is defined by means of (F), the Frank t-norms cannot be replaced by any other family of t-norms. Indeed, (f2–) implies

$$F^{-1}_{1-u,1-v}(1-u-v+w) = F^{-1}_{u,v}(w).$$

Denote this value by s. This means that

$$T_s(u,v) = w,$$

$$T_s(1-u,1-v) = 1-u-v+w.$$

Using the dual t-conorm, S_s, in the latter equality,

$$1 - S_s(u,v) = 1-u-v+w,$$

we obtain the formula

$$T_s(u,v) + S_s(u,v) = u + v.$$

This is characteristic for the Frank t-norms (Th. 2.1).

4 Uncertainty in quantum logics

In quantum logic, the phenomenon of noncompatibility requires a more general structure than Boolean algebras. Recall that two events are called *noncompatible* if they are not simultaneously observable. In the corresponding mathematical model, two events are noncompatible if they do not belong to a Boolean subalgebra of the event structure (for more detailed exposition, see [1, 9]).

The event structure of a quantum system is often supposed to be an *orthomodular lattice* (OML), L. In comparison with a Boolean algebra, in OMLs we relax the absorption laws

$$a \wedge (a' \vee b) = a \wedge b, \quad a \vee (a' \wedge b) = a \vee b$$

as well as the distributivity law. We replace them by the *orthomodular law*

$$a \leq b \Longrightarrow b = a \vee (a' \wedge b).$$

Formally, an OML is a bounded lattice L with a unary operation $' : L \to L$ (*orthocomplementation*) such that it satisfies the following conditions

1. $a'' = a$,
2. $a \leq b \Longrightarrow b' \leq a'$,
3. $a \vee a' = 1$

and the orthomodular law. A *state* on an OML can be defined by conditions (s1), (s2) in the same way as in Section 3. Thus, we say that $s : L \to [0,1]$ is a state if

(s1) $m(0) = 0, \quad m(1) = 1$

(s2) $\forall x, y \in L: \; m(x \vee y) + m(x \wedge y) = m(x) + m(y)$

(It should be noted that these "modular-like" states have been investigated by several authors, see e. g. [2, 14, 18, 20, 23], etc.).

We shall make use of free algebras. Let L be an OML and let $a, b \in L$. The subalgebra (sub-OML) $L_{a,b}$ of L generated by a, b is a homomorphic image of the free OML F_{OML} with 2 free generators. In order to describe the state space of $L_{a,b}$, it is sufficient to characterize the state space of F_{OML}. Let us denote by x, y the free generators of F_{OML}. The structure of F_{OML} is described in [1, 9]. The *commutator*,

$$c(x,y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y'),$$

plays an important role in the description. (In a Boolean algebra, the commutator is always zero.) It is known that F_{OML} is isomorphic to the direct product $B \times M$, where $B = \{z \in F_{\mathrm{OML}} : z \le c(x,y)'\}$ and $M = \{z \in F_{\mathrm{OML}} : z \le c(x,y)\}$. (The intervals B, M are given the natural OML structure inherited from L.) The factor B is a Boolean algebra isomorphic to 2^4, freely generated by $x \wedge c(x,y)'$, $y \wedge c(x,y)'$. The factor M is the modular ortholattice $\{0, c(x,y), x \wedge c(x,y), x' \wedge c(x,y), y \wedge c(x,y), y' \wedge c(x,y)\}$, known as MO2. (Its maximal element is $1_M = c(x,y)$.)

Let m be a state on F_{OML}. It is of the form

$$m(z) = (1 - q_m(x,y)) \cdot m_B(z \wedge c(x,y)') + q_m(x,y) \cdot m_M(z \wedge c(x,y))$$

for some $q_m(x,y) \in [0,1]$ and states m_B, m_M, on B, M, resp. As m_B is a classical state, m_M can be interpreted as a "purely non-Boolean state". A nonzero value of the commutator $c(x,y)$ indicates the presence of quantum effects in the observations of x and y, and $m(c(x,y))$ represents their probability. As the projections $x \wedge c(x,y), y \wedge c(x,y)$ (of x, y to M) have the same probability proportional to $m(c(x,y))$ (see the theorem below), we may consider the parameter $q_m(x,y) = m(c(x,y))$ as the *degree of quantum dependence* between x and y at state m.

Theorem 4.1 (see also [5, 10, 14, 21]): *The OML M (i. e. MO2) admits only one state, m_M. This state attains the value $1/2$ on all the elements $x \wedge c(x,y)$, $x' \wedge c(x,y)$, $y \wedge c(x,y)$, $y' \wedge c(x,y)$.*

PROOF: Let u, v be arbitrary two different elements of $\{x \wedge c(x,y), x' \wedge c(x,y), y \wedge c(x,y), y' \wedge c(x,y)\} = M \setminus \{0,1\}$. We apply the valuation property (s2) to u, v. As $u \vee v = 1_M$, $u \wedge v = 0$, we obtain

$$m(u) + m(v) = m(1_M) + m(0) = 1.$$

This equality for all combinations of $u, v \in M \setminus \{0,1\}$ implies that m attains $1/2$ on all these elements. □

Let us comment on the previous result. If we require the condition (s2) to be satisfied only for $x \le y'$, we obtain a weaker definition of a state. This definition is, however, often adopted in quantum mechanics (see [8, 17]). In the case of Boolean algebras, both definitions coincide. With this weaker definition, the values $m(x \wedge c(x,y))$, $m(y \wedge c(x,y))$, may be any numbers from the interval $[0, m(c(x,y))]$ and we obtain two more degrees of freedom for weaker states.

The state m_B is a classical state subject to the characterization in the preceding section. As a consequence, each state m on F_{OML} is uniquely determined by $m(x)$, $m(y)$, $m(x \wedge y)$ and $m(c(x,y))$. The parameter $m(c(x,y)) = q_m(x,y)$ is the degree of quantum dependence. Again, the parameter $m(x \wedge y)$ can be replaced by a more appropriate degree of probability dependence, $p_m(x,y)$, defined now as the degree of probability dependence of the classical state m_B. Explicitly, $p_m(x,y) = p_{m_B}(x \wedge c(x,y)', y \wedge c(x,y)')$.

5 Uncertainty in fuzzy logic — a perspective

Despite many studies in recent years, the notion of fuzzy logic does not seem to be standardized. Various mathematical structures — MV-algebras [12], T-tribes [3], fuzzy quantum spaces [6, 16], etc. — are suggested to represent events in a fuzzy logic. Also states on fuzzy logics — fuzzy measures — allow for different generalizations (some of them being analogous to the definition of Section 3). Therefore our basic question on degrees of dependence leads us to a series of questions on various definitions of fuzzy logics and states. What is typical for most of them is the absence of the law of contradiction and the excluded middle law. Thus

$$a \wedge a' = 0, \quad a \vee a' = 1$$

need not hold. (Even if these laws are satisfied in some mathematical models, all fuzzy models admit that there is a nonzero element a such that $a \leq a'$. This is, however, excluded in both classical and quantum logic.) For most of the definitions of fuzzy measure, m, the value $m(a \wedge a')$ may be nonzero. If this is the case, the values of m on a fuzzy sublogic generated by a, b depend on new degrees of freedom (expressed, e. g., by $m(a \wedge a')$ and $m(b \wedge b')$) which are typical for a fuzzy logic.

Among the mathematical structures describing fuzzy logic, only few are equational classes (e. g., MV-algebras). Hence not all fuzzy logics allow for the existence of a "free fuzzy logic with 2 free generators". Those fuzzy logics which do — like e. g. MV-algebras — could be treated analogously to our previous analysis. In general, the question of dependence of two events in a fuzzy logic needs a further study. It is conceivable that it may help to clarify some properties of various concepts of fuzzy logic. Also, a contribution to the investigation of fuzzy quantum logics, which are common generalizations of quantum and fuzzy logics — see [4, 11, 19], could be achieved.

6 Conclusion

In a classical (=Boolean) logic, we have defined a degree of probabilistic dependence between two events, a, b. It corresponds to one degree of freedom in the description of a state m with fixed values $m(a), m(b)$. A reasonable degree of dependence can be obtained using the Frank family of t-norms — a family of connectives which were typical for fuzzy logical considerations but were not applied yet in classical or quantum logics.

In a quantum logic, a new degree of freedom appears, corresponding to a degree of quantum dependence. Thus, two events in a quantum logic have two independent degrees of dependence — probabilistic and quantum.

In fuzzy logic, new degrees of dependence may appear. They seem to require another generalization of the degree of probabilistic dependence. This is largely open to further research.

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