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Taylor coefficients of functions in Fock spaces ☆

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Abstract

Analogues of the Hausdorff–Young and Hardy–Littlewood theorems are proved for functions in Fock spaces. Estimates of Taylor coefficients are shown to be sharp in the sense of Duren–Taylor.

Keywords: Fock space; Entire function; Taylor coefficients

1. Introduction

Let dA be the Lebesgue area measure. For $0 < p < \infty$ and $\alpha > 0$, the *Fock space* F_α^p consists of all entire functions f for which

$$\|f\|_{p,\alpha} = \left(\frac{\alpha p}{2\pi} \int_{\mathbb{C}} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p \, dA(z)\right)^{\frac{1}{p}} < \infty.$$

The space F_α^∞ consists of all entire functions f for which

$$\|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}} \left|f(z)\right| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

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The spaces L^p_α (respectively L^∞_α) consist of all measurable functions f for which $\|f\|_{p,\alpha}$ (respectively $\|f\|_{\infty,\alpha}$) is finite. In the definition of L^∞_α the essential supremum is taken.

For $r > 0$ and any analytic function f, the *integral mean* is

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} \left|f\left(re^{i\theta}\right)\right|.$$

The definition of $\|f\|_{p,\alpha}$ can be rewritten as

$$\|f\|_{p,\alpha} = \left(\alpha p \int_0^\infty M_p(r, f)^p e^{-\frac{\alpha p}{2}r^2} r\, dr\right)^{\frac{1}{p}},$$

$$\|f\|_{\infty,\alpha} = \sup_{r\geqslant 0} M_\infty(r, f) e^{-\frac{\alpha}{2}r^2}.$$

For $p = 2$, we have a precise way to determine whether an analytic function $f(z) = \sum a_n z^n$ is in F^2_α by its Taylor coefficients. A calculation shows

$$\|f\|^2_{2,\alpha} = 2\alpha \int_0^\infty \left(\sum_{n=0}^\infty |a_n|^2 r^{2n}\right) e^{-\alpha r^2} r\, dr = \sum_{n=0}^\infty |a_n|^2 \frac{n!}{\alpha^n}; \tag{1}$$

thus $f \in F^2_\alpha$ if and only if $\sum |a_n|^2 n!/\alpha^n < \infty$. However, for $p \neq 2$ it is difficult to find a single condition in terms of the Taylor coefficients that is both necessary and sufficient for f to belong to F^p_α.

The purpose of this paper is to obtain conditions, some sufficient and others necessary, for membership in Fock spaces F^p_α. We will utilize the available techniques for obtaining similar results in Hardy and Bergman spaces.

An analytic function f in the unit disk $\mathbb{D}$ is said to belong to the *Hardy space* H^p if

$$\|f\|_{H^p} = \sup_{r<1} M_p(r, f) < \infty.$$

The function f is in the *Bergman space* A^p if

$$\|f\|_{A^p} = \left(\frac{1}{\pi}\int_{\mathbb{D}} \left|f(z)\right|^p dA(z)\right)^{\frac{1}{p}} < \infty.$$

For $p = 2$ we have necessary and sufficient conditions for membership in H^2 or A^2:

$$f \in H^2 \iff \sum_{n=0}^\infty |a_n|^2 < \infty; \qquad f \in A^2 \iff \sum_{n=0}^\infty \frac{|a_n|^2}{n+1} < \infty.$$

For other values of p, similar necessary or sufficient conditions are known.

Theorem A *(Hausdorff–Young theorem). Let $1 < p < \infty$, let $q = p/(p-1)$ be its conjugate exponent, and let $f(z) = \sum a_n z^n$ be analytic in the unit disk.*

(i) *For* $1 < p \leqslant 2$,

$$f \in H^p \implies \sum_{n=0}^{\infty} |a_n|^q < \infty; \qquad f \in A^p \implies \sum_{n=1}^{\infty} n^{1-q} |a_n|^q < \infty.$$

(ii) *For* $2 \leqslant p < \infty$,

$$\sum_{n=0}^{\infty} |a_n|^q < \infty \implies f \in H^p; \qquad \sum_{n=1}^{\infty} n^{1-q} |a_n|^q < \infty \implies f \in A^p.$$

Theorem B *(Hardy–Littlewood theorem for H^p and A^p). Let $f(z) = \sum a_n z^n$ be analytic in the unit disk.*

(i) *For* $0 < p \leqslant 2$,

$$f \in H^p \implies \sum_{n=1}^{\infty} n^{p-2} |a_n|^p < \infty;$$

$$\sum_{n=1}^{\infty} n^{-1} |a_n|^p < \infty \implies f \in A^p \implies \sum_{n=1}^{\infty} n^{p-3} |a_n|^p < \infty.$$

(ii) *For* $2 \leqslant p < \infty$,

$$\sum_{n=1}^{\infty} n^{p-2} |a_n|^p < \infty \implies f \in H^p;$$

$$\sum_{n=1}^{\infty} n^{p-3} |a_n|^p < \infty \implies f \in A^p \implies \sum_{n=1}^{\infty} n^{-1} |a_n|^p < \infty.$$

The H^p portion of Theorem A is the well-known Hausdorff–Young theorem, adapted to the Hardy space setting; the A^p analogue of the Hausdorff–Young theorem is due to Horowitz [6]. Hardy and Littlewood [7] proved Theorem B for H^p spaces, whereas Nakamura, Ohya, and Watanabe [10] obtained the A^p analogue. Proofs of the theorems for H^p spaces can be found in [3, Chapter 6]. For A^p spaces see [5, Chapter 3]. In Sections 2 and 3 of this paper we will prove analogues of both theorems for Fock spaces.

The Hardy–Littlewood theorem can be put in quantitative form. The original H^p portion can be stated as follows.

Theorem B′. *Let $f(z) = \sum a_n z^n$ be analytic in the unit disk.*

(i) *For* $0 < p \leqslant 2$,

$$\left(\sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p \right)^{\frac{1}{p}} \leqslant C \| f \|_{H^p}.$$

(ii) *For* $2 \leqslant p < \infty$,

$$\|f\|_{H^p} \leqslant C\left(\sum_{n=0}^{\infty}(n+1)^{p-2}|a_n|^p\right)^{\frac{1}{p}}.$$

For functions in H^p and A^p, some asymptotic properties are known for individual coefficients a_n as $n \to \infty$. Proofs can be found in [3] and [5].

Theorem C. *Let* $f(z) = \sum a_n z^n$ *be analytic in the unit disk.*

(i) *For* $0 < p \leqslant 1$,

$$f \in H^p \quad \Longrightarrow \quad a_n = o\big(n^{\frac{1}{p}-1}\big); \qquad f \in A^p \quad \Longrightarrow \quad a_n = o\big(n^{\frac{2}{p}-1}\big).$$

(ii) *For* $1 \leqslant p < \infty$,

$$f \in H^p \quad \Longrightarrow \quad a_n = o(1); \qquad f \in A^p \quad \Longrightarrow \quad a_n = o\big(n^{\frac{1}{p}}\big).$$

In Section 4 we will show that analogous estimates in F_α^p follow easily from Hausdorff–Young and Hardy–Littlewood theorems. We also show that the estimates are optimal.

2. Hausdorff–Young theorem

In order to obtain an analogue of the Hausdorff–Young theorem for Fock spaces, we will make use of a special case of the Riesz–Thorin interpolation theorem (see Zygmund [12, Chapter XII] or Katznelson [9, p. 97]). For $1 \leqslant p \leqslant \infty$, we let $q = p/(p-1)$ denote the conjugate index.

Theorem *(Riesz–Thorin). Let* (X, μ) *and* (Y, ν) *be two measure spaces, and let* $1 \leqslant p_1 < p_2 \leqslant \infty$*, and let* q_1, q_2 *be the respective conjugate indices. Let* T *be a linear operator defined on all simple functions on* X*. Suppose* T *extends to a bounded linear operator from* $L^{p_j}(X, \mu)$ *to* $L^{q_j}(Y, \nu)$ *for* $j = 1, 2$*. Then for any* p *in the interval* $p_1 < p < p_2$*,* T *can be extended a bounded linear operator from* $L^p(X, \mu)$ *to* $L^q(Y, \nu)$.

Theorem 1. *Let* $1 \leqslant p \leqslant \infty$*, let* q *be its conjugate index, and let* $f(z) = \sum a_n z^n$ *be an entire function.*

(i) *For* $1 < p \leqslant 2$,

$$f \in F_\alpha^p \quad \Longrightarrow \quad \sum_{n=1}^{\infty}|a_n|^q\left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} < \infty.$$

For the case $p = 1$*, we have*

$$f \in F_\alpha^1 \quad \Longrightarrow \quad \sup_{n \geqslant 0} |a_n|\sqrt{\frac{n!}{\alpha^n}}\, n^{\frac{1}{4}} < \infty.$$

(ii) *For* $2 \leqslant p \leqslant \infty$,

$$\sum_{n=1}^{\infty} |a_n|^q \left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} < \infty \quad \Longrightarrow \quad f \in F_\alpha^p.$$

Note when $p = q = 2$, the sum becomes $\sum |a_n|^2 n!/\alpha^n$, and we recover the necessary and sufficient condition for membership in F_α^2 given by Eq. (1).

By Stirling's formula, we see that

$$\frac{n!}{\alpha^n} \sim C\left(\frac{n}{\alpha e}\right)^n \sqrt{n}, \quad n \to \infty.$$

We will use these two expressions interchangeably.

Proof. We apply the Riesz–Thorin theorem to the spaces $L^p(\mathbb{C}) = L^p(\mathbb{C}, dA)$ and to the spaces $\ell^p(\mu) = L^p(\mathbb{N}, \mu)$, where μ is the discrete measure with

$$\mu(\{0\}) = 1, \qquad \mu(\{n\}) = \frac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

(ii) Let $2 \leqslant p \leqslant \infty$, so that $1 \leqslant q \leqslant 2$. For a sequence $b = \{b_n\}$ with $b_n = O(\sqrt{n})$, define the operator

$$T(b) = \left(b_0 + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha^n}{n!\sqrt{n}}} b_n z^n\right) e^{-\frac{\alpha}{2}|z|^2}.$$

The sum inside the parentheses converges locally uniformly to an entire function. Every sequence $b \in \ell^q(\mu)$ has the property $b_n = O(\sqrt{n})$, so T is well defined on the spaces $\ell^q(\mu)$. We will show that T is a bounded operator from $\ell^q(\mu)$ to $L^p(\mathbb{C})$.

We first show that T is a bounded operator of $\ell^2(\mu)$ into $L^2(\mathbb{C})$. Using Eq. (1), we have, for every $b \in \ell^2(\mu)$,

$$\begin{aligned}\|T(b)\|_{L^2(\mathbb{C})}^2 &= \int_{\mathbb{C}} \left| \left(b_0 + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha^n}{n!\sqrt{n}}} b_n z^n\right) e^{-\frac{\alpha}{2}|z|^2} \right|^2 dA(z) \\ &= |b_0|^2 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!\sqrt{n}} |b_n|^2 \frac{n!}{\alpha^n} = \|b\|_{\ell^2(\mu)}^2.\end{aligned}$$

Next we show that T maps boundedly from $\ell^1(\mu)$ into $L^\infty(\mathbb{C})$. Note that the real-valued function $x \mapsto x^n e^{-\frac{\alpha}{2}x^2}$ has maximum $(\frac{n}{\alpha e})^{\frac{n}{2}}$ on the interval $[0, \infty)$. Thus by Stirling's formula,

$$\begin{aligned}\|T(b)\|_{L^\infty(\mathbb{C})} &= \left\| \left(b_0 + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha^n}{n!\sqrt{n}}} b_n z^n\right) e^{-\frac{\alpha}{2}|z|^2} \right\|_{L^\infty(\mathbb{C})} \\ &\leqslant |b_0| + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha^n}{n!\sqrt{n}}} |b_n| \left(\frac{n}{\alpha e}\right)^{\frac{n}{2}} \leqslant C\|b\|_{\ell^1(\mu)}.\end{aligned}$$

We now apply the Riesz–Thorin interpolation theorem to conclude that for every $2 \leqslant p < \infty$ and for every $b \in \ell^q(\mu)$,

$$\|T(b)\|_{L^p(\mathbb{C})} \leqslant C\|b\|_{\ell^q(\mu)}.$$

Let $f(z) = \sum a_n z^n$ be an entire function, with $\sum_{n=0}^{\infty} |a_n|^q (\frac{n!}{\alpha^n})^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} < \infty$. Then the sequence $b = \{b_n\}$ defined by

$$b_0 = a_0, \qquad b_n = \sqrt{\frac{n!\sqrt{n}}{\alpha^n}}\, a_n, \quad n = 1, 2, \ldots,$$

is in $\ell^q(\mu)$. Thus by what we have proved,

$$\begin{aligned}
\|f\|_{p,\alpha}^p &= C \int_{\mathbb{C}} \left| \left(\sum_{n=0}^{\infty} a_n z^n \right) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \\
&= C \int_{\mathbb{C}} \left| \left(b_0 + \sum_{n=1}^{\infty} \sqrt{\frac{\alpha^n}{n!\sqrt{n}}}\, b_n z^n \right) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \\
&= C \|T(b)\|_{L^p(\mathbb{C})}^p \leqslant C\|b\|_{\ell^q(\mu)}^p \\
&= C \left(\sum_{n=0}^{\infty} |a_n|^q \left(\frac{n!}{\alpha^n} \right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} \right)^{\frac{p}{q}} < \infty.
\end{aligned}$$

This shows $f \in F_\alpha^p$.

(i) Let $1 \leqslant p \leqslant 2$, so that $2 \leqslant q \leqslant \infty$. Let $Q : L^p(\mathbb{C}) \to L_\alpha^p$ be the operator

$$Qg(z) = g(z) e^{\frac{\alpha}{2}|z|^2}, \quad g \in L^p(\mathbb{C}),$$

and let $P : L_\alpha^p \to F_\alpha^p$ be the Bergman projection

$$Pf(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(\zeta) e^{\alpha z \bar{\zeta}} e^{-\alpha|\zeta|^2}\, dA(\zeta), \quad f \in L_\alpha^p.$$

It is easy to check that for any $0 < p \leqslant \infty$, Q is an isometry from $L^p(\mathbb{C})$ to L_α^p, up to a constant multiple. We also need the fact that for $1 \leqslant p \leqslant \infty$, the Bergman projection P is a bounded operator [8, Theorem 7.1]. Thus for every $g \in L^p(\mathbb{C})$, $1 \leqslant p \leqslant \infty$, the function PQg is in F_α^p, with

$$\|PQg\|_{p,\alpha} \leqslant C\|Qg\|_{p,\alpha} = C\|g\|_{L^p(\mathbb{C})}.$$

For a function $g \in L^p(\mathbb{C})$, define the operator T by $T(g) = b = \{b_n\}$, where

$$b_0 = \int_{\mathbb{C}} PQg(z) e^{-\alpha|z|^2}\, dA(z),$$

$$b_n = \sqrt{\frac{\alpha^n \sqrt{n}}{n!}} \int_{\mathbb{C}} \bar{z}^n PQg(z) e^{-\alpha|z|^2}\, dA(z), \quad n = 1, 2, \ldots.$$

Since PQg is in F_α^p, we see that the integrals are well defined. First we show that T is a bounded operator from $L^1(\mathbb{C})$ into $\ell^\infty(\mu)$. Indeed,

$$|b_0| \leqslant C\|PQg\|_{1,\alpha} \leqslant C\|g\|_{L^1(\mathbb{C})},$$

and for every index $n \geqslant 1$, by Stirling's formula,

$$\begin{aligned}
|b_n| &\leqslant \sqrt{\frac{\alpha^n\sqrt{n}}{n!}} \int_{\mathbb{C}} |z|^n \big|PQg(z)\big| e^{-\alpha|z|^2}\,dA(z) \\
&\leqslant C\sqrt{\frac{\alpha^n\sqrt{n}}{n!}} \Big(\sup_{z\in\mathbb{C}} |z|^n e^{-\frac{\alpha}{2}|z|^2}\Big)\|PQg\|_{1,\alpha} \\
&= C\sqrt{\frac{\alpha^n\sqrt{n}}{n!}} \left(\frac{n}{\alpha e}\right)^{\frac{n}{2}} \|PQg\|_{1,\alpha} \leqslant C\|g\|_{L^1(\mathbb{C})}.
\end{aligned}$$

Next we show that T is a bounded operator from $L^2(\mathbb{C})$ into $\ell^2(\mu)$. Let $g \in L^2(\mathbb{C})$, so that PQg is in F_α^2. Let $PQg = \sum c_m z^m$. For $\{b_n\} = T(g)$, we find

$$\begin{aligned}
b_0 &= \int_{\mathbb{C}} \left(\sum_{n=0}^{\infty} c_m z^m\right) e^{-\alpha|z|^2}\,dA(z) = 2\pi c_0 \int_0^\infty e^{-\alpha r^2} r\,dr = \frac{\pi}{\alpha} c_0, \\
b_n &= \sqrt{\frac{\alpha^n\sqrt{n}}{n!}} \int_{\mathbb{C}} \bar{z}^n \left(\sum_{m=0}^{\infty} c_m z^m\right) e^{-\alpha|z|^2}\,dA(z) \\
&= 2\pi c_n \sqrt{\frac{\alpha^n\sqrt{n}}{n!}} \int_0^\infty r^{2n} e^{-\alpha r^2} r\,dr = \frac{\pi}{\alpha} c_n \sqrt{\frac{n!\sqrt{n}}{\alpha^n}}.
\end{aligned}$$

Thus

$$\big\|T(g)\big\|_{\ell^2(\mu)}^2 = C\sum_{n=0}^{\infty} |c_n|^2 \frac{n!}{\alpha^n} = C\|PQg\|_{2,\alpha}^2 \leqslant C\|g\|_{L^2(\mathbb{C})}^2.$$

We can now apply the Riesz–Thorin theorem to conclude that

$$\big\|T(g)\big\|_{\ell^q(\mu)} \leqslant C\|g\|_{L^p(\mathbb{C})}, \quad g \in L^p(\mathbb{C}),$$

for $1 \leqslant p \leqslant 2$.

Now let $1 < p \leqslant 2$, $f \in F_\alpha^p$, $f(z) = \sum a_n z^n$. Then $f = Qg$ for some $g \in L^p(\mathbb{C})$, with $\|f\|_{p,\alpha} = \|g\|_{L^p(\mathbb{C})}$, and $PQg = Pf = f$. Thus we have $T(g) = \{b_n\}$, where

$$b_0 = \frac{\pi}{\alpha} a_0, \qquad b_n = \frac{\pi}{\alpha} a_n \sqrt{\frac{n!\sqrt{n}}{\alpha^n}}.$$

Thus

$$\left(\sum_{n=0}^{\infty}|a_n|^q\left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}}n^{\frac{q}{4}-\frac{1}{2}}\right)^{\frac{1}{q}}=C\left(\sum_{n=0}^{\infty}|b_n|^q\frac{1}{\sqrt{n}}\right)^{\frac{1}{q}}$$
$$=C\left\|T(g)\right\|_{\ell^q(\mu)}\leqslant C\|g\|_{L^p(\mathbb{C})}=C\|f\|_{p,\alpha}<\infty,$$

whenever $f\in F^p_\alpha$. The case $p=1$ follows similarly. □

3. Hardy–Littlewood theorem

In a recent paper, Blasco and Galbis [1] obtained some result similar to ours. The spaces of entire functions under their consideration are different from ours, but one of their theorem says, in particular, that

$$f\in F^1_2\quad\Longrightarrow\quad\sum_{n=1}^{\infty}|a_n|\sqrt{\frac{n!}{2^n}}\,n^{-\frac{1}{4}}<\infty.$$

Using their method, we can now establish the following generalization.

Theorem 2. *Let $f(z)=a_nz^n$ be an entire function. For $\alpha>0$ and $1\leqslant p<\infty$,*

$$f\in F^p_\alpha\quad\Longrightarrow\quad\sum_{n=0}^{\infty}|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}}n^{-\frac{p}{4}}<\infty.$$

Proof. By Cauchy's integral formula and Hölder's inequality, it is easy to see that

$$|a_n|^pr^{np}\leqslant M_p(r,f)^p$$

for every $n\in\mathbb{N}$ and $r>0$.

Note that

$$\int_{\sqrt{\frac{n}{\alpha}}}^{\sqrt{\frac{n+1}{\alpha}}}r^{np}e^{-\frac{\alpha p}{2}r^2}r\,dr\geqslant C\left(\frac{n}{\alpha e}\right)^{\frac{np}{2}}\geqslant C\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}}n^{-\frac{p}{4}}.$$

Thus for every n,

$$|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}}n^{-\frac{p}{4}}\leqslant C\int_{\sqrt{\frac{n}{\alpha}}}^{\sqrt{\frac{n+1}{\alpha}}}M_p(r,f)^pe^{-\frac{\alpha p}{2}r^2}r\,dr,$$

and the result follows by summing over n. □

The following integral appears frequently in the proof of the theorem.

Lemma 3. *For every positive integer n,*

$$c\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} \leqslant \int_0^\infty r^{np} e^{-\frac{\alpha p}{2}r^2} r\,dr \leqslant C\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}}.$$

Proof. By a change of variable and Stirling's formula,

$$\int_0^\infty r^{np} e^{-\frac{\alpha p}{2}r^2} r\,dr = \left(\frac{2}{\alpha p}\right)^{\frac{np}{2}} \Gamma\left(\frac{np}{2}+1\right) \sim \left(\frac{n}{\alpha e}\right)^{\frac{np}{2}} \sqrt{2\pi n}.$$

The result follows by applying Stirling's formula again, with

$$\left(\frac{n}{\alpha e}\right)^{\frac{np}{2}} \sqrt{2\pi n} \sim C\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}}. \qquad \square$$

Theorem 4. *Let $f(z) = \sum a_n z^n$ be an entire function, and fix $\alpha > 0$.*

(i) *For $0 < p \leqslant 2$,*

$$\sum_{n=0}^\infty |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty \quad \Rightarrow \quad f \in F_\alpha^p \quad \Rightarrow \quad \sum_{n=1}^\infty |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty.$$

(ii) *For $2 \leqslant p < \infty$,*

$$\sum_{n=0}^\infty |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty \quad \Rightarrow \quad f \in F_\alpha^p \quad \Rightarrow \quad \sum_{n=1}^\infty |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty.$$

As in the case for the Hausdorff–Young theorem, the results here include the necessary and sufficient condition for F_α^2 given by Eq. (1). We also see that, curiously, the above necessary conditions are better than Theorem 2 for $p \geqslant \frac{3}{2}$.

Proof. (i) Fix $0 < p \leqslant 2$, and let $f(z) = \sum a_n z^n$ be an entire function. We first suppose that $\sum_{n=0}^\infty |a_n|^p (\frac{n!}{\alpha^n})^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty$. Note that since $\frac{p}{2} \leqslant 1$, we have $(a+b)^{\frac{p}{2}} \leqslant a^{\frac{p}{2}} + b^{\frac{p}{2}}$ for nonnegative a, b; combining this with Hölder's inequality gives

$$\begin{aligned}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta &\leqslant C\left(\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta\right)^{\frac{p}{2}} = C\left(\sum_{n=0}^\infty |a_n|^2 r^{2n}\right)^{\frac{p}{2}} \\ &\leqslant C\sum_{n=0}^\infty |a_n|^p r^{np}.\end{aligned}$$

Multiplying both sides by $e^{-\frac{\alpha p}{2}r^2} r$, integrating from $r = 0$ to ∞, and applying the monotone convergence theorem and Lemma 3, we see that

$$\|f\|_{p,\alpha}^p \leqslant C\sum_{n=0}^{\infty}|a_n|^p\int_0^{\infty} r^{np}e^{-\frac{\alpha}{2}r^2}r\,dr \leqslant C\sum_{n=0}^{\infty}|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} < \infty,$$

which shows $f \in F_\alpha^p$.

To prove the second implication, let $f \in F_\alpha^p$. For every $r > 0$, the dilation $f_r(z) = f(rz)$ is in H^p. Then Theorem B′ shows that for every $r > 0$,

$$\sum_{n=0}^{\infty}(n+1)^{p-2}\left|a_n r^n\right|^p \leqslant C\|f_r\|_{H^p}^p = C\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta.$$

Integrating from $r = 0$ to ∞ with respect to the appropriate weight, we see that

$$\sum_{n=0}^{\infty}(n+1)^{p-2}|a_n|^p\int_0^{\infty} r^{np}e^{-\frac{p}{2}r^2}r\,dr \leqslant C\|f\|_{p,\alpha}^p < \infty.$$

By Lemma 3, the left-hand side is bounded below by

$$c\sum_{n=1}^{\infty}|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}},$$

which gives the desired result.

(ii) Let $2 \leqslant p \leqslant \infty$. We first let $\sum_{n=0}^{\infty}|a_n|^p(\frac{n!}{\alpha^n})^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty$. Theorem B′ shows that for every $r > 0$,

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta \leqslant C\sum_{n=0}^{\infty}(n+1)^{p-2}\left|a_n r^n\right|^p.$$

Integrating and applying Lemma 3, we have

$$\|f\|_{p,\alpha}^p \leqslant C\sum_{n=0}^{\infty}|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} < \infty.$$

For the second implication, let $f \in F_\alpha^p$. Note that since $p \geqslant 2$,

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta \geqslant C\left(\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta\right)^{\frac{p}{2}} = C\left(\sum_{n=0}^{\infty}|a_n|^2 r^{2n}\right)^{\frac{p}{2}}$$
$$\geqslant C\sum_{n=0}^{\infty}|a_n|^p r^{np};$$

so

$$|a_n|^p\left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} \leqslant C\|f\|_{p,\alpha}^p < \infty,$$

as required. □

Remark. For the case $1 < p < 2$, the results in Theorems 1 and 4 give us two necessary conditions for $f \in F^p_\alpha$. We observe that neither condition is better than the other. Let

$$a_n = \left(\frac{\alpha^n}{n!}\right)^{\frac{1}{2}} n^{-\frac{3}{4}+\frac{1}{2p}} (\log n)^{-\frac{1}{p}}.$$

Then

$$\sum_{n=2}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} = \sum_{n=2}^{\infty} \frac{1}{n\log n} = \infty,$$
$$\sum_{n=2}^{\infty} |a_n|^q \left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\frac{q}{p}}} < \infty.$$

On the other hand, the example

$$a_n = \begin{cases} \left(\frac{\alpha^n}{n!}\right)^{\frac{1}{2}}, & n = 2^k \text{ for some positive integer } k, \\ 0, & \text{otherwise,} \end{cases}$$

shows that

$$\sum_{n=1}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{\frac{3p}{4}-\frac{3}{2}} = \sum_{k=1}^{\infty} \left(2^{\frac{3p}{4}-\frac{3}{2}}\right)^k < \infty,$$
$$\sum_{n=1}^{\infty} |a_n|^q \left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} = \sum_{k=1}^{\infty} \left(2^{\frac{q}{4}-\frac{1}{2}}\right)^k = \infty.$$

Similar examples can be constructed to show that for the case $2 < p < \infty$, neither of the two sufficient conditions for $f \in F^p_\alpha$ is better.

4. Coefficient estimates

The *order* of an entire function f is

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_\infty(r, f)}{\log r}.$$

If $0 < \rho < \infty$, the *type* of f is

$$\tau = \limsup_{r\to\infty} \frac{\log M_\infty(r, f)}{r^\rho}.$$

The function f is said to be *of growth* (ρ, τ) if it is of order not exceeding ρ and of type not exceeding τ if of order ρ. We know from classical theory that if an entire function $f(z) = \sum a_n z^n$ is of order ρ and type τ, then

$$e\tau\rho = \limsup_{n\to\infty} n|a_n|^{\frac{\rho}{n}}.$$

(See [2, p. 11].) Now let $0 < p < \infty$, $\alpha > 0$. If $f(z) = \sum a_n z^n$ belongs to F_α^p, then f is of growth $(2, \frac{\alpha}{2})$ [11, Theorem 2], so the coefficients a_n satisfy

$$\limsup_{n\to\infty} n|a_n|^{\frac{2}{n}} \leqslant \alpha e,$$

or

$$a_n = O\left(\left(\frac{\alpha e}{n}\right)^{\frac{n}{2}}\right) = O\left(\sqrt{\frac{\alpha^n}{n!}}\, n^{\frac{1}{4}}\right), \quad n \to \infty.$$

This estimate can be improved by appeal to our Hausdorff–Young and Hardy–Littlewood theorems. The improved result turns out to be best possible in the sense of Duren–Taylor [4].

Corollary 5. *Let* $1 < p < \infty$, $\alpha > 0$. *If* $f \in F_\alpha^p$, *then*

$$a_n = o\left(\sqrt{\frac{\alpha^n}{n!}}\, n^{\frac{1}{4}-\frac{1}{2p}}\right), \quad n \to \infty.$$

For the case $p = 1$, *we have*

$$a_n = O\left(\sqrt{\frac{\alpha^n}{n!}}\, n^{-\frac{1}{4}}\right), \quad n \to \infty.$$

Proof. The case $p = 1$ follows directly from part (i) of Theorem 1. Now let $1 < p \leqslant 2$, let q be its conjugate index, and let $f \in F_\alpha^p$. Again by part (i) of Theorem 1, the convergence of the sum implies that

$$|a_n|^q \left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} = \left(|a_n|\sqrt{\frac{n!}{\alpha^n}}\, n^{-\frac{1}{4}+\frac{1}{2p}}\right)^q \to 0,$$

as $n \to \infty$.

For the case $2 \leqslant p < \infty$, we use the second implication of part (ii) of Theorem 4. For $f \in F_\alpha^p$, the convergence of the sum implies that

$$|a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} = \left(|a_n|\sqrt{\frac{n!}{\alpha^n}}\, n^{-\frac{1}{4}+\frac{1}{2p}}\right)^p \to 0,$$

as $n \to \infty$. $\square$

Theorem 6. *Let* $1 \leqslant p < \infty$, $\alpha > 0$. *For any sequence* $\{\delta_n\}$ *with* $\delta_n \searrow 0$, *there is a function* $f(z) = \sum a_n z^n \in F_\alpha^p$ *such that*

$$a_n \neq O\left(\sqrt{\frac{\alpha^n}{n!}}\, n^{\frac{1}{4}-\frac{1}{2p}}\, \delta_n\right), \quad n \to \infty.$$

Proof. Given $\delta_n \searrow 0$, pick indices $n_0 < n_1 < n_2 < \cdots$ such that $\delta_{n_k} \leqslant 2^{-k}$, $k = 0, 1, 2, \ldots$. Let $f(z) = \sum a_n z^n$, where

$$a_n = \begin{cases} k\sqrt{\frac{\alpha^{n_k}}{(n_k)!}}\, n_k^{\frac{1}{4}-\frac{1}{2p}}\, \delta_{n_k}, & n = n_k \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $a_n \neq O\big(\sqrt{\frac{\alpha^n}{n!}}\, n^{\frac{1}{4}-\frac{1}{2p}}\delta_n\big)$.

For $1 \leqslant p \leqslant 2$, we calculate

$$\sum_{n=0}^{\infty} |a_n|^p \left(\frac{n!}{\alpha^n}\right)^{\frac{p}{2}} n^{-\frac{p}{4}+\frac{1}{2}} \leqslant \sum_{k=0}^{\infty} k\delta_{n_k}^p \leqslant \sum_{k=0}^{\infty} k\left(\frac{1}{2^p}\right)^k < \infty,$$

so by part (i) of Theorem 4, $f \in F_\alpha^p$.

For $2 \leqslant p < \infty$, let q be the conjugate index. Then

$$\sum_{n=1}^{\infty} |a_n|^q \left(\frac{n!}{\alpha^n}\right)^{\frac{q}{2}} n^{\frac{q}{4}-\frac{1}{2}} \leqslant \sum_{k=0}^{\infty} k n_k^{\frac{q}{4}-\frac{1}{2}+\frac{q}{4}-\frac{q}{2p}} \delta_{n_k}^q \leqslant \sum_{k=0}^{\infty} k\left(\frac{1}{2^q}\right)^k < \infty,$$

so by part (ii) of Theorem 1, $f \in F_\alpha^p$. □

Remark. We now can show that for $1 \leqslant p_1 < p_2$, $F_\alpha^{p_1}$ is not the same space as $F_\alpha^{p_2}$. Let $\delta_n = n^{\frac{1}{2p_2}-\frac{1}{2p_1}}$; then the above theorem says we can find a function $f \in F_\alpha^{p_2}$ whose coefficients satisfy

$$a_n \neq O\left(\sqrt{\frac{\alpha^n}{n!}}\, n^{\frac{1}{4}-\frac{1}{2p_1}}\right).$$

By Corollary 5, f is not in $F_\alpha^{p_1}$.

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