

Freiling's axioms of symmetry in a general setting and some applications.

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Abstract

We formulate C. Freiling's axioms of symmetry for general second-order structures with respect to a certain ideal of small sets contained in them and find several equivalent formulations of the principles. Then we focus on particular models, namely saturated and recursively saturated ones, and show that they are symmetric with respect to appropriate classes of small sets when their second-order part consists of definable sets. Some asymmetric models are also exhibited as well as partial asymmetric ones constructed by forcing.

1 Introduction

C. Freiling [1] has proposed certain axioms for the continuum of the real numbers intended to express the symmetric behavior of small subsets, like the countable ones, sets of cardinality less than the continuum, or sets of measure zero. For each such class we have corresponding symmetry axiom(s). Typical is the following axiom (concerning countable subsets):

$$(A_{\aleph_0}) \quad (\forall f : \mathbb{R} \rightarrow \mathbb{R}_{\aleph_0})(\exists x, y)(x \notin f(y) \ \& \ y \notin f(x)), \quad (1)$$

where \mathbb{R}_{\aleph_0} is the set of countable sets of reals. The intuition behind this principle is the following: Suppose we assign to each real number x a countable set of reals $f(x)$ (e.g. the rational multiples of x). Then if we throw two darts at \mathbb{R} , landing at x, y respectively, then the second dart will miss (with probability 1) the set $f(x)$. Then, by symmetry ("the real line does not know which dart is thrown first or second"), the first dart should also miss $f(y)$.

The nice thing is that A_{\aleph_0} is equivalent to $\neg\text{CH}$ over ZFC, the proof being short and elementary. Using n darts and generalizing A_{\aleph_0} to $A_{\aleph_0}^n$ one can prove in ZFC that $A_{\aleph_0}^n \iff 2^{\aleph_0} \geq \aleph_n$. Later G. Weiskamp [7] showed that ZFC proves $A_{\aleph_0}(\Sigma_1^1)$, i.e., A_{\aleph_0} with f being Σ_1^1 (analytic), as well as that $A_{\aleph_0}(\Sigma_2^1)$ and $A_{\aleph_0}(\Pi_1^1)$ are equivalent.

The collection of countable sets, *Count*, provides here a notion of *smallness* (with respect to \mathbb{R}). Other such notions (in the context of classical set theory) are the finite sets (*Fin*), the sets of cardinality $< 2^{\aleph_0}$, the zero measure sets (*Null*), and the meagre sets (*Meagre*). Some of them give rise to analogous principles. For example $A_{<2^{\aleph_0}} \Rightarrow \neg\text{AC}$, while, over $\text{ZFC}+2^{\aleph_0} = \aleph_2$, A_{null} is equivalent to the existence of a nonmeasurable set of cardinality less than 2^{\aleph_0} plus \mathbb{R} is not equal to the \aleph_1 -union of null sets. In nonstandard models of arithmetic a natural notion of smallness is provided by the class of *thin* sets $\text{Thin} = \{X : (\exists x)(X \subseteq \{(x)_n : n \in \mathbb{N}\})\}$, where \mathbb{N} represents the initial segment of standard natural numbers.

In this paper we formulate dart axioms in a very general setting and prove some simple facts clarifying their symmetric content. Then we apply them to saturated and recursively saturated models. The paper is organized as follows. In section 2 we formulate general axioms of symmetry for a second-order structure and prove certain equivalents of them. In section 3 we concentrate on saturated models and sets with small cardinality and show they are strongly symmetric with respect to the universe of infinitary-definable sets. In section 4 we consider countable recursively saturated models of PA and we take the small sets to be the *thin* sets, i.e., the subsets of coded sequences of length \mathbb{N} (the standard segment). These models too are shown to be symmetric in the universe of infinitary-definable subsets. In section 5 we exhibit some asymmetric models (with respect to the preceding notion of smallness), either definably asymmetric, or asymmetric in a generic extension of a definable universe. In section 6 we show that the hierarchy of axioms A_{thin}^n is of strictly increasing strength, by constructing generic models satisfying $\neg A_{thin}^{n+1} + A_{thin}^n$, for every n .

2 Symmetry axioms in a general setting.

Let M be an infinite first-order structure, and let \mathfrak{M} be a class of subsets of M , closed under some comprehension principle. (For example \mathfrak{M} is the set of definable subsets of M , with respect to some notion of definability.) We assume that the set \mathbb{N} of nonnegative integers, together with their usual operations, belongs to \mathfrak{M} . The fact that a set X has n elements is uniformly definable in \mathbb{N} and is denoted $|X| = n$. We set $M_n = \{X \subset M : |X| = n\}$ and $M_{fin} = \bigcup_{n \in \mathbb{N}} M_n$.

A *notion of smallness* is a definable class $Small \subset \mathfrak{M}$ with the following properties: (i) $Small$ is closed under finite unions and subsets, (ii) $M \notin Small$ and (iii) $\mathbb{N} \in Small$.

Then, clearly, $M_{fin} \subseteq Small$. If $X \in Small$ and $|X| \neq n$ for all $n \in \mathbb{N}$, we write $|X| = \infty$. Finally, putting $M_\infty = \{X \in Small : |X| = \infty\}$, we have $Small = M_{fin} \cup M_\infty$.

Henceforth all objects, referred to by lower-case letters x, y, a, b, \dots , are elements of M , and all sets, referred to by capital letters X, Y, A, \dots , are elements of \mathfrak{M} . Also truth of formulas refers to the model (M, \mathfrak{M}) . Letters m, n, i, j, \dots range over \mathbb{N} . Dart axioms in this setting are the principles A_{small}^n , $n = 2, 3, \dots, \infty$ defined below:

$$A_{small}^n \quad (\forall F : M_{n-1} \rightarrow Small)(\exists X \in M_n)(\forall x \in X)(x \notin F(X \setminus \{x\})).$$

In particular we write A_{small} instead of A_{small}^2 . This latter axiom can be written

$$A_{small} \quad (\forall F : M \rightarrow Small)(\exists x, y : x \neq y)(x \notin F(y) \ \& \ y \notin F(x)),$$

while

$$A_{small}^\infty \quad (\forall F : Small \rightarrow Small)(\exists X \in M_\infty)(\forall x \in X)(x \notin F(X \setminus \{x\})).$$

Note that A_{small}^n are Π_1^1 -formulas, so if $Small \subset \mathfrak{M} \subseteq \mathfrak{N}$ and $(M, \mathfrak{N}) \models A_{small}^n$, then $(M, \mathfrak{M}) \models A_{small}^n$. Therefore we are seeking families \mathfrak{M} as large as possible satisfying the preceding axioms, though it is natural to confine ourselves to families of definable sets, with respect to some class of formulas.

We shall first reformulate slightly the above axioms replacing functions by relations. This simple change will prove useful in grasping their symmetric content.

Definition 2.1 Let $n \in \mathbb{N}, n \geq 2$. By an n -ary relation we shall understand any relation $R \subseteq M \times M_{n-1}$. R is said to be ∞ -ary if $M \subseteq M \times M_\infty$. The n -ary

(∞ -ary) relation R is said to be *total* if for any $X \in M_n$ (resp. $X \in M_\infty$), there is a $x \in X$ such that $R(x, X \setminus \{x\})$. The relation R (either n -ary or ∞ -ary) is said to be *asymmetric* (with respect to *Small*), if for every X , $\{x : R(x, X)\} \in \text{Small}$.

Given an n -ary relation R we put for brevity

$$R^X = \{x : R(x, X)\}.$$

Note that if R is binary, identifying M with M_1 , we may assume that $R \subseteq M \times M$. R is total iff for every $x \neq y$ xRy or yRx , and is asymmetric iff for all y , $R^y \in \text{Small}$. The following gives a simple characterization of the dart axioms:

Proposition 2.2 For $n \in \mathbb{N} \cup \{\infty\}$, $n \geq 2$,

$$A_{\text{small}}^n \iff \text{there is no } n\text{-ary total asymmetric (t.a.) relation.}$$

Proof. Suppose $\neg A_{\text{small}}^n$ holds and F be a function satisfying it. For every $x \in M$ and every $X \in M_{n-1}$ let $R(x, X) := x \in F(X)$. Then R is n -ary and $R^X = F(X) \in \text{Small}$, hence R is asymmetric. Let $|X| = n$. By $\neg A_{\text{small}}^n$, there is $x \in X$ such that $x \in F(X \setminus \{x\})$, hence $R(x, X \setminus \{x\})$. Therefore R is total.

Conversely, if R is n -ary, total and asymmetric, putting $F(X) = R^X$ if $|X| = n - 1$ and $F(X) \in \text{Small}$ arbitrary, F satisfies $\neg A_{\text{small}}^n$. \square

Concerning the relative strength of A_{small}^n , we have the following:

Lemma 2.3 For every $n \in \mathbb{N}$, $n \geq 2$, $A_{\text{small}}^{n+1} \Rightarrow A_{\text{small}}^n$.

Proof. Suppose $\neg A_{\text{small}}^n$ holds and let R be an n -ary t.a. relation. Define the $(n + 1)$ -ary relation S by putting, for any $X \in M_n$, $S^X = \bigcup \{R^Y : Y \subset X \text{ \& } Y \in M_{n-1}\}$. Clearly $S^X \in \text{Small}$, hence S is asymmetric. To show that it is total, let $X \in M_{n+1}$ and let $Y \subset X$, $Y \in M_n$. By the totality of R , there is $x \in Y$ such that $x \in R^Y \setminus \{x\}$. Since $Y \setminus \{x\} \subset X \setminus \{x\}$, it follows that $x \in S^X \setminus \{x\}$. Hence S is $(n + 1)$ -ary t.a. and A_{small}^{n+1} fails. \square

One may guess that $A_{\text{small}}^\infty \Rightarrow A_{\text{small}}^n$ holds also. However for this implication, as well as for improving the characterizations for A_{small} , we need a further closure condition for *Small*. This condition resembles the σ -ideal closure property and for the case of the ideal M_{\aleph_0} it is identical to that. The condition says that the union of a small family of small sets is small. We denote it $\text{SC}(\text{Small})$ and say that *Small* is *self-closed*:

$$\text{SC}(\text{Small}): \quad (\forall X \in \text{Small})(\forall F : X \rightarrow \text{Small})(\cup F''X \in \text{Small}).$$

For example $\text{SC}(\text{Fin})$ is true, $\text{SC}(\text{Count})$ holds if we assume the countable axiom of choice (CAC), $\text{SC}(< 2^{\aleph_0})$ holds iff 2^{\aleph_0} is a regular cardinal, while $\text{SC}(\text{Null})$ and $\text{SC}(\text{Meagre})$ are false. For example if C is the Cantor set, $F : C \rightarrow \mathbb{R}$ is a bijection, and $G : \mathbb{R} \ni x \mapsto \{x\} \in \text{Null}$, then $C \in \text{Null}$, GF maps $C \in \text{Null}$ into Null but $\cup GF''C = \mathbb{R}$. In this paper we shall deal only with ideals satisfying SC. The following contains some consequences of $\text{SC}(\text{Small})$.

Lemma 2.4 Let $\text{SC}(\text{Small})$ hold. Then

- (i) *Small* is closed under subsets.
- (ii) For every function F with $\text{dom}(F) \in \text{Small}$, $\text{rang}(F) \in \text{Small}$.
- (iii) If $X \in \text{Small}$ and X_{fin} is the set of finite subsets of X (codable in a certain way in \mathfrak{M}), then $X_{\text{fin}} \in \text{Small}$.

Proof. (i) Let $X \in \text{Small}$ and $Y \subseteq X$, $Y \neq \emptyset$. Choose $x_0 \in Y$ and let $F(x) = \{x\}$ if $x \in Y$, and $F(x) = \{x_0\}$ if $x \notin Y$. Clearly $\cup F''X = Y$, hence Y is small.

(ii) $\text{range}(F) = \bigcup \{F(x) : x \in \text{dom}(F)\}$, and since each $\{F(x)\}$ is small, the claim is immediate by $\text{SC}(\text{Small})$.

(iii) To show that $X_{fin} \in \text{Small}$ observe that $X_{fin} = \bigcup_n X_n$, where X_n is the set of subsets of X of cardinality $\leq n$. So it suffices to show that each X_n is small. By induction on n . Clearly X_1 is small and let $X_n \in \text{Small}$. For every $x \in X$ let $F_x : X_n \rightarrow \text{Small}$ be the function such that $F_x(Y) = Y \cup \{x\}$. Then $X_{n+1} = \bigcup \{\text{range}(F_x) : x \in X\}$. By the hypothesis and (ii), each $\text{range}(F_x)$ is small and hence, by $\text{SC}(\text{Small})$, X_{n+1} is small. \square

First let us give the supplement of lemma 2.3.

Lemma 2.5 *If $\text{SC}(\text{Small})$, then for every $n \geq 2$, $A_{small}^\infty \Rightarrow A_{small}^n$.*

Proof. The proof is similar to that of 2.3. Suppose $\neg A_{small}^n$ and let R be a t.a. n -ary relation. Define again the ∞ -ary relation S by putting, for $X \in M_\infty$:

$$S^X = \bigcup \{R^Y : Y \in M_{n-1} \ \& \ Y \subset X\}.$$

Since, by 2.4 (iii), the set of finite subsets of X is small, by $\text{SC}(\text{Small})$, $S^X \in \text{Small}$, hence S is asymmetric. That S is total is proved as in 2.3. \square

In presence of $\text{SC}(\text{Small})$ we can give also better characterizations of A_{small} . A binary relation which is reflexive and transitive is said to be a *preordering*.

Theorem 2.6 *If $\text{SC}(\text{Small})$, the following are equivalent:*

- (i) A_{small} .
- (ii) There is no binary t.a. relation of M .
- (iii) There is no t.a. preordering of M .

If in addition \mathfrak{M} contains a total (=linear) ordering of M , the preceding clauses are equivalent also to the following:

- (iv) There is no t.a. ordering of M .

Proof. The equivalence of (i), (ii) follows from 2.2, while (ii) \Rightarrow (iii) and (iii) \Rightarrow (iv) are trivial.

(iii) \Rightarrow (ii). Suppose there is a binary t.a. relation R . It suffices to extend it to an asymmetric preordering \bar{R} . Let $R^=$ be the reflexive closure of R and let \bar{R} be the transitive closure of $R^=$. Obviously \bar{R} is reflexive, transitive and total. It remains to prove that all initial segments of \bar{R} are small. Let $a \in M$. We have to show that $A = \{x : x\bar{R}a\}$ is small. For every $x \in M$, we write $d(x, a) = n \in \mathbb{N}$, if there is a sequence of least length n such that $xR^=x_1, x_1R^=x_2, \dots, x_nR^=a$, and let $A_n = \{x : x\bar{R}a, d(x, a) = n\}$. Then, clearly,

$$\begin{aligned} A_0 &= R^= \cup \{a\}, \\ A_{n+1} &= \bigcup \{R^=x \cup \{x\} : x \in A_n\}, \\ A &= \bigcup_{n \in \mathbb{N}} A_n. \end{aligned}$$

Every A_n is small. By induction on n . This is true for A_0 because R is asymmetric, and suppose A_n is small. Now A_{n+1} is a union of a small set of small sets, hence by the assumption of self-closure about Small , the claim follows. Finally by the same principle, and since N is small, A is small.

(iv) \Rightarrow (iii). Let R be a total asymmetric preordering, and suppose $<$ is total (proper) ordering of M . Define the relation $<_0$ as follows:

$$x <_0 y \text{ iff } xRy \ \& \ \neg(yRx) \text{ or } xRy \ \& \ yRx \ \& \ x < y.$$

It is easy to see that $<_0$ is a (proper) total ordering. Now for every x , clearly,

$\{y : y <_0 x\} \subseteq R^x$. Since the last set is small, so is also $\{y : y <_0 x\}$. This shows that $<_0$ is asymmetric, and this completes the proof. \square

Remark. It is important to stress here that whenever $M \models \neg A_{small}$, its asymmetry isn't due just to the total preordering (or ordering) R , but to the fact that R is *asymmetric*. Because in this case, each point $x \in M$ divides M into two very unequal parts determined by x , a small initial segment and a nonsmall final segment. If the ordering is not asymmetric, no asymmetry arises. For example the natural ordering of reals produces no asymmetry (with respect to any notion of smallness). Even if R is a wellordering, we do not have necessarily an asymmetric situation in the above sense. It depends upon its initial segments. E.g. if R is a wellordering of \mathbb{R} of type ω_1 , then R is an asymmetric relation with respect to *Count*, but if it is a wellordering of type $> \omega_1$, then we have uncountable initial segments and asymmetry fails.

Recalling that CAC implies $SC(\text{Count})$, “ 2^{\aleph_0} is regular” implies $SC(< 2^{\aleph_0})$, and that \mathbb{R} contains a natural total ordering, we have the following immediate corollary.

Corollary 2.7 (i) In $ZF + CAC$, A_{\aleph_0} holds iff there is no total ordering of the reals all initial segments of which are countable.

(ii) In $ZF + “2^{\aleph_0}$ is regular”, $A_{< 2^{\aleph_0}}$ holds iff there is no total ordering of the reals all initial segments of which have cardinality less than the continuum.

3 Saturation and symmetry

In this section we shall prove that the various forms of saturation imply symmetry with respect to certain sorts of small sets.

Let L be a countable language and κ, λ be infinite cardinals. As usual $L_{\kappa\lambda}$ ($L_{\infty\lambda}$) is the set of formulas of L in the construction of which we allow $< \kappa$ (resp. any number of) conjunctions, and $< \lambda$ quantifiers and free variables. Given an L -structure M , $Def_{\kappa\lambda}(M)$, $Def_{\infty\lambda}(M)$ are the classes of subsets of M definable by $L_{\kappa\lambda}$ and $L_{\infty\lambda}$ formulas respectively. We shall prove the following:

Theorem 3.1 Let M be a κ -saturated L -structure, where $\omega \leq \kappa \leq |M|$ and κ regular.

(i) If $\kappa > \omega$ and $Small = \{X \subseteq M : |X| < \kappa\}$, then $(M, Def_{\infty\kappa}(M)) \models A_{small}^{\infty}$.

(ii) If $\kappa = \omega$ and $Small = M_{fin}$, then $(M, Def_{\infty\omega}(M)) \models A_{small}^n$, for every $n \in \mathbb{N}$.

First we recall some notions from model theory (see e.g. [3]). Let M be an L -structure. For every formula $\phi(v, \vec{u})$ of L and every tuple of parameters $\vec{a} \in M$, let $\phi^M(v, \vec{a})$ denote the extension of $\phi(v, \vec{a})$ in M , i.e. the set $\{x : M \models \phi(x, \vec{a})\}$. Let $A \subseteq M$. An element $x \in M$ is said to be *algebraic over A* in M , if there is a formula $\phi(v, \vec{u})$ of L and a tuple $\vec{a} \in A$ such that $x \in \phi^M(v, \vec{a})$ and $\phi^M(v, \vec{a})$ is finite. x is said to be *definable over A* if for some ϕ and some $\vec{a} \in A$, $\phi^M(v, \vec{a}) = \{x\}$. The set $acl_M(A) = \{x : x \text{ is algebraic over } A\}$ is the *algebraic closure* of A in M . The type of x with respect to A is denoted $tp(x; A)$ and $tp^M(x; A)$ is the set of elements of M indiscernible from x with respect to A . $Aut(M)$ is the group of automorphisms of M .

Lemma 3.2 Let M be κ -saturated. Then:

(i) For every $x, y \in M$ and A , with $|A| < \kappa$,

$$y \in tp^M(x; A) \Leftrightarrow (\exists f \in Aut(M))(f \upharpoonright A = id \ \& \ f(x) = y).$$

(ii) For every $x \in M$ and $A \subseteq M$ with $|A| < \kappa$,

$$x \notin \text{alc}_M(A) \Rightarrow |tp^M(x; A)| \geq \kappa.$$

Proof. (i) is a standard consequence of κ -saturation.

(ii) Let $x \notin \text{alc}_M(A)$. Then, clearly, for every $\phi \in L$ and $\vec{a} \in A$, if $x \in \phi^M(v, \vec{a})$, then $\phi^M(v, \vec{a})$ is infinite. Assume that $|tp^M(x; A)| < \kappa$, and let $tp^M(x; A) = B = \{y_\alpha : \alpha < \lambda\}$, $\lambda < \kappa$. Consider the type:

$$p(v) = \{\phi(v, \vec{a}) \Leftrightarrow \phi(x, \vec{a}) : \phi \in L, \vec{a} \in A\} \cup \{v \neq y_\alpha : \alpha < \lambda\}.$$

This is a type over $A \cup B$, where $|A \cup B| < \kappa$ and by the fact that every $\phi^M(v, \vec{a})$ is infinite, we easily see that it is consistent. Hence it is satisfiable in M . But if $p(y)$, then $y \in tp^M(x; A)$ and $y \notin B = tp^M(x; A)$, a contradiction. \square

Lemma 3.3 *For every $A \subset M$ and $|A| < \kappa$, there is an infinite C , $|C| < \kappa$ such that $(\forall c \in C)(c \notin \text{alc}_M(A \cup C \setminus \{c\}))$.*

Proof. Call a set C having the property stated above *independent over A* . We shall construct C as a sequence $\{c_1, c_2, \dots\}$ such that for every $n \in \mathbb{N}$, $\{c_1, \dots, c_n\}$ is independent over A . Then clearly C will be independent over A . C is defined inductively as follows: Choose $c_1 \notin \text{alc}_M(A)$ (this is possible since, clearly, for $|A| < \kappa$, $|\text{alc}_M(A)| < \kappa$, while by κ -saturation, $|M| \geq \kappa$). Then $\{c_1\}$ is independent over A . Suppose c_1, \dots, c_n has been chosen to be independent over A . Consider the type:

$$\begin{aligned} p(v) = & \{\neg\phi(v, \vec{a}, c_1, \dots, c_n) : \phi^M(v, \vec{a}, c_1, \dots, c_n) \text{ is finite, } \vec{a} \in A\} \cup \\ & \{\phi(\vec{a}, c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n) \rightarrow \neg\phi(\vec{a}, c_1, \dots, c_{i-1}, c_i, c_{i+1}, \dots, c_n) : \\ & \phi^M(\vec{a}, c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n) \text{ is finite, } \vec{a} \in A, i \leq n\}. \end{aligned}$$

$p(v)$ is a type on the parameters $A \cup \{c_1, \dots, c_n\}$ whose cardinality is $< \kappa$. Clearly if y satisfies $p(v)$, then $\{c_1, \dots, c_n, y\}$ is independent over A . So it suffices to show that $p(v)$ is finitely satisfiable in M . Assume the contrary. Then there are ϕ_i , $\vec{a}_i \in A$, for $i \leq n$, and ψ_j , $\vec{b}_j \in A$, \vec{c}_j for $j \leq m$, where \vec{c}_j is a tuple of elements of $\{c_1, \dots, c_n\}$ of length $\leq n - 1$, such that $\phi_i^M(v, \vec{a}_i, c_1, \dots, c_n)$, $\psi_j^M(\vec{b}_j, \vec{c}_j, v)$, for all $i \leq n, j \leq m$, are finite, and

$$(\forall v) \left(\bigwedge_{i \leq n} \neg\phi_i(v, \vec{a}_i, c_1, \dots, c_n) \Rightarrow \bigvee_{j \leq m} (\psi_j(\vec{b}_j, \vec{c}_j, v) \ \& \ \psi_j(\vec{b}_j, c_1, \dots, c_n)) \right),$$

holds in M . But the lefthand formula of the above implication defines a co-finite set in M , while the righthand one defines a finite set. This is a contradiction that completes the proof. \square

Proof of theorem 3.1. We prove (i), the proof of (ii) being similar. Suppose now that $(M, \text{Def}_{\infty\kappa}(M)) \models \neg A_{\text{small}}^\infty$, and let R be an ∞ -ary t.a. relation on M , definable in $L_{\infty\kappa}$ by a formula Φ whose set of parameters is A , with $|A| < \kappa$. By lemma 3.3 there is an infinite set C , with $|C| < \kappa$, independent over A . Hence $C \in \text{Small}$ and by lemma 3.2(ii), $|tp^M(c; A \cup C \setminus \{c\})| \geq \kappa$ for all $c \in C$. Since R is total, for some $c \in C$, $c \in R^{C \setminus \{c\}}$. By lemma 3.2(i), for every $c' \in tp^M(c; A \cup C \setminus \{c\})$, there is an automorphism f of M such that $f \upharpoonright A \cup C \setminus \{c\} = \text{id}$ and $f(c) = c'$. But such an f clearly fixes both R and $C \setminus \{c\}$, hence $tp^M(c; A \cup C \setminus \{c\}) \subseteq R^{C \setminus \{c\}}$. Since $|tp^M(c; A \cup C \setminus \{c\})| \geq \kappa$, while $R^{C \setminus \{c\}}$ is small, i.e. of cardinality $< \kappa$, this is a contradiction. \square

4 Recursive saturation and symmetry.

For $\kappa = \omega$, the preceding result is not satisfactory first because M_{fin} is a rather trivial notion of smallness and second because quite few theories have countable ω -saturated models. Rich theories with uncountably many types, as for example Peano Arithmetic (PA), lack such models. For such theories, the much weaker notion of *countable recursively saturated* models, or crs models for short, (which always exist) is a good substitute. For the rest of this section L will be the language of PA and M will be a crs model of PA. Every such model is nonstandard and the reader is referred to [4] for background information.

\mathbb{N} denotes the standard part of M . The letters m, n, k, \dots range over elements of \mathbb{N} . Capital letters X, Y, Z range over subsets of M . A natural class of small subsets of M consists of the so called “thin” sets defined below. In M we possess devices for coding definable bounded sets, finite sequences, \mathbb{N} -sequences, n -ary relations etc. The precise definition of coding does not matter. Specifically,

$(x)_y = z$ means: z is the y -th element of the sequence coded by x .

$len(x)$ is the length of the sequence coded by x .

$\langle x_0, \dots, x_a \rangle = y$ if $len(y) = a + 1$ and $(y)_i = x_i$ for $i \leq a$.

$X_{(x)} = \{y : \langle x, y \rangle \in X\}$.

$D_x = \{y : (x)_y = 1\}$. D_x is said to be the *set coded by x* . We often write $y \in x$ instead of $y \in D_x$.

$\hat{x} = \{(x)_n : n \in \mathbb{N}\}$. \hat{x} is said to be a *coded sequence*.

Definition 4.1 A set X is said to be *thin* if $X \subseteq \hat{x}$ for some x . *Thin* denotes the set of all thin sets.

Obviously *Thin* is closed under subsets. Closure under unions is also easily checked. Given \hat{x}, \hat{y} we find z such that $\hat{z} = \hat{x} \cup \hat{y}$ by putting $(z)_{2a} = (x)_a$ and $(z)_{2a+1} = (y)_a$ for every a . We shall prove the following:

Theorem 4.2 For every crs model M , $(M, Def_{\infty\omega}(M)) \models A_{thin}^\infty$.

Working in models of PA (having definable Skolem functions), for every $X \subseteq M$, we consider (instead of $acl_M(X)$), the Skolem hull of X , in M denoted $K(M; X)$. That is

$$K(M; X) = \{t(\vec{x}) : \vec{x} \in X \text{ \& } t \text{ is a definable Skolem term of } L\} \prec M.$$

If M is crs, it is a standard fact that for every finite X , $K(M; X)$ is a coded sequence, hence a thin set (cf. e.g. [4]).

Lemma 4.3 Let M be crs. For every $n \in \mathbb{N}$, there is a set X such that $|X| = n$ and for every $x \in X$, $x \notin K(M; X \setminus \{x\})$. We call such a set *independent*. More generally, for any $a \notin K(M; \emptyset)$, and for any n there is a set X , $|X| = n$, such that $X \cup \{a\}$ is *independent*.

Proof. We prove the first claim the other being similar. By induction on n . For $n = 1$ the claim holds trivially, and suppose it holds for $n \geq 2$. Let $X = \{x_1, \dots, x_n\}$ be an independent set of cardinality n . We shall extend it to an independent set of cardinality $n + 1$. Let \vec{x} be the n -tuple of x_1, \dots, x_n and for every $i = 1, \dots, n$, let \vec{x}_i be the $(n - 1)$ -tuple of $X \setminus \{x_i\}$. Let T_n be the set of Skolem terms $t(\vec{v})$ in n -variables. Consider the following type

$$p(v) = \{v \neq t(\vec{x}) : t \in T_n\} \cup \{x_1 \neq t(v, \vec{x}_1) : t \in T_n\} \cup \dots \\ \cup \{x_n \neq t(v, \vec{x}_n) : t \in T_n\}.$$

Clearly $p(v)$ is recursive and if x realizes $p(v)$, then the set $\{x_1, \dots, x_n, x\}$ is independent. So it suffices to show that $p(v)$ is finitely realizable in M . Assume the contrary. Then there are terms t_{ij} , $i = 0, 1 \dots n$, $j \leq k_i$ such that

$$M \models (\forall v) \left[\bigvee_{j \leq k_0} (v = t_{0j}(\vec{x})) \vee \bigvee_{j \leq k_1} (x_1 = t_{1j}(v, \vec{x}_1)) \vee \dots \vee \bigvee_{j \leq k_n} (x_n = t(v, \vec{x}_n)) \right].$$

But then taken an $m \in N$ such that $m \neq t_{0j}(\vec{x})$ for all $j = 1, \dots, k_0$, we should have $x_i = t_{lr}(m, \vec{x}_i)$ for some $l \geq 1$ and some $r \leq k_l$. This says that $x_i \in K(M; X \setminus \{x_i\})$, which is a contradiction. \square

Lemma 4.4 *Let M be crs. There is an infinite independent coded sequence $\hat{x} \subseteq M$.*

Proof. Let T_n be the set of Skolem terms in n variables and for every $n + 1$ and $m \leq n$ let \vec{v}_{nm} denote the n -tuple $((v)_1, \dots, (v)_{m-1}, (v)_{m+1}, \dots, (v)_{n+1})$. Let $p(v)$ be the type

$$\bigcup_{m \leq n} \{(v)_m \neq t(\vec{v}_{nm}) : t \in T_n\}.$$

Clearly $p(v)$ is recursive. Suppose c realizes $p(v)$. We claim that \hat{c} is independent. If not, there would be an m such that $(c)_m \in K(M; \hat{c} \setminus \{(c)_m\})$. Hence there is an initial segment $X = \{(c)_1, \dots, (c)_n\}$, $n > m$, such that $(c)_m \in K(M; X \setminus \{(c)_m\})$. Hence $(c)_m = t(\vec{v}_{mn})$ for some t , which contradicts the fact that c realizes $p(v)$. It suffices to show the consistency of $p(v)$. In the opposite case there should be terms t_1, \dots, t_k , $t_i \in T_{m_i}$, and indices $j_i < m_i$ such that

$$M \models (\forall v) [(v)_{i_1} = t_1(\vec{v}_{i_1 m_1}) \vee \dots \vee (v)_{i_k} = t_k(\vec{v}_{i_k m_k})].$$

If $r = \max\{m_1, \dots, m_k\}$, the preceding formula implies that for every $x \in M$, the set $\{(x)_1, \dots, (x)_r\}$ is not independent. But this obviously contradicts lemma 4.3. This completes the proof. \square

Lemma 4.5 *Let M be crs, \hat{c} be a coded sequence, $x \in M$ and $tp(x; \hat{c})$ be the type of x with parameters from \hat{c} . If $tp^M(x; \hat{c})$ is the corresponding set of elements of M realizing $tp(x; \hat{c})$, then:*

- (a) $tp^M(x, \hat{c}) = \{x\}$ iff $x \in K(M; \hat{c})$. Moreover if $x \notin K(M; \hat{c})$, then $tp^M(x; \hat{c})$ includes an infinite coded set.
- (b) If for all $f \in \text{Aut}(M)$, $f \upharpoonright \hat{c} = \text{id}$ implies $f(x) = x$, then $x \in K(M; \hat{c})$.

Proof. (a) Let F_n be the set of all formulas with n free variables. The type $tp(x; \hat{c})$ is written

$$tp(x; \hat{c}) = \bigcup_n \{\phi(v, (c)_1, \dots, (c)_{n-1}) : \phi \in F_n \ \& \ M \models \phi(x, (c)_1, \dots, (c)_{n-1})\}.$$

It is recursive and, clearly, $tp^M(x; \hat{c}) \neq \{x\}$ iff $tp(x; \hat{c}) \cup \{v \neq x\}$ is (finitely) realizable and clearly the latter holds iff $x \notin K(M; \hat{c})$.

Now if $x \notin K(M; \hat{c})$, the type

$$q(v) = \{v \subset \phi : \phi \in tp(x; \hat{c})\} \cup \{|v| > n : n \in \mathbb{N}\}$$

is also recursive and realizable, so if y realizes $q(v)$, $y \subseteq tp^M(x, \hat{c})$ and $|y| > \mathbb{N}$.

(b) Let $x \notin K(M; \hat{c})$. It suffices to show that there is an $f \in \text{Aut}(M)$ such that $f \upharpoonright \hat{c} = \text{id}$ and $f(x) \neq x$. By (a) there is $y \neq x$, $y \in tp^M(x, \hat{c})$. Clearly, the mapping

f_0 such that $f_0((c)_n) = (c)_n$ for all $n \in \mathbb{N}$ and $f_0(x) = y$ is a partial isomorphism. Now by an easy back and forth, using recursive saturation, we can extend f_0 to an automorphism f . Thus $f \upharpoonright \hat{c} = id$ and $f(x) \neq x$. \square

Proof of theorem 4.2. Assume the contrary, hence there is an ∞ -ary t.a. relation $R \in Def_{\infty\omega}(M)$. We assume for simplicity that R is $L_{\omega_1\omega}$ -definable without parameters, the other case being similar.

Claim. For every x, \hat{c} , $R(x, \hat{c}) \Rightarrow x \in K(M; \hat{c})$.

Proof of the claim. For every $f \in Aut(M)$, $f \upharpoonright \hat{c} = id$ implies $f''R^{\hat{c}} = R^{\hat{c}}$. Therefore for every $x \in R^{\hat{c}}$, $tp^M(x; \hat{c}) \subseteq R^{\hat{c}}$. Suppose $x \notin K(M; \hat{c})$. It follows from lemma 4.5 that there is an infinite $y \subseteq tp^M(x; \hat{c}) \subseteq R^{\hat{c}}$. But this contradicts the fact that $R^{\hat{c}}$ is thin.

Now, for every infinite thin X there is $x \in X$ such that $x \in R^{X \setminus \{x\}}$. In particular, for every c , there is a $x \in \hat{c}$ such that $x \in R^{\hat{c} \setminus \{x\}}$. Hence by the previous claim, for every c , there is an $x \in \hat{c}$ such that $x \in K(M; \hat{c} \setminus \{x\})$. This means that no coded sequence is an independent set, but this contradicts lemma 4.4. \square

Theorem 4.2 seems to reinforce the results of [5] and the conjecture expressed there that for a crs M , $Def_{\omega_1\omega}(M)$ does not contain any object whose construction makes substantial use of a wellordering of M .

Concerning now the principle $SC(Thin)$, recall that it says that for every X and c , if $X_{((c)_n)}$ is thin for all n , then $\bigcup_n X_{((c)_n)}$ is thin. This is written as follows:

$$SC(Thin) \quad \text{For every } X, c,$$

$$(\forall n)(\exists x)(X_{((c)_n}) \subseteq \hat{x}) \Rightarrow (\exists x)(\forall n)(X_{((c)_n}) \subseteq \hat{x}).$$

We see that $SC(Thin)$ takes the form of a weak *saturation* axiom in the sense e.g. of [2].

Lemma 4.6 *If M is crs, $SC(Thin)$ holds in $(M, Def_{\infty\omega}(M))$.*

Proof. Let $X \in Def_{\infty\omega}(M)$ and c be such that $X_{((c)_n)}$ is thin for all n . If a is the parameter occurring in the definition of X , for every f fixing a and c $f''X_{((c)_n)} = X_{((c)_n)}$. Since the latter are thin, we conclude by the argument used already previously that f is the identity on these sets, hence $X_{((c)_n}) \subseteq K(M; a, c)$. Therefore $\bigcup_n X_{((c)_n)} \subseteq K(M; a, c)$. Since the last set is a coded sequence, we are done. \square

In view of the preceding lemma and the general result 2.6 we get immediately:

Corollary 4.7 *If M is crs, there is no $L_{\infty\omega}$ -definable asymmetric preordering of M .*

5 Some asymmetry results.

When talking of symmetry or asymmetry of a model M , we always refer to a universe \mathfrak{M} of subsets of M . And obviously these notions are most natural when \mathfrak{M} consists of definable subsets. Because in this case the symmetry or asymmetry is an inherent feature of the structure, i.e., based on what can be defined inside it.

For instance the standard model \mathbb{N} of PA is inherently asymmetric, since its natural ordering is a wellordering.

On the other hand, asymmetric objects can be added externally by forcing. Given any countable nonstandard $M \models \text{PA}$, it is easy to construct by forcing a total ordering \preceq of M whose all initial segments are thin, essentially in the way described in [6], §2. Let (M, \mathbb{N}) be M expanded by the unary predicate \mathbb{N} . Observe that $\text{Thin} \subseteq \text{Def}_{\omega\omega}(M, \mathbb{N})$. Let us write $\text{Def}(K)$ instead of $\text{Def}_{\omega\omega}(K)$.

Theorem 5.1 *Let M be countable nonstandard. There is a generic extension $(M, \text{Def}(M, \mathbb{N})[G])$ of $(M, \text{Def}(M, \mathbb{N}))$ such that $(M, \text{Def}(M, \mathbb{N})[G]) \models \neg \text{A}_{\text{thin}}$.*

Proof (Sketch). Let $K \in \text{Def}(M, \mathbb{N})$ be a code of all coded sequences of M i.e., $K = \{\langle x, y \rangle : y \in \hat{x}\}$, hence $K_{(x)} = \hat{x}$ for every $x \in \text{dom}(K)$. Let $\text{WO}(V)$ be the predicate “ V is a wellordering”, and put $P = \{p \in \text{dom}(K) : \text{Def}(M, \mathbb{N}) \models \text{WO}(K_{(p)})\}$, and $H = K \upharpoonright P$. P is roughly the set of forcing conditions ordered by:

$$p \leq q \iff H_{(p)} \text{ extends } H_{(q)}.$$

Since M is countable there is a generic $G \subset P$ and putting $\preceq = \bigcup \{H_{(p)} : p \in G\}$ we easily see (details can be found in [6] theorem 2.5) that $(M, \text{Def}(M, \mathbb{N})[G]) \models \text{WO}(\preceq)$. Of course \preceq need not be a real wellordering, but it is certainly a total ordering of M with thin initial segments, hence $(M, \text{Def}(M, \mathbb{N})[G]) \models \neg \text{A}_{\text{thin}}$ according to 2.6. \square

Concerning inherently asymmetric models of PA the trivial example is, as mentioned above, the standard model \mathbb{N} , whose natural ordering is asymmetric with respect to Fin (hence also with respect to Thin). For nonstandard models, it is unknown to us if we can improve theorem 4.2 by weakening the condition of recursive saturation. For example the following questions are open:

Are all (countable) homogeneous models, or models with $|\text{Aut}(M)| > \aleph_0$ symmetric?

Are all models with $|\text{Aut}(M)| \leq \aleph_0$ nonsymmetric?

Concerning the last question we can answer it in the affirmative for a particular kind of models with countably many automorphisms, namely the *simple* ones.

Let $M \models \text{PA}$ be countable nonstandard. A *simple* submodel of M is any model of the form $K(M; a)$ for some $a \in M$. Let $\Delta_1^1(K)$ be the set of Δ_1^1 -definable subsets of K . Since \mathbb{N} is not recursively saturated, $\mathbb{N} \in \Delta_1^1(K)$.

Theorem 5.2 *Let $K = K(M; a)$ denote a simple submodel of a nonstandard model M . Then $(K, \Delta_1^1(K)) \models \neg \text{A}_{\text{thin}}$.*

Proof. Let T be the set of (Gödel-numbers of) all L -definable Skolem terms. Then $K = \{t(a) : t \in T\}$. Let $T_n = T \cap \Sigma_n$, where Σ_n is the set of Σ_n -formulas of L . For every $x \in K$, let $\text{rank}(x) = \text{least}\{n : (\exists t \in T_n)(t(a) = x)\}$, and for $x, y \in K$ let xRy iff $\text{rank}(x) \leq \text{rank}(y)$. Clearly R is a total preordering. To see that R is asymmetric, i.e., for each $x \in K$, $R^x \in \text{Thin}$, take some $x \in K$ with $\text{rank}(x) = n$. Since $\Sigma_m \subset \Sigma_n$ for $m < n$, $R^x = \{t(a) : t \in T_n\}$. Now it is well-known that every nonstandard model has a definable Σ_n -satisfaction class. Since T_n is a coded set of Σ_n -formulas, it follows easily that R^x is a coded sequence.

Now concerning the complexity of R , observe that using a Δ_1^1 -definable satisfaction class for K , and the fact that \mathbb{N} is Δ_1^1 -definable, we get that $R \in \Delta_1^1(K)$. \square

6 Models of partial symmetry.

By partial symmetry we mean the situation where a model satisfies A_{small}^n for some $n \geq 2$ but not A_{small}^{n+1} . In this section we use forcing to construct a model (M, \mathfrak{M}) , satisfying A_{thin}^n and $\neg A_{thin}^{n+1}$ for any fixed n . This shows that each A_{thin}^{n+1} is strictly stronger than A_{thin}^n . We start with a crs $M \models \text{PA}$ and a countable family $\mathfrak{M} \subseteq P(M)$ satisfying some comprehension principle and containing \mathbb{N} . (E.g. the ramified hierarchy $RA(M)$ or the class of analytically definable sets are such families. Note that these \mathfrak{M} are subfamilies of $Def_{\infty\omega}(M)$, hence they satisfy A_{thin}^∞ .)

Let p be a function $p : M_n \rightarrow M$. This induces a function $p' : M_n \rightarrow \text{Thin}$, by putting $p'(X) = p(\hat{X})$. Henceforth we identify p and p' . From now on we fix an independent set $A = \{a_1, \dots, a_n\}$ of cardinality n . The set P of forcing conditions consists of all finite functions $p : M_n \rightarrow M$ with the following properties:

(i) If $X \in M_{n+1}$ and $\{X \setminus \{x\} : x \in X\} \subseteq \text{dom}(p)$, then there is a $x \in X$ such that $x \in p(\hat{X} \setminus \{x\})$.

(ii) p is closed under all partial isomorphisms (p.i.) $e : \text{Field}(p) \rightarrow \text{Field}(p)$, compatible with p . Namely, if $e : \text{Field}(p) \rightarrow \text{Field}(p)$ is a p.i. such that $e''p$ is a function, then $e''p = p$.

(iii) Suppose $X \in \text{dom}(p)$, and $X \subseteq K(M; A)$, then $p(X) \notin A$.

Let P be ordered as usual by reverse inclusion $p \leq q := p \supseteq q$. Condition (i) guarantees that if G is a generic subset of P and $F = \cup G$, then F is a total function from M_n to Thin such that for every $X \in M_{n+1}$ there is a $x \in X$ such that $x \in F(\hat{X} \setminus \{x\})$. Therefore:

Lemma 6.1 *If P is the preceding set of forcing conditions and G is a generic subset of P , then $(M, \mathfrak{M}[G]) \models \neg A_{thin}^{n+1}$.*

Lemma 6.2 *Let G be a P -generic set and let $p \in G$. If e is a partial isomorphism such that $e''p = p$, then e can be extended to an automorphism f of M such that $f''G = G$.*

Proof. By a back and forth argument. Let a_1, a_2, \dots be an enumeration of M , and let $a_n \notin \text{dom}(e) = \text{Field}(p)$. Let

$$D = \{q : q \leq p \ \& \ a_n \in \text{Field}(q) \ \& \ (\exists d)(d \supseteq e \ \& \ d''q = q)\}.$$

It is easy to see that D is dense in P . Hence there is a $q \in G \cap D$. This shows that for each n there is a $q_n \in G$ extending p and a d_n extending e such that $a_n \in \text{dom}(d_n) \cap \text{rang}(d_n)$ and $d_n''q_n = q_n$. Let $f = \cup_n d_n$. Then f is an automorphism and we claim that $f''G = G$. Assume the contrary. Then for some q , $f(q) \notin G$. But there is $q_n \leq q$ and since $f(q) = \{f(x) : x \in q\}$ and $f \upharpoonright q_n = d_n$, we have: $f(q) \subseteq f(q_n) = f''q_n = d_n''q_n = q_n$, a contradiction. Therefore f fixes G . \square

Lemma 6.3 *Let M be a crs model and let $a, b, c \in M$ such that $tp(a) = tp(b)$. The following are equivalent:*

- (i) $(\forall f \in \text{Aut}(M))(f(a) = b \Rightarrow f(c) = c)$.
- (ii) *There is a term t such that $t(a) = t(b) = c$.*

Proof. (ii) \Rightarrow (i) is obvious. Assume (ii) is false. We find f such that $f(a) = b$ and $f(c) \neq c$. Consider the type:

$$p(v) = \{\phi(a, c) \Leftrightarrow \phi(b, v) : \phi \in L\} \cup \{v \neq c\}.$$

Since $p(v)$ is recursive and every d satisfying $p(v)$ proves the claim, it suffices to show that $p(v)$ is consistent. Assume it is not. Then for some $\phi \in L$,

$$M \models \phi(a, c) \ \& \ (\forall v)(v \neq c \Rightarrow \neg \phi(b, v)). \quad (2)$$

Let $d = \min\{v : M \models \phi(a, v)\}$. For every h such that $h(a) = b$, $h(d) = \min\{v : M \models \phi(b, v)\}$. But by (2) the only element satisfying $\phi(b, v)$ is c . Therefore $h(d) = c$. On the other hand $d \in K(M; a)$, hence $d = t(a)$ for some term t . Hence $h(d) = c = t(h(a)) = t(b)$, i.e., $c = t(b)$. Now c is the unique element satisfying $\phi(b, v)$, hence $d = h^{-1}(c)$ is the unique element satisfying $\phi(a, v)$. But by assumption $\phi(a, c)$ holds, hence $c = d$. Thus $c = t(b) = d = t(a)$. \square

Lemma 6.4 *Let $A = \{a_1, \dots, a_n\}$ be an independent set. Let $A_i = A \setminus \{a_i\}$. If G is P -generic, for every i and for every $b \in tp^M(a_i; A_i)$, there is an automorphism f such that $f \upharpoonright A_i = id$, $f(a_i) = b$ and $f''G = G$.*

Proof. For every i and b as above consider the set

$$D_{i,b} = \{q : (\exists p.i. e)(e''q = q \ \& \ e \upharpoonright A_i = id \ \& \ e(a_i) = b)\}.$$

It is not difficult to show that $D_{i,b}$ is dense in P . Let $p \in P$. Now $a_i \notin K(M; A_i)$, hence there is always an elementary e fixing A_i and sending a_i to b . The only case in which we would not be able to extend p to an element $q \in D_{i,b}$ would be that in which for some $X \in \text{dom}(p)$, $p(X) = a_i$ and every automorphism f fixing A_i and sending a_i to b , fixes X pointwise. In that case, if $e''q = q$, $e \upharpoonright A_i = id$ and $e(a_i) = b$, then $e(X) = X$, hence $q(X) = b$. But in this case q would be incompatible with p . We claim that this case is impossible because of the condition (iii) imposed on the forcing conditions.

Indeed let X be such that for every $f \in \text{Aut}(M)$, $f \upharpoonright A_i = id$ and $f(a_i) = b \Rightarrow f \upharpoonright X = id$. Equivalently, for every f such that $f(\langle A_i, a_i \rangle) = \langle A_i, b \rangle$ (A_i is considered here as a single element via coding), $f \upharpoonright X = id$. By lemma 6.3, for every $x \in X$, there is a term t such that $t(\langle A_i, a_i \rangle) = t(\langle A_i, b \rangle) = x$. But then $X \subseteq K(M; A)$. By (iii), $p(X) \notin A$, i.e. we cannot have $p(X) = a_i$. This shows that p is always extendible to an element of $D_{i,b}$.

Therefore there is a $q \in D_{i,b} \cap G$. If e is the corresponding isomorphism, this fixes q . By lemma 6.2, this can be extended to an automorphism f fixing G with the same other properties, hence the claim is proved. \square

For $X \subseteq M$, let $\text{Aut}(M)_{\{X\}}$ be the set of automorphisms fixing X setwise. Let G be a P -generic set and let $(M, \mathfrak{M}[G])$ be the corresponding extension. Let $\mathfrak{S} = \{X \subseteq M : \text{Aut}(M)_{\{G\}} \subseteq \text{Aut}(M)_{\{X\}}\}$.

Theorem 6.5 $(M, \mathfrak{S}) \models \neg A_{\text{thin}}^{n+1} + A_{\text{thin}}^n$.

Proof. Clearly $G \in \mathfrak{S}$ hence the latter contains an $(n+1)$ -ary t.a. relation. Suppose \mathfrak{S} contains also an n -ary t.a. relation R . Let A be an independent set with $|A| = n$. Then there is $a_i \in A$ such that $a_i \in R^{A_i}$. Let $b \in tp^M(a_i; A_i)$. By lemma 6.4 there is an automorphism f of M such that $f''G = G$, $f \upharpoonright A_i = id$ and $f(a_i) = b$. But then $f''R = R$ by the definition of \mathfrak{S} . It follows that $f(a_i) \in (f''R)^{f''A_i}$, or $b \in R^{A_i}$. Therefore $tp^M(a_i; A_i) \subseteq R^{A_i}$. But $tp^M(a_i; A_i)$ is not thin, a contradiction. \square

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