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Gaps between zeros of Dedekind zeta-functions of quadratic number fields

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ABSTRACT

Let K be a quadratic number field, and let $\zeta_K(s)$ denote the Dedekind zeta-function attached to K. Using the mixed second moments of derivatives of $\zeta_K(\frac{1}{2}+it)$, we prove the existence of gaps between consecutive zeros of $\zeta_K(s)$ on the critical line which are at least $\sqrt{6} = 2.44949\ldots$ times the average spacing.

Let K be a number field and $\mathcal{O}_K$ its ring of integers. The Dedekind zeta-function attached to K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\subset\mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}\subset\mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \quad \Re(s) > 1,$$

where $\mathfrak{a}$ and $\mathfrak{p}$ run over the nonzero ideals and prime ideals of $\mathcal{O}_K$, respectively. Let K be a quadratic field with discriminant d, and let χ_d be the Kronecker symbol of d. Then the Dedekind zeta-function factors as

$$\zeta_K(s) = \zeta(s)L(s,\chi_d), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s,\chi_d)$ is the Dirichlet L-function associated to χ_d.

This note studies the vertical distribution of the zeros of $\zeta_K(s)$ in the critical strip, which we denote by $\rho_K = \beta + i\gamma$. It has been shown that for an imaginary quadratic number field K, the vertical distribution of the nontrivial zeros of $\zeta_K(s)$ is related to the existence or non-existence of Landau–Siegel zeros and hence the size of the class number of K. This correspondence is described in the work of Conrey and Iwaniec [10]; see also Montgomery and Weinberger [22]. This circle of ideas is often referred to as the Deuring–Heilbronn

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phenomenon. For a very nice overview of the Deuring–Heilbronn phenomenon and its implications, see Stopple's survey article [27].

For a real or imaginary quadratic number field of discriminant d, it is known [19, Theorem 5.31] that for $T \geqslant 2$, we have

$$N_K(T) := \sum_{0<\gamma\leqslant T} 1 = \frac{T}{\pi} \log \frac{\sqrt{|d|}T}{(2\pi e)^2} + O\big(\log\big(\sqrt{|d|}T\big)\big).$$

Consider the sequence $0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots$ of consecutive ordinates of the nontrivial zeros of $\zeta_K(s)$, and note that the average size of $\gamma_{n+1} - \gamma_n$ is $\pi/\log(\sqrt{|d|}\gamma_n)$. Normalizing, let

$$\mu_K := \liminf_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}$$

and

$$\lambda_K := \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}.$$

By definition we have $\mu_K \leqslant 1 \leqslant \lambda_K$, however it is conjectured that $\mu_K = 0$ and $\lambda_K = \infty$. In other words, we expect that there are arbitrarily small and large normalized gaps between consecutive nontrivial zeros of Dedekind zeta-functions of quadratic number fields. While we expect $\mu_K = 0$, this is not due to the presumption of coincident nontrivial zeros of $\zeta(s)$ and $L(s, \chi_d)$. On the contrary, we expect that the zeros of $\zeta_K(s)$ are simple. Conrey, Ghosh, and Gonek [8] have shown that the number of simple zeros of $\zeta_K(s)$ with $0 < \gamma \leqslant T$ exceeds $T^{6/11}$ for sufficiently large T. In [9], the same authors show, assuming the generalized Riemann hypothesis for Dirichlet L-functions, that a positive proportion of the zeros of $\zeta_K(s)$ are simple. In general, it is conjectured that any two distinct primitive L-functions should have no shared zero.

That $\mu_K < 1 < \lambda_K$ is an open question, and in particular there do not seem to be any quantitative results concerning the sizes of μ_K or λ_K. This is in contrast to the distribution of the zeros of the Riemann zeta-function, where there is an abundance of results, both unconditional and assuming various unproved hypotheses. See, for instance, [1–7,11–14,20,21,24,26,29].

The object of this note is to provide a nontrivial lower bound for λ_K. Towards this goal, we prove the following unconditional theorem.

Theorem 1. *Let $T \geqslant 2$ and $\varepsilon > 0$. Let K be a quadratic number field of discriminant d with $|d| \leqslant T^{\frac{7}{9}-\varepsilon}$. There exists a subinterval of $[T, 2T]$ having length at least*

$$\sqrt{6} \cdot \frac{\pi}{\log\sqrt{|d|}T}\big(1 + O\big(|d|^{\varepsilon} \log^{-1} T\big)\big)$$

for which the function $t \mapsto \zeta_K(1/2 + it)$ is free of zeros.

Theorem 1 does not, *a fortiori*, state anything about the quantity λ_K. However, if we assume the generalized Riemann hypothesis for $\zeta_K(s)$, then Theorem 1 immediately implies the following inequality for λ_K.

Corollary 2. *Assume the generalized Riemann hypothesis for $\zeta_K(s)$. Then $\lambda_K \geqslant \sqrt{6}$. In particular, there are infinitely many normalized gaps between consecutive zeros of $\zeta_K(s)$ which are greater than $\sqrt{6} - \varepsilon$ times the average spacing for any $\varepsilon > 0$.*

The constant $\sqrt{6}$ in Corollary 2 is larger than one might expect since the same method of proof applied to the Riemann zeta-function only exhibits gaps between nontrivial zeros of $\zeta(s)$ of size $\sqrt{3}$ times the average spacing. (See [13].) Moreover, in contrast to Theorem 1 and its corollary, establishing a nontrivial upper bound on μ_K seems to be more difficult due to the connection to the Deuring–Heilbronn phenomenon and the class number problem for imaginary quadratic fields mentioned above.

We prove Theorem 1 by combining the mixed second moments of derivatives of $\zeta_K(s)$ and an argument of R.R. Hall [13]. In 1926, Ingham [18] proved that for $s = 1/2 + it$ and $|\alpha|, |\beta| < 1/2$, we have

$$\int_0^T \zeta(s+\alpha)\zeta(1-s+\beta)\,dt = \int_0^T \left(\zeta(1+\alpha+\beta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\zeta(1-\alpha-\beta)\right)\left(1+O\left(t^{-\frac{1}{2}+\varepsilon}\right)\right)dt.$$

This 'shifted' moment reveals a beautiful underlying structure which allows one to deduce lower order terms and moments of derivatives of $\zeta(s)$ via differentiation and Cauchy's integral formula. For instance, Ingham's theorem can be used to show that, for fixed $\mu, \nu \in \mathbb{N}$,

$$\int_0^T \zeta^{(\mu)}\left(\frac{1}{2}+it\right)\zeta^{(\nu)}\left(\frac{1}{2}-it\right)dt = \frac{(-1)^{\mu+\nu}}{\mu+\nu+1}T(\log T)^{\mu+\nu+1} + O\left(T(\log T)^{\mu+\nu}\right),$$

where $\zeta^{(\mu)}(s)$ denotes the μth derivative of $\zeta(s)$. We make use of a similar shifted moment result for a Dedekind zeta-function of a quadratic number field due to Heap [16] to obtain the mixed second moments of derivatives of $\zeta_K(s)$ on the critical line. In particular, the proof of Theorem 1 requires asymptotic estimates of the mixed second moments of $\zeta_K(\frac{1}{2}+it)$ and $\zeta_K'(\frac{1}{2}+it)$ with a uniform error. We obtain these by way of the following theorem.

Theorem 3. *Let K be the quadratic number field with discriminant d. Let $T \geqslant 2$, and μ, ν be non-negative integers. We have*

$$\int_T^{2T} \zeta_K^{(\mu)}\left(\frac{1}{2}+it\right)\zeta_K^{(\nu)}\left(\frac{1}{2}-it\right)dt = \frac{(-1)^{\mu+\nu}(2^{\mu+\nu+1}-1)}{(\mu+\nu+2)(\mu+\nu+1)}2C_dT(\log T)^{\mu+\nu+2} + O\left(\mu!\nu!|d|^{\varepsilon}C_dT(\log T)^{\mu+\nu+1}\right),$$

where the constant

$$C_d := \frac{6}{\pi^2}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi_d). \tag{2}$$

Special cases of Theorem 3 are known by the work of Motohashi [23] and Weinstein [28], however we require the more general case to prove Theorem 1. We deduce Theorem 3 from the following recent result of Heap [16].

Theorem 4 *(Heap). Let K be the quadratic number field with discriminant d. Let $s = 1/2 + it$ and $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|, |\beta| \ll 1/\log(\sqrt{|d|}T)$. Then we have*

$$\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt$$

$$= \int_T^{2T} \left\{\prod_p\left(1-\frac{1}{p^{2+2\alpha+2\beta}}\right)\prod_{p|d}\left(1+\frac{1}{p^{1+\alpha+\beta}}\right)^{-1}\zeta_K^2(1+\alpha+\beta)\right.$$

$$+ \left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\frac{6}{\pi^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|d}\left(1-\frac{1}{p^{1+\alpha+\beta}}\right)L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)$$

$$+\frac{1}{d^{\alpha+\beta}}\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\frac{6}{\pi^2}\prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|d}\left(1-\frac{1}{p^{1-\alpha-\beta}}\right)L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)$$
$$+\frac{1}{d^{\alpha+\beta}}\left(\frac{t}{2\pi}\right)^{-2\alpha-2\beta}\prod_{p}\left(1-\frac{1}{p^{2-2\alpha-2\beta}}\right)\prod_{p|d}\left(1+\frac{1}{p^{1-\alpha-\beta}}\right)^{-1}\zeta_K^2(1-\alpha-\beta)\Bigg\}\,dt$$
$$+O\big(|d|^{\varepsilon}C_dT\log T\big), \tag{3}$$

where the constant C_d is defined in (2).

Proof. This is a consequence of [16, Theorem 1], letting $h=k=1$. □

Prior to the work of Heap [16], the author independently derived Theorem 4 using a method of Ramachandra [25]. Using different techniques, Heap computes the second moment of a Dedekind zeta-function of a quadratic field times an arbitrary Dirichlet polynomial of length $T^{1/11-\varepsilon}$.

Proof of Theorem 3. Let $\varepsilon>0$ be an arbitrary constant, $s=1/2+it$, and $T\geqslant 2$ be fixed. We first simplify the integral on the right-hand side of (3) by considering each factor of each term of the integrand. Since $\alpha+\beta\ll 1/\log(\sqrt{|d|}T)$, it follows that $d^{-\alpha-\beta}=1+O((\alpha+\beta)|d|^{\varepsilon})$. The Euler products on the right-hand side of (3) can be simplified as

$$\prod_{p}\left(1-\frac{1}{p^{2\pm(\alpha+\beta)}}\right)=\prod_{p}\left(1-\frac{1}{p^2}\right)\big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\big)=\frac{6}{\pi^2}\big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\big),$$
$$\prod_{p|d}\left(1+\frac{1}{p^{1\pm(\alpha+\beta)}}\right)^{-1}=\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}\big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\big),$$

and

$$\prod_{p|d}\left(1-\frac{1}{p^{1\pm(\alpha+\beta)}}\right)=\prod_{p|d}\left(1-\frac{1}{p}\right)\big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\big).$$

The factorization given in (1) implies that

$$\zeta_K\big(1\pm(\alpha+\beta)\big)=L(1,\chi_d)\zeta\big(1\pm(\alpha+\beta)\big)\big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\big).$$

Furthermore, since $t\in[T,2T]$, we have that $(t/2\pi)^{-\alpha-\beta}=T^{-\alpha-\beta}(1+O(1/\log T))$. Using these estimates, we find that

$$\int_T^{2T}\zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt=\int_T^{2T}\left\{\frac{6}{\pi^2}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi_d)\zeta^2(1+\alpha+\beta)\right\}dt$$
$$+2\int_T^{2T}\left\{\frac{6}{\pi^2}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)T^{-\alpha-\beta}\right\}dt$$
$$+\int_T^{2T}\left\{\frac{6}{\pi^2}\prod_{p|d}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi_d)\zeta^2(1-\alpha-\beta)T^{-2\alpha-2\beta}\right\}dt$$
$$+O\big(|d|^{\varepsilon}C_dT\log T\big)$$
$$:=I_1+2I_2+I_3+O\big(|d|^{\varepsilon}C_dT\log T\big),$$

say. Since $\zeta(1-s) = 1/s + O(1)$, we can express the three integrals as

$$I_1 = (\alpha+\beta)^{-2} C_d T + O\bigl(|d|^{\varepsilon} C_d T \log T\bigr),$$
$$I_2 = -(\alpha+\beta)^{-2} C_d T^{-\alpha-\beta+1} + O\bigl(|d|^{\varepsilon} C_d T \log T\bigr),$$

and

$$I_3 = (\alpha+\beta)^{-2} C_d T^{-2\alpha-2\beta+1} + O\bigl(|d|^{\varepsilon} C_d T \log T\bigr).$$

Finally, noting that

$$T^{-\delta(\alpha+\beta)} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta^n (\alpha+\beta)^n (\log T)^n}{n!},$$

we simplify $I_1 + 2I_2 + I_3$ to conclude that

$$\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt = F(\alpha+\beta;T) + O\bigl(|d|^{\varepsilon} C_d T \log T\bigr),$$

where

$$F(\alpha+\beta;T) := 2C_d T \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+\beta)^n (\log T)^{n+2}}{(n+2)!} \bigl\{2^{n+1} - 1\bigr\}. \tag{4}$$

We now follow an argument of Ingham [18] to complete the proof. Let

$$R(\alpha,\beta;T) := \int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt - F(\alpha+\beta;T). \tag{5}$$

Then $R(\alpha,\beta;T)$ is an analytic function of two complex variables α and β when $\Re(\alpha), \Re(\beta) < 1/2$, and

$$R(\alpha,\beta;T) = O\bigl(|d|^{\varepsilon} C_d T \log T\bigr) \tag{6}$$

holds by Theorem 4. Differentiating (5), it follows that

$$\int_T^{2T} \zeta_K^{(\mu)}(s+\alpha)\zeta_K^{(\nu)}(1-s+\beta)\,dt = \frac{\partial^{\mu+\nu} F(\alpha+\beta;T)}{\partial\alpha^{\mu}\partial\beta^{\nu}} + R_{\mu,\nu}(\alpha,\beta;T), \tag{7}$$

where μ and ν are fixed nonnegative integers and

$$R_{\mu,\nu}(\alpha,\beta;T) := \frac{\partial^{\mu+\nu} R(\alpha,\beta;T)}{\partial\alpha^{\mu}\partial\beta^{\nu}}.$$

Let $\mathscr{C} = \{w \in \mathbb{C};\ |w-\alpha| = 1/\log T\}$. By the Cauchy integral formula and (6), we have

$$\begin{aligned}\frac{\partial^{\mu}}{\partial\alpha^{\mu}} R(\alpha,\beta;T) &= \frac{\mu!}{2\pi i} \int_{\mathscr{C}} \frac{R(w,\beta;T)}{(w-\alpha)^{\mu+1}}\,dw \\ &= O\bigl(\mu! |d|^{\varepsilon} C_d T (\log T)^{\mu+1}\bigr).\end{aligned}$$

Appealing to the Cauchy integral formula once more, we deduce that

$$R_{\mu,\nu}(\alpha,\beta;T) := \frac{\partial^{\mu+\nu}}{\partial\alpha^\mu\partial\beta^\nu}R(\alpha,\beta;T) = O\big(\mu!\nu!|d|^\varepsilon C_d T(\log T)^{\mu+\nu+1}\big).$$

Thus (7), with $\alpha = \beta = 0$, gives

$$\int_0^T \zeta_K^{(\mu)}\left(\frac{1}{2}+it\right)\zeta_K^{(\nu)}\left(\frac{1}{2}-it\right)dt = \left[\frac{\partial^{\mu+\nu}F(\alpha+\beta;T)}{\partial\alpha^\mu\partial\beta^\nu}\right]_{\alpha=\beta=0} + O\big(\mu!\nu!|d|^\varepsilon C_d T(\log T)^{\mu+\nu+1}\big), \tag{8}$$

and it remains only to calculate the first term on the right-hand side. By differentiating (4) with respect to α and β and simplifying, we determine that

$$\left[\frac{\partial^{\mu+\nu}F(\alpha+\beta;T)}{\partial\alpha^\mu\partial\beta^\nu}\right]_{\alpha=\beta=0} = \frac{(-1)^{\mu+\nu}(2^{\mu+\nu+1}-1)}{(\mu+\nu+2)(\mu+\nu+1)}2C_d T(\log T)^{\mu+\nu+2}. \tag{9}$$

Theorem 3 now follows upon inserting (9) into (8). □

We now demonstrate how to obtain the lower bound in Theorem 1. The proof is a variation of a method of R.R. Hall [13] using some ideas of Bredberg [1]. We begin by defining the function

$$f(t) := e^{ivt\log T}\zeta_K\left(\frac{1}{2}+it\right), \tag{10}$$

where v is a real constant that will be chosen later. By Stirling's formula, $f(t)$ mimics the analogue of the Hardy Z-function for $\zeta_K(s)$. Fix K, and let $\widetilde{\gamma}$ denote an ordinate of a zero of $\zeta_K(s)$ on the critical line $\Re(s) = 1/2$. Note that $f(t)$ has the same zeros as $\zeta_K(\frac{1}{2}+it)$, that is, $f(t) = 0$ if and only if $t = \widetilde{\gamma}$. Let $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \dots, \widetilde{\gamma}_N\}$ denote the set of distinct zeros of $f(t)$ in the interval $[T, 2T]$ arranged in non-decreasing order and ignoring multiplicity. Furthermore, let

$$\kappa_T = \max\{\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n\colon\ T+1 \leqslant \widetilde{\gamma}_n \leqslant 2T-1\},$$

and note that $\lambda_K \geqslant \limsup_{T\to\infty}\kappa_T$. Without loss of generality, we may assume that

$$\widetilde{\gamma}_1 - T \ll 1 \quad \text{and} \quad 2T - \widetilde{\gamma}_N \ll 1, \tag{11}$$

as otherwise there exist zeros $\widetilde{\gamma}_0 \leqslant \widetilde{\gamma}_1$ and $\widetilde{\gamma}_{N+1} \geqslant \widetilde{\gamma}_N$ such that $\widetilde{\gamma}_0 - \widetilde{\gamma}_1$ and $\widetilde{\gamma}_{N+1} - \widetilde{\gamma}_N$ are $\gg 1$, and Theorem 1 holds for this reason. In order to obtain a lower bound on κ_T, we require the following lemma.

Lemma 5. *Let $y : [a,b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then*

$$\int_a^b \big|y(x)\big|^2\,dx \leqslant \left(\frac{b-a}{\pi}\right)^2\int_a^b \big|y'(x)\big|^2\,dx.$$

Proof. This is a variation of a well-known inequality of Wirtinger [15, Theorem 256] due to Bredberg [1, Corollary 1]. □

With this setup, we now prove Theorem 1.

Proof of Theorem 1. Let $\varepsilon > 0$ be a small positive constant which may vary from line to line, and let $f(t)$ be the function defined in (10). By the definition of κ_T, for each pair of consecutive zeros of $f(t)$ in the interval $[T, 2T]$, we have

$$\int_{\tilde{\gamma}_n}^{\tilde{\gamma}_{n+1}} \left|f(t)\right|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2} \int_{\tilde{\gamma}_n}^{\tilde{\gamma}_{n+1}} \left|f'(t)\right|^2 dt. \tag{12}$$

Summing both sides of the equation in (12) over n for $n = 1, 2, \ldots, N-1$, it follows that

$$\int_{\tilde{\gamma}_1}^{\tilde{\gamma}_N} \left|f(t)\right|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2} \int_{\tilde{\gamma}_1}^{\tilde{\gamma}_N} \left|f'(t)\right|^2 dt.$$

By Weyl's bound for the zeta-function, $\zeta(\frac{1}{2}+it) \ll t^{\frac{1}{6}+\varepsilon}$, and the subconvexity bound $L(\frac{1}{2}+it, \chi_d) \ll |td|^{\frac{3}{16}+\varepsilon}$ due to Heath-Brown [17], we see that $|f(t)| \ll t^{\frac{17}{48}+\varepsilon}|d|^{\frac{3}{16}+\varepsilon}$ for $T \leqslant t \leqslant 2T$ and $\varepsilon > 0$. Therefore, by the assumption in (11), we have

$$\int_T^{2T} \left|f(t)\right|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2} \int_T^{2T} \left|f'(t)\right|^2 dt + O\left(|d|^{\frac{3}{8}+\varepsilon} T^{\frac{17}{24}+\varepsilon}\right). \tag{13}$$

Note that $|f(t)|^2 = |\zeta_K(\frac{1}{2}+it)|^2$ and

$$\left|f'(t)\right|^2 dt = \left|\zeta_K'\left(\frac{1}{2}+it\right)\right|^2 + v^2 \log^2 T \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^2 + 2v \log T \cdot \operatorname{Re}\left(\zeta_K'\left(\frac{1}{2}+it\right)\overline{\zeta_K\left(\frac{1}{2}+it\right)}\right). \tag{14}$$

Theorem 3 implies that

$$\int_T^{2T} \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^2 dt = C_d T \log^2 T + O\left(|d|^{\varepsilon} C_d T \log T\right), \tag{15}$$

$$\int_T^{2T} \zeta_K'\left(\frac{1}{2}+it\right)\overline{\zeta_K\left(\frac{1}{2}+it\right)}\, dt = -C_d T \log^3 T + O\left(|d|^{\varepsilon} C_d T \log^2 T\right), \tag{16}$$

and

$$\int_T^{2T} \left|\zeta_K'\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{7}{6} C_d T \log^4 T + O\left(|d|^{\varepsilon} C_d T \log^3 T\right), \tag{17}$$

where C_d is the constant in (2). By combining the estimates in (13)–(17), we find that

$$\frac{\kappa_T^2}{\pi^2} \geqslant \frac{6}{6v^2 - 12v + 7} \frac{1}{\log^2 T}\left(1 + O\left(|d|^{\varepsilon} \log^{-1} T\right)\right),$$

uniformly for $|d| \leqslant T^{\frac{7}{9}-\varepsilon}$. The choice of $v = 1$ minimizes $6v^2 - 12v + 7$, the minimum value being 1. We conclude that

$$\kappa_T \geqslant \frac{\sqrt{6}\pi}{\log(\sqrt{|d|}T)}\left(1 + O\left(|d|^{\varepsilon} \log^{-1} T\right)\right).$$

This completes the proof of Theorem 1. □

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