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MULTIVALENTLY CLOSE-TO-CONVEX FUNCTIONS

TOSHIO UMEZAWA

1. Introduction. The class of close-to-convex functions in $|z| < 1$ introduced by Kaplan [1] is characterized by the property that there exists a function $\phi(z)$ analytic, schlicht and convex in $|z| < 1$ for which $R[f'(z)/\phi'(z)] > 0$, $|z| < 1$. Such functions are necessarily schlicht.

Let

$$(1.1) \quad f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be a function of this class. Then it has been pointed out by Reade [2] that the inequalities

$$(1.2) \quad |a_n| \leq n, \quad n = 2, 3, \dots$$

hold. This result extends (1.2) to a wider class than was previously considered [3].

In this paper we extend these results to the case of functions p -valent in the unit circle.

2. Preliminaries. Hereafter we consider mainly functions of the form

$$(2.1) \quad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \quad 1 \leq q \leq p.$$

DEFINITION 1. The function $\phi(z)$ of the form (2.1) is said to be an element of the class $C(p)$, p a positive integer, if it is regular in $|z| < 1$ and if there is a $\rho < 1$ such that for all r in the interval $\rho < r < 1$

$$(2.2) \quad G(r, \theta) = 1 + R \left(r e^{i\theta} \frac{\phi''(r e^{i\theta})}{\phi'(r e^{i\theta})} \right) > 0, \quad 0 \leq \theta \leq 2\pi,$$

and

$$(2.3) \quad \int_0^{2\pi} G(r, \theta) d\theta = 2\pi p.$$

This definition is due to Goodman [4]. It is shown in [4] that if $\phi(z) \in C(p)$, it maps $|z| \leq r < 1$ onto a locally convex region and that $\phi(z)$ is at most p -valent in $|z| < 1$.

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REMARK. Since

$$1 + R \frac{z\phi''(z)}{\phi'(z)} = \frac{d \arg d\phi(z)}{d \arg dz}$$

(2.2) is equivalent to

$$(2.4) \quad d \arg d\phi(z) > 0, \quad |z| = r < 1$$

and (2.3) means that $\phi(z)$ has $p - 1$ critical points in $|z| < 1$.

DEFINITION 2. The function $f(z)$ is said to be an element of the class $K(p)$, p a positive integer, if it is regular in $|z| < 1$ and if there exists a function $\phi(z)$ of the class $C(p)$ for which $R[f'(z)/\phi'(z)] > 0$, $|z| < 1$ holds.

Such functions will be called close-to-convex p -valent functions. We shall prove that such functions are at most p -valent in $|z| < 1$. For the purpose we need the following lemma due to the author [5].

LEMMA 1. Let $f(z)$ be regular for a simply connected closed domain D whose boundary Γ consists of a regular curve and $f'(z) \neq 0$ on Γ . Suppose that

$$(2.5) \quad \int_{\Gamma} d \arg df(z) = 2p\pi.$$

If we have for an arbitrary arc C on the boundary Γ

$$(2.6) \quad \int_C d \arg df(z) < (2p + 1)\pi$$

or

$$(2.7) \quad \int_C d \arg df(z) > -\pi,$$

then $f(z)$ is at most p -valent in D .

To handle the coefficient problem and the deformation problem for the class $K(p)$ we need the following lemma due to [4] and [6].

LEMMA 2. Let $\phi(z) \in C(p)$ of the form (2.1) have $p - q$ critical points $\beta_1, \beta_2, \dots, \beta_{p-q} \neq 0$ in $|z| < 1$. Then

$$(2.8) \quad \phi'(z) \ll qF(z)/z \quad \text{for } |z| < 1$$

and

$$(2.9) \quad |\phi'(re^{i\theta})| \leq qF(r)/r \quad \text{for } r < 1$$

where $F(z)$ is defined by

$$(2.10) \quad F(z) = \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q} \left(1 + \frac{z}{|\beta_j|}\right) (1 + z |\beta_j|).$$

3. The main theorems.

THEOREM 1. *If $f(z) \in K(p)$, then $f(z)$ is at most p -valent in $|z| < 1$.*

PROOF. By Definition 2 there exists a function $\phi(z)$ of the class $C(p)$ such that, for $|z| < 1$,

$$(3.1) \quad R[f'(z)/\phi'(z)] = R[df(z)/d\phi(z)] > 0.$$

Since $\phi'(z)$ has exactly $p-1$ roots in $|z| < 1$ by Definition 1, $f'(z)$ must have the same roots as $\phi'(z)$ by (3.1). Hence we have

$$(3.2) \quad \int_{|z|=\rho} d \arg df(z) = 2p\pi \quad \text{for } \rho < r < 1.$$

Since $df/d\phi \neq 0$ in $|z| < 1$ by (3.1), $\arg(df/d\phi)$ is one-valued in $|z| < 1$. On the other hand we have

$$\arg df = \arg \frac{df}{d\phi} + \arg d\phi.$$

Hence $\arg df$ is also one-valued if we take a suitable branch of $\arg d\phi(z)$.

By taking these branches and by the assumption (3.1), we have

$$-\pi/2 < \arg df(z_i) - \arg d\phi(z_i) < \pi/2$$

and

$$-\pi/2 < -\arg df(z_j) + \arg d(\phi z_j) < \pi/2$$

for arbitrary points $z_i, z_j, i > j$ on $|z| = r$, sufficiently near to 1. Hence we have

$$-\pi < \arg df(z_i) - \arg df(z_j) - (\arg d\phi(z_i) - \arg d\phi(z_j)) < \pi$$

where $2p\pi \geq \arg d\phi(z_i) - \arg d\phi(z_j) > 0$ since $\phi(z) \in C(p)$. Thus we have

$$-\pi < \arg df(z_i) - \arg df(z_j) < (2p + 1)\pi,$$

which is equivalent to

$$(3.3) \quad -\pi < \int_{z_j}^{z_i} d \arg df(z) < (2p + 1)\pi.$$

(3.2) and (3.3) show that $f(z)$ is at most p -valent in $|z| < 1$ by Lemma 1.

THEOREM 2. *Let*

$$(3.4) \quad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$$

be a function of the class $K(p)$. Suppose that in addition to the $(q-1)$ th order critical points at $z=0$ the function $f(z)$ has exactly $p-q$ critical points $\beta_1, \beta_2, \dots, \beta_{p-q}$ such that $0 < |\beta_j| < 1, j=1, 2, \dots, p-q$. Then

$$(3.5) \quad |a_n| \leq C_n, \quad n = q + 1, q + 2, \dots,$$

$$(3.6) \quad |f(re^{i\theta})| \leq F(r) \quad \text{for } r < 1$$

and

$$(3.7) \quad |f'(re^{i\theta})| \leq F'(r) \quad \text{for } r < 1$$

where C_n and $F(r)$ are defined by

$$(3.8) \quad \begin{aligned} F(z) &= q \int_0^z \frac{t^{q-1}(1+t)}{(1-t)^{2p+1}} \sum_{j=1}^{p-1} \left(1 + \frac{t}{|\beta_j|}\right) (1+t|\beta_j|) dt \\ &= z^q + \sum_{n=q+1}^{\infty} c_n z^n. \end{aligned}$$

PROOF. By our hypothesis, there exists a function of the class $C(p)$ which has the same form and the same critical points as $f(z)$ such that $R[f'(z)/\phi'(z)] > 0$ for $|z| < 1$. Hence by Carathéodory's theorem

$$\frac{f'(z)}{\phi'(z)} \ll \frac{1+z}{1-z}$$

and by Ströhäcker's theorem [7]

$$(3.9) \quad \frac{1-r}{1+r} \leq \left| \frac{f'(re^{i\theta})}{\phi'(re^{i\theta})} \right| \leq \frac{1+r}{1-r} \quad \text{for } r < 1.$$

On the other hand we have (2.8) and (2.9) for $\phi'(z)$. Accordingly we have (3.7) and

$$f'(z) \ll \frac{z^{q-1}(1+z)}{(1-z)^{2p+1}} \prod_{j=1}^{p-q} \left(1 + \frac{z}{|\beta_j|}\right) (1+z|\beta_j|).$$

Hence we have (3.5) and (3.6) with (3.8). q.e.d.

COROLLARY 1. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in K(p)$, then $f(z)$ is p -valent for $|z| < 1$ and the inequalities

$$(3.10) \quad |a_n| \leq \binom{n+p-1}{2p-1}, \quad n = p+1, p+2, \dots,$$

$$(3.11) \quad \frac{r^p}{(1+r)^{2p}} \leq |f(re^{i\theta})| \leq \frac{r^p}{(1-r)^{2p}} \quad \text{for } r < 1$$

and

$$(3.12) \quad \frac{pr^{p-1}(1-r)}{(1+r)^{2p+1}} \leq |f'(re^{i\theta})| \leq \frac{pr^{p-1}(1+r)}{(1-r)^{2p+1}} \quad \text{for } r < 1$$

hold.

PROOF. (3.10) and the upper bounds in (3.11) and (3.12) are obvious. Let us prove the lower bounds in them.

In this special case, for the corresponding convex p -valent function $\phi(z)$, we have

$$\frac{pr^{p-1}}{(1+r)^{2p}} \leq |\phi'(re^{i\theta})| \quad r < 1$$

since $z\phi'(z)$ is p -valently starlike in $|z| < 1$, [4]. Hence we have the lower bound for $|f'(z)|$ in (3.12) from (3.9).

Let w be one of the nearest points from the origin to the image curve of $|z| = r < 1$ under $f(z)$. Further let L be the curve on the z -plane corresponding to the line segment $0w$. Then we have

$$\begin{aligned} |f(z)| &= \int_L |f'(z)| |dz| \geq \int_0^{|z|} |f'(z)| d|z| \\ &\geq \int_0^{|z|} \frac{p|z|^{p-1}(1-|z|)}{(1+|z|)^{2p-1}} d|z| = \frac{|z|^p}{(1+|z|)^{2p}}, \end{aligned}$$

which gives the lower bound in (3.10).

But we remark that the inequalities (3.11) follow at once from the p -valency of $f(z)$ by the result due to W. K. Hayman [8].

COROLLARY 2. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be regular for $|z| < 1$. Suppose that $f(z)$ satisfies one of the following conditions:

- (1) $R[zf'(z)/f(z)] > 0$ for $|z| < 1$.
- (2) $R[f'(z)/z^{p-1}] > 0$ for $|z| < 1$.
- (3) $R[(1-z)^p f'(z)/z^{p-1}] > 0$ for $|z| < 1$.
- (4) $R[(1-z^p)f'(z)/z^{p-1}] > 0$ for $|z| < 1$.

- (5) $R[(1 - z)^{2p}f'(z)/z^{p-1}] > 0$ for $|z| < 1$.
 (6) $R[(1 - z^p)^2f'(z)/z^{p-1}] > 0$ for $|z| < 1$.
 (7) $R[(1 - z^2)^pf'(z)/z^{p-1}] > 0$ for $|z| < 1$.
 (8) $R[(1 - z^{2p})f'(z)/z^{p-1}] > 0$ for $|z| < 1$.

Then $f(z)$ is p -valent for $|z| < 1$, and (3.10), (3.11) and (3.12) hold.

PROOF. Under the condition (1), $\phi(z) = p \int_0^z (f(z)/z) dz \in C(p)$ and $R[f'(z)/\phi'(z)] = R[zf'(z)/f(z)]/p > 0$ for $|z| < 1$. Hence $f(z) \in K(p)$.

In the other cases, $f(z) \in K(p)$, since the following functions

$$(2) z^p, \quad (3) \int_0^z \frac{z^{p-1}}{(1-z)^p} dz, \quad (4) \int_0^z \frac{z^{p-1}}{1-z^p} dz, \quad (5) \int_0^z \frac{z^{p-1}}{(1-z)^{2p}} dz,$$

$$(6) \int_0^z \frac{z^{p-1}}{(1-z^p)^2} dz, \quad (7) \int_0^z \frac{z^{p-1}}{(1-z^2)^p} dz, \quad (8) \int_0^z \frac{z^{p-1}}{1-z^{2p}} dz$$

are easily seen to be p -valently convex in $|z| < 1$.

Hence our conclusion follows in every case by Corollary 1.

Here we note that in almost every case in Corollary 2 the inequalities can be sharpened considerably by direct use of Carathéodory's and Strohäcker's Theorems.

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